Unit 2

7.6 Evaluating improper integral exactly

7.7 Using a comparison test to determine whether an improper integral converges or diverges.

8.1 Areas and Volumes

8.2 Arclength, Solids of Revolution, Footprints.

8.4 Density

1. Book Problem #5.
   Determine whether the integral converges
   $$\int_1^\infty \frac{1}{5x+2}\,dx$$

1. Book Problem #6.
   Determine whether the integral converges
   $$\int_1^{\infty} \frac{1}{(x+2)^2}\,dx$$

- Book Problem # 13. Determine whether the integral converges

$$\int_{-\infty}^{0} \frac{e^z}{1+e^z}\,dz$$

- Book Problem # ##. Determine whether the integral converges

$$\int_0^1 \frac{\ln(x)}{x^2} dy$$

- Book Problem # 30. Determine whether the integral converges
$$\int_1^\infty \frac{1}{x^2-1}dy$$

- Book Problem # 7.
  Determine whether the integral converges
  $\int_0^1 \ln(x) dx$

- Book Problem # 8.
  Determine whether the integral converges
  $\int_0^\infty e^{-\sqrt{x}} dx$

1. Book Problem #12.
   Determine whether the integral converges
   $$\int_1^{\infty} \frac{x}{4+x^2}\,dx$$

1. Book Problem # 15.
   Determine whether the integral converges
   $$\int_0^4 \frac{1}{\sqrt{x}} dx$$

1. Book Problem #23.
   Determine whether the integral converges
   $$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$$

1. Determine whether the integral below converges. (You may calculate the limits by appealing to the dominance of one function over another, or by l'Hopital's rule.)

$$\int_{\pi/4}^{\pi/2} \frac{\sin(s)}{\sqrt{\cos(s)}}\, ds$$

Section 7.6 and 7.7

- Section 7.6 -
  - Direct approach
  - Use definition re-write for finite bounds
  - Find actual antiderivative
  - Take limit as $b \to \infty$
- Section 7.7 -
  - Indirect approach
  - Know what parameters cause certain integrals to converge or diverge.
  - Compare given integral to known integral

A Theorem - Write this down!

- Prove that $\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \leq 1$.

A Theorem - Write this down!

- Prove that $\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \geq 1$.

Concepts for improper integrals:

- Let's consider $\int_1^\infty \frac{1}{x^2}\,dx$.

Concepts for improper integrals:

- Now let's consider $\int_0^1 \frac{1}{x^2}\,dx$.

$$\int_0^1 \frac{x^4+5}{x}\, dx$$

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$$\int_1^\infty \frac{x^4+5}{x}\,dx$$

Quiz for 2-11-16

1. Use integration by parts and l'Hospital's Rule to show that $\displaystyle\int_0^1 \ln(t)\, dt$ converges.

Quiz for 2-16-16

1. Determine whether the integral converges by a comparison:

$$\int_0^\infty \frac{1}{7 + e^{2y}}\, dy.$$

Decide if the integral converges or diverges and prove it using the Comparison Test. If the integral converges, find its value or give a bound on its value.

$$\int_1^{\infty} \frac{x}{x+1}\, dx$$

A Theorem - Write this down!

- Prove that $\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \geq 1$.

Determine for which p the integral $\int_0^1 \frac{-1}{x^p}\,dx$ converges/diverges.

Determine for which p the integral $\int_0^1 \frac{-1}{x^p}\,dx$ converges/diverges.

Answer: it converges for $p < 1$ and diverges for $p \geq 1$ (same behavior as the positive integrand case). But why???

The Comparison Test for $\int_a^\infty f(x)\,dx$ with positive integrand

Assume $f(x)$ is positive. Making a comparison involves two stages:

1. Guess, by looking at the behavior of the integrand for large x, whether the integral converges or not. (This is the "behaves like" principle.)
2. Confirm the guess by comparison with a positive function $g(x)$:
   - If $f(x) \le g(x)$ and $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.
   - If $g(x) \le f(x)$ and $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

Decide if the integral converges or diverges. If it converges, find its value, or give a bound on its value.

$$\int_0^1 (\sin(x))^{-3/2}\, dx$$

Book Problem # 48

- The gamma function is defined for all $x > 0$ by the rule

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

1. Find $\Gamma(1)$ and $\Gamma(2)$
2. Integrate by parts with respect to t to show that, for positive n, $\Gamma(n+1) = n\Gamma(n)$
3. Find a simple expression for $\Gamma(n)$ for positive integers n.

Example Problem

- Use the comparison test to determine if the improper integral converges or diverges.

$$\int_1^\infty \frac{x}{x^3 - 1}\,dx$$

Support your answer by clearly stating which function(s) your are using for your comparison.

Quiz - September 27, 2016

- Determine if the improper integral converges or diverges.

$$\int_5^\infty \frac{x^3}{x^4 - 1}\,dx$$

- Determine if the improper integral converges or diverges.

$$\int_1^2 \frac{1}{\sqrt{y - 1}}\,dy$$

Quiz - September 27, 2016

- Determine if the improper integral converges or diverges.

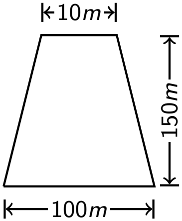
$$\int_5^\infty \frac{x^3}{2x^4 - 1} dx$$

Use a comparison test and show:

Quiz - September 29, 2016

- Sketch the graphs of $y = \sqrt{x}, \quad y = x^3, \quad y = 0.4 \quad y = 0.8$
- Calculate the area of the region enclosed.

- A dam has a rectangular base 1400 meters long and 160 meters wide. Its cross section is shown below. By slicing horizontally, set up and evaluate a definite integral giving the volume of material used to build this dam.



Arc length problems

1. Problem # 63
   Write an integral that represents the arc length of the portion of the graph of $f(x) = -x(x-4)$ that lies above the x-axis. Do not evaluate the integral.

Arc length problems

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   Write an integral that represents the arc length of the portion of the graph of $f(x) = -x(x-4)$ that lies above the x-axis. Do not evaluate the integral.
2. With x and b in meters, a chain hangs in the shape of the catenary for $-b \leq x \leq b$. If the chain is 10 meters long, how far apart are its ends?

Arc length problems

1. Problem # 67
   With x and b in meters, a chain hangs in the shape of the catenary for $-b \leq x \leq b$. If the chain is 10 meters long, how far apart are its ends?

Solids of Revolution

- Let $\mathscr{R}$ be the region bounded by $y = e^x$, the x-axis, and the line $x = 1$. Sketch $\mathscr{R}$.
- Find the volume of the solid obtained by rotating $\mathscr{R}$ about the x-axis.
- Find the volume of the solid obtained by rotating $\mathscr{R}$ about the y-axis.

Some announcements

- Note that we skip section 8.3. Polar coordinates will be covered in Calc III, Math 223.
- Reading for Tuesday: we don't cover center of mass, so you only need to read from the beginning of section 8.4 through the first four examples.

- Practice Test session on Saturday, October 8. Time and location TBA.
- Exam 2 on Tuesday, October 11. Covers sections 7.6, 7.7, 8.1, 8.2, 8.4.

"Footprint" Problems

- The region $\mathscr{R}$, which is bounded by $y = x^2$, $y = 1$ and the y-axis for $x \geq 0$. Find the volume of the resulting solids.
- The solid whose base is the region $\mathscr{R}$ and whose cross-sections perpendicular to the x-axis are semicircles.

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- The solid whose base is the region $\mathscr{R}$ and whose cross-sections perpendicular to the y-axis are equilateral triangles.

"Footprint" Problems

- Let $\mathscr{R}$ be the region bounded by $y = e^x$, and the lines $y = 1$ and $x = 1$ and . Sketch $\mathscr{R}$.
- Find the volume of the solid whose base is $\mathscr{R}$ and whose cross-sections perpendicular to the x-axis are equilateral triangles.

“Footprint” Problems

- Let $\mathscr{R}$ be the region bounded by $y = e^x$, and the lines $y = 1$ and $x = 1$ and . Sketch $\mathscr{R}$.
- Find the volume of the solid whose base is $\mathscr{R}$ and whose cross-sections perpendicular to the x-axis are equilateral triangles.
- Find the volume of the solid whose base is $\mathscr{R}$ and whose cross-sections perpendicular to the y-axis are semi-circles.

Quiz - October 4, 2016

- Set up and integral that represents the volume of the solid obtained by rotating $\mathscr{R}$ about the line $y = -1$.
- Set up and integral that represents the volume of the solid whose base is the region $\mathscr{R}$ and whose cross-section perpendicular to the y-axis are squares.

Solids of Revolution

- Set up, but do not evaluate, an integral that represents the volume obtained when the region in the first quadrant is rotated about the line $y = 3$. The region is bounded by $y = \sqrt[3]{x}$ and $x = 4y$.

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Density

- A rod of length 1 meter has density $\delta(x) = 1 + kx^2$ grams/meter, where k is a positive constant and x is the distance from one end of the rod. Find the mass of the rod.

Density

- The density of oil in a circular oil slick on the surface of the ocean at a distance r km from the center fo the slick is given by the table

r	0	1	2	3	4	5
δ	60	30	20	15	12	10

- Find upper and lower estimates for the total mass of the oil slick.

Density

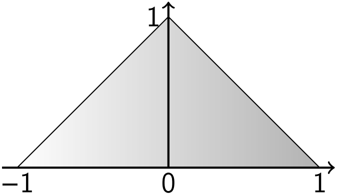
1. Circle City, a typical metropolis, is densely populated near its center, and its population gradually thins out toward the city limits. In fact, its population density is $10{,}000(3 - r)$ people per square mile at distance r miles from the center.
   1.1 Assuming that the population density at the city limits is zero, find the radius of the city.
   1.2 What is the total population of the city?

Density (Book #6)

- 1. Find a Riemann sum which approximates the total mass of a 3×5 rectangular sheet, whose density per unit area at a distance x from one of the sides of length 5 is $\frac{1}{1+x^4}$.
  2. Calculate the mass.

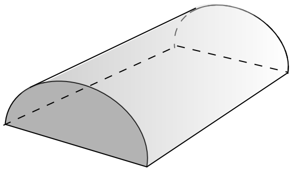
Continued

- Find the total mass of the triangular region below given that the density is $\delta(x) = 1 + x$ grams



Density (Book #19)

- A storage shed is the shape of a half-cylinder of radius r and length l.



Continued

1. Use horizontal slices to find the volume of the shed.
2. Use vertical slices perpendicular to the length to find the volume of the shed.
3. Use vertical slices perpendicular to the width to find the volume of the shed.
4. The shed is filled with sawdust whose density (mass/unit volume) at any point is proportional to the distance of that point from the floor. The constant of proportionality is k. Calculate the total mass of sawdust in the shed.

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Review for Exam 2

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Review Problem - 7.6

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Review Problem 7.7

- For $f(x) = 1/\sqrt[3]{x^3 + 7x^5}\,dx$, describe the behaviour of $f(x)$ as (a) $x \to 0$ and (b) $x \to \infty$.
- Use an appropriate comparison test to determine whether

$$a) \int_0^1 \frac{1}{\sqrt[3]{x^3 + 7x^5}}\,dx \qquad b) \int_1^\infty \frac{1}{\sqrt[3]{x^3 + 7x^5}}\,dx$$

  converges or diverges.
- Clearly state why your comparison converges or diverges.
- Make sure your argument is water-tight!
  (I.e. show that your comparison function is either gre)

Review Problem 7.7

- In Planck's Radiation Law, we encounter the integral $\int_1^\infty \frac{dx}{x^5(e^{1/x}-1)}$.

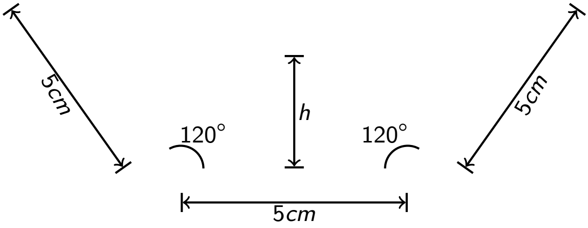
- Use the comparison test to determine whether the integral converges or diverges?

Review Problem 8.1

- Write down definite integrals which give the area underneath the curve $y = e^{2x}$, above the x-axis, and between $x = -1$ and $x = 5$ by slicing as instructed. **Do not evaluate the integrals.** (Hint: Make a reasonable sketch of the region.)
  1. Vertical slices.
  2. Horizontal slices.

Review Problem 8.1

- A 100 cm long gutter is made of three strips of metal, each 5 cm. wide; the figure below shows a cross section.



1. Find the volume of water in the gutter when the depth is h cm.

Continued

1. Find the volume of water in the gutter when the depth is h cm.
2. What is the maximum value of h?
3. What is the maximum volume of water that the gutter can hold?
4. If the gutter is filled with half the maximum volume of water, is the depth larger or smaller than half of the answer to part(b)? Explain how you can answer without any calculation.
5. Find the depth of the water when the gutter contains half the maximum possible volume.

Review 8.1

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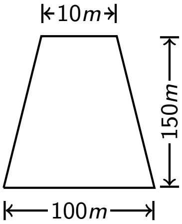

Review Problem 8.2

- Arclength
- Solids of Revolution
- "Footprint" Problems

Review Problem 8.2

- Find a curve whose arc length is

$$\int_3^6 \sqrt{1+e^{6t}}\,dt$$

Review Problem 8.2

- A cylindrical centrifuge of radius 1 m and height 2 m is filled with water to a depth of 1 m. As the centrifuge accelerates, the water level rises along the wall and drops in the center; the cross section is a parabola. (See pg 485 for figures.)
- Find the equation of the parabola in terms of h, the depth of water at its lowest point
- As the centrifuge rotates faster and faster, either water will be spilled out the top, or the bottom of the centrifuge will be exposed. Which happens first?