

Fall 2016

Math 129 - 01 Calculus II

Exam # 2, Sections 7.6 - 8.4

Time allocated: 60 minutes

Calculators: Permitted

Instructor's Name: Colin Clark

Student's Name (please print): \_\_\_\_\_

By signing my name below, I agree that I am following all rules and regulations set forth by the Code of Academic Integrity. Furthermore, I agree that I am following all rules set by my instructor and by the course policy for this exam. This includes ensuring that all calculator programs have been deleted.

Signature: \_\_\_\_\_ Date: \_\_\_\_\_

1. Determine if the following improper integrals converge or diverge. Mark your answer with an X.

Suppose  $f$  is a continuous function and that  $0 < f(x) < 1/x^2$  on  $0 \leq x \leq 1$ .

$$\int_0^1 f(x) dx \quad \text{_____ converges} \quad \text{_____ diverges} \quad \text{_____ impossible to tell}$$

Suppose  $g$  is a continuous function and that  $\int_1^\infty g(x) dx$  converges.

$$\int_1^\infty g(2x) dx \quad \text{_____ converges} \quad \text{_____ diverges} \quad \text{_____ impossible to tell}$$

$$\int_0^1 \frac{e^{-x}}{x^4} dx \quad \text{_____ converges} \quad \text{_____ diverges} \quad \text{_____ impossible to tell}$$

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2. Which of the following gives the arc length of the curve  $f(x) = \sqrt{x}$  from  $x = 1$  to  $7$ ? Mark your answer with an X.

$$\text{_____ } \int_1^7 \sqrt{1 + \frac{1}{4x}} dx$$

$$\text{_____ } \int_1^7 \sqrt{\frac{1}{2\sqrt{x}}} dx$$

$$\text{_____ } \int_1^7 \sqrt{1 + \frac{1}{2x}} dx$$

$$\text{_____ } \int_1^7 \sqrt{1 + \frac{1}{2\sqrt{x}}} dx$$

3. Evaluate the integral below or prove that it diverges. The point of this problem is to demonstrate your understanding of improper integrals by treating every step with care, using correct notation, and fully justifying your work.

$$\int_0^{\pi/2} \tan(x) dx$$

4. Figure 1 shows  $y = x$  and  $y = \sin(x)$  for your convenience.  
 Figure 2. shows the graphs of  $y = 1/\sin(x)$  and will be used in part(b).

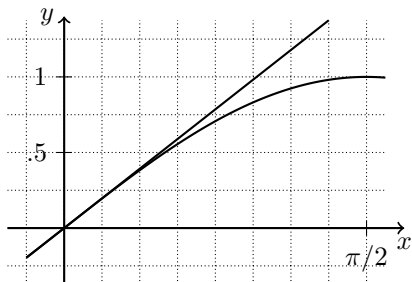


Figure 1: Graphs of  $y = x$  and  $y = \sin(x)$ .

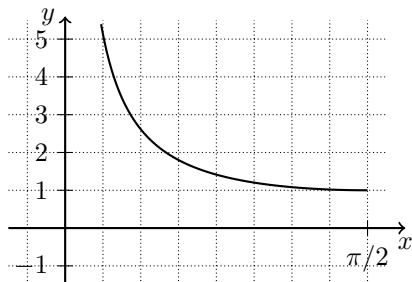


Figure 2: Graph of  $y = f(x)$ . Sketch  $y = g(x)$ .

We wish to determine whether the integral  $\int_0^{\pi/2} \frac{1}{\sin(x)} dx$  converges or diverges.

- (a) Let  $f(x) = 1/\sin(x)$ . Find the most appropriate  $g(x)$  to use as a comparison.  
 (That is:  $f(x)$  behaves like \_\_\_\_\_ as  $x \rightarrow 0$ ?)

- (b) For your chosen  $g(x)$ , make an argument whether  $f(x) \leq g(x)$  or  $f(x) \geq g(x)$ .  
 Sketch  $y = g(x)$  the figure 2.

- (c) For your chosen  $g(x)$ , state whether  $\int_0^{\pi/2} g(x) dx$  converges or diverges.  
 Clearly state why this is the case.

- (d) Clearly state how your answers to (a), (b) and (c) show that the integral  $\int_0^{\pi/2} \frac{1}{\sin(x)} dx$  either converges or diverges.

5. Consider the region  $\mathcal{R}$  bounded between the  $x$ -axis, and the graphs  $y = 4 - x/2$ , and  $y = \sqrt{x}$ . We wish to find the volume of the solid whose base is the region  $\mathcal{R}$  and whose cross-sections perpendicular to the  $y$ -axis are squares. [ 15 points ]

(a) Make an accurate sketch of the graphs on the figure below and shade the region  $\mathcal{R}$ .

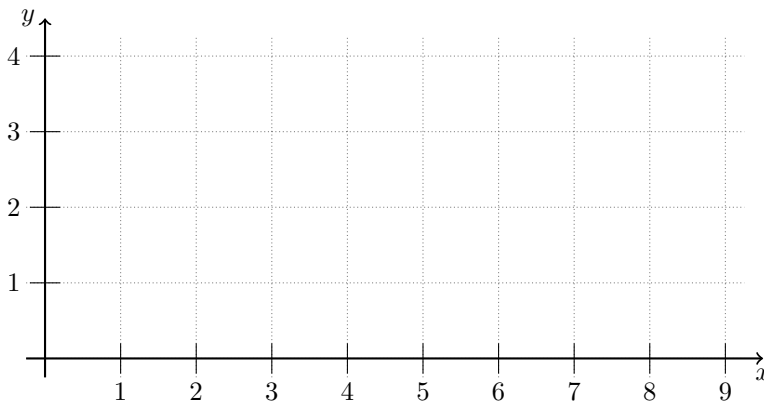
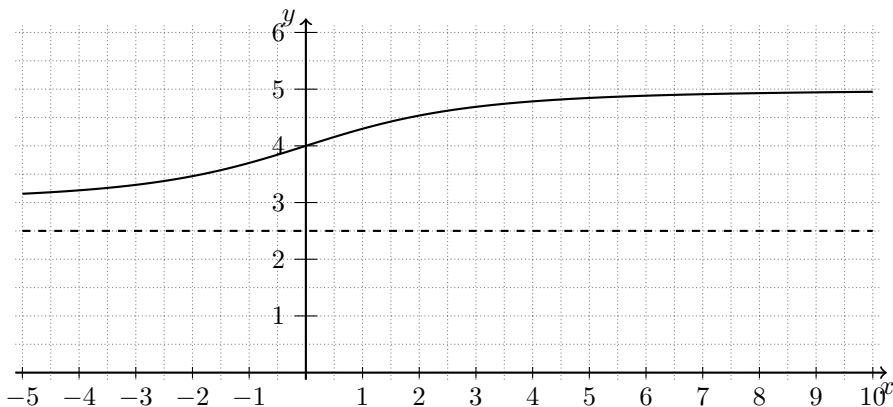


Figure 3: Graphs of  $y = 4 - x/2$  and  $y = \sqrt{x}$

(b) Make a **large** sketch of a slice of the solid and (c) give an expression for the volume of the slice. *Clearly indicate all relevant dimensions on your sketch, and simplify the expression for the volume.*

(d) Set up, but do not evaluate, an integral that gives the volume of the solid.

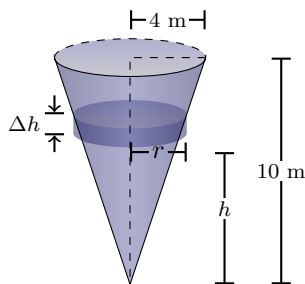
6. The curve  $y = 4 + x/\sqrt{x^2 + 10}$  for  $-5 \leq x \leq 10$  is rotated about the line  $y = 2.5$  to produce a solid. [ 15 points ]



- (a) Make a **large** sketch of a slice of the solid and (c) give an expression for the volume of the slice. *Clearly indicate all relevant dimensions on your sketch, and simplify the expression for the volume.*

- (d) Set up, but do not evaluate, an integral that gives the volume of the solid.

7. An industrial disposal container is in the shape of an inverted cone of height 10m and maximum radius 4m.



The cone is filled with toxic waste whose density varies according to the height above the vertex,

$$\delta(h) = 2e^{-h/10} \text{ kg / m}^3$$

Find the area and the mass of each slice, and use this information to set up an integral that represents the total mass of toxic waste in the disposal container. **Do not evaluate the integral.** [ 15 points ]

## A Short Table of Indefinite Integrals

### I. Basic Functions

1.  $\int x^n dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1$
2.  $\int \frac{1}{x} dx = \ln|x| + C$
3.  $\int a^x dx = \frac{1}{\ln a} a^x + C$
4.  $\int \ln x dx = x \ln x - x + C, \quad x > 0$
5.  $\int \sin x dx = -\cos x + C$
6.  $\int \cos x dx = \sin x + C$
7.  $\int \tan x dx = -\ln|\cos x| + C$

### II. Products of $e^x$ , $\cos x$ , and $\sin x$

8.  $\int e^{ax} \sin(bx) dx = \frac{1}{a^2 + b^2} e^{ax} [a \sin(bx) - b \cos(bx)] + C$
9.  $\int e^{ax} \cos(bx) dx = \frac{1}{a^2 + b^2} e^{ax} [a \cos(bx) + b \sin(bx)] + C$
10.  $\int \sin(ax) \sin(bx) dx = \frac{1}{b^2 - a^2} [a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + C, \quad a \neq b$
11.  $\int \cos(ax) \cos(bx) dx = \frac{1}{b^2 - a^2} [b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + C, \quad a \neq b$
12.  $\int \sin(ax) \cos(bx) dx = \frac{1}{b^2 - a^2} [b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + C, \quad a \neq b$

### III. Product of Polynomial $p(x)$ with $\ln x$ , $e^x$ , $\cos x$ , $\sin x$

13.  $\int x^n \ln x dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C, \quad n \neq -1, \quad x > 0$
14.  $\int p(x) e^{ax} dx = \frac{1}{a} p(x) e^{ax} - \frac{1}{a} \int p'(x) e^{ax} dx$   
 $= \frac{1}{a} p(x) e^{ax} - \frac{1}{a^2} p''(x) e^{ax} + \frac{1}{a^3} p'''(x) e^{ax} - \dots$   
 (+ - + - ...) (signs alternate)
15.  $\int p(x) \sin ax dx = -\frac{1}{a} p(x) \cos ax + \frac{1}{a} \int p'(x) \cos ax dx$   
 $= -\frac{1}{a} p(x) \cos ax + \frac{1}{a^2} p'(x) \sin ax + \frac{1}{a^3} p''(x) \cos ax - \dots$   
 (- + - + ...) (signs alternate in pairs after first term)
16.  $\int p(x) \cos ax dx = \frac{1}{a} p(x) \sin ax - \frac{1}{a} \int p'(x) \sin ax dx$   
 $= \frac{1}{a} p(x) \sin ax + \frac{1}{a^2} p'(x) \cos ax - \frac{1}{a^3} p''(x) \sin ax - \dots$   
 (+ - + - ...) (signs alternate in pairs)

### IV. Integer Powers of $\sin x$ and $\cos x$

17.  $\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx, \quad n \text{ positive}$
18.  $\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx, \quad n \text{ positive}$
19.  $\int \frac{1}{\sin^m x} dx = \frac{-1}{m-1} \frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\sin^{m-2} x} dx, \quad m \neq 1, m \text{ positive}$
20.  $\int \frac{1}{\sin x} dx = \frac{1}{2} \ln \left| \frac{(\cos x) - 1}{(\cos x) + 1} \right| + C$
21.  $\int \frac{1}{\cos^m x} dx = \frac{1}{m-1} \frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\cos^{m-2} x} dx, \quad m \neq 1, m \text{ positive}$
22.  $\int \frac{1}{\cos x} dx = \frac{1}{2} \ln \left| \frac{(\sin x) + 1}{(\sin x) - 1} \right| + C$
23.  $\int \sin^m x \cos^n x dx$ : If  $m$  is odd, let  $w = \cos x$ . If  $n$  is odd, let  $w = \sin x$ . If both  $m$  and  $n$  are even and non-negative, convert all to  $\sin x$  or all to  $\cos x$  (using  $\sin^2 x + \cos^2 x = 1$ ), and use IV-17 or IV-18. If  $m$  and  $n$  are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21. The case in which both  $m$  and  $n$  are even and negative is omitted.

### V. Quadratic in the Denominator

24.  $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C, \quad a \neq 0$
25.  $\int \frac{bx+c}{x^2+a^2} dx = \frac{b}{2} \ln|x^2+a^2| + \frac{c}{a} \arctan \frac{x}{a} + C, \quad a \neq 0$
26.  $\int \frac{1}{(x-a)(x-b)} dx = \frac{1}{a-b} (\ln|x-a| - \ln|x-b|) + C, \quad a \neq b$
27.  $\int \frac{cx+d}{(x-a)(x-b)} dx = \frac{1}{a-b} [(ac+d) \ln|x-a| - (bc+d) \ln|x-b|] + C, \quad a \neq b$

### VI. Integrands Involving $\sqrt{a^2 + x^2}$ , $\sqrt{a^2 - x^2}$ , $\sqrt{x^2 - a^2}$ , $a > 0$

28.  $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$
29.  $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C$
30.  $\int \sqrt{a^2 \pm x^2} dx = \frac{1}{2} \left( x \sqrt{a^2 \pm x^2} + a^2 \int \frac{1}{\sqrt{a^2 \pm x^2}} dx \right) + C$
31.  $\int \sqrt{x^2 - a^2} dx = \frac{1}{2} \left( x \sqrt{x^2 - a^2} - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} dx \right) + C$