

Chapter 1

Linear Functions

SEC. 1.1: SLOPES AND EQUATIONS OF LINES

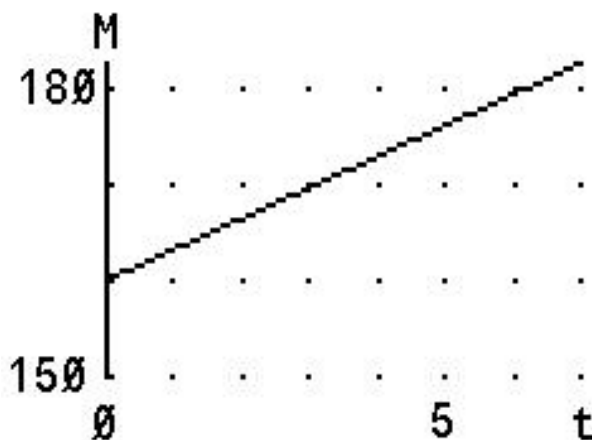
Date _____

Lines play a very important role in Calculus where we will be approximating complicated functions with lines. We need to be experts with lines to do well in Calculus. In this section, we review slope and equations of lines.

SLOPE OF A LINE: *The slope of a nonvertical line is defined as the vertical change (the “rise”) over the horizontal change (the “run”) as one travels along the line. In symbols, taking two different points (x_1, y_1) and (x_2, y_2) on the line, the slope is*

$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example: World milk production rose at an approximately constant rate between 1996 and 2003 as shown in the following graph:



where M is in million tons and t is the years since 1996. Estimate the slope and interpret it in terms of milk production.

See Video - <http://www.showme.com/sh/?h=Pa3eVmq>

CLICKER QUESTION 1: Find the slope of the line through the following pair of points

$(-2, 11)$ and $(3, -4)$.

The slope is

- (A) Less than -2 (B) Between -2 and 0 (C) Between 0 and 2
(D) More than 2 (E) Undefined

See Video - <http://www.showme.com/sh/?h=hFGm4lk>

It will be helpful to recall the following facts about intercepts.

INTERCEPTS:

x-intercepts The points where a graph touches the x -axis. If we have an equation, we can find them by setting $y = 0$.

y-intercepts The points where a graph touches the y -axis. If we have an equation, we can find them by setting $x = 0$.

In addition, you should be familiar with the following forms of the equation of a line.

EQUATIONS OF A LINE:

Slope-Intercept Form If a line has slope m and y -intercept b , then

$$y = mx + b.$$

Point-Slope Form If a line has slope m and passes through the point (x_1, y_1) , then

$$y - y_1 = m(x - x_1).$$

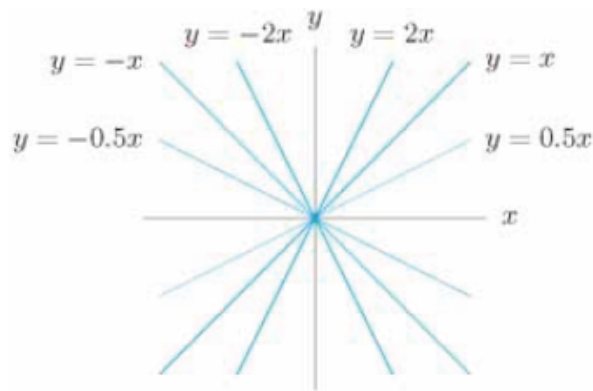
Vertical Line The line with undefined slope and x -intercept k has the form

$$x = k.$$

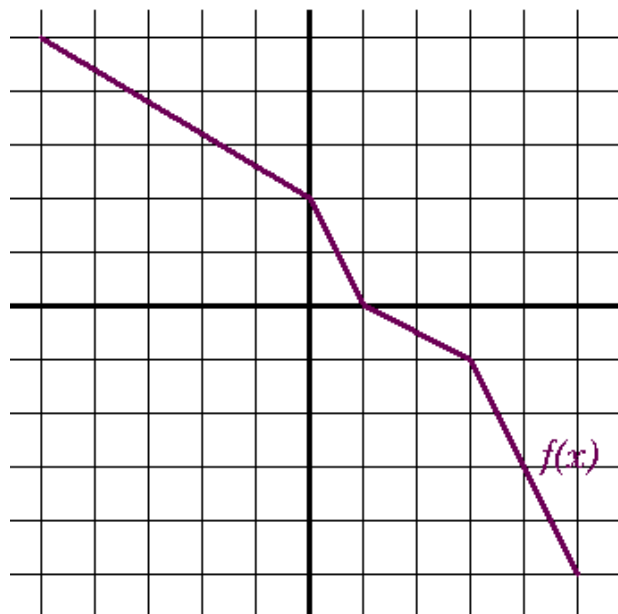
Horizontal Line The line with zero slope and y -intercept k has the form

$$y = k.$$

Below we see lines going through the origin with different slopes. You will want to have a “feel” for approximating the slope of a line. Be able to identify immediately whether the slope of a line is positive or negative. Is the slope close to zero? or one? between zero and one? larger than one? Is the line straight up and down (undefined slope)?



Example: Estimate the slope of the function graphed below at the following points:



$$x = -3$$

$$x = \frac{1}{2}$$

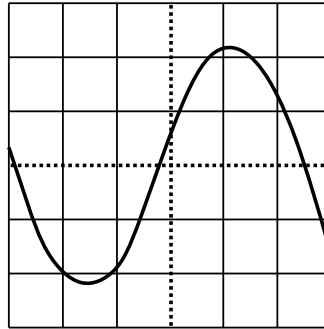
$$x = 2$$

$$x = 4$$

See Video - <http://www.showme.com/sh/?h=tR4pmsa>

In the next chapter, we will ask the same question, but with non-linear graphs, such as:

Example: Estimate the slope of the function graphed below at the following points:



$x = -2$

$x = 0$

$x = \frac{1}{2}$

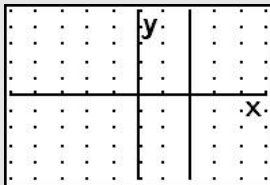
$x = 2$

CLICKER QUESTION 2:

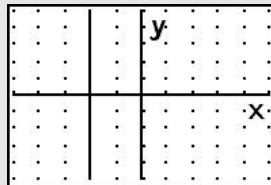
(i) Graph the equation $x = -2$ in the window $[-5, 5] \times [-5, 5]$.

Choose the correct graph

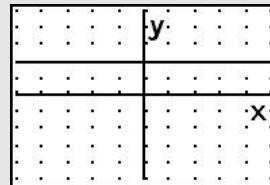
(A)



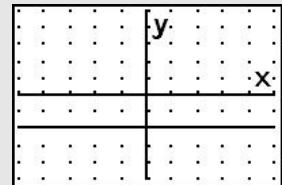
(B)



(C)



(D)



(ii) Find the slope of the line $x = -2$.

The slope is

(A) Less than -2 (B) Between -2 and 0 (C) Between 0 and 2

(D) More than 2 (E) Undefined

See Video - <http://www.showme.com/sh/?h=73wQbPk>

CLICKER QUESTION 3: Find an equation of the line that contains the following pair of points

$(-2, 6)$ and $(7, 6)$.

(A) $y = \frac{4}{3}x + \frac{26}{3}$ (B) $y = \frac{12}{5}x + \frac{54}{5}$ (C) $x = -2$

(D) $y = 6$ (E) None of these

See Video - <http://www.showme.com/sh/?h=BciS6ng>

Example:

(a) Find the equation of the line that goes through the point $(4, 5)$ and has a slope of -3 .

See Video - <http://www.showme.com/sh/?h=nLD3eGe>

(b) Graph the line.

When you include a graph on a homework assignment, or an exam, be sure to include the following:

- A set of axes - and make sure they are labeled with the correct variables! (And label the positive axis with the positive variable).
- A scale on each axis.
- The graph itself should be labeled - especially if you have more than one graph on a set of axes.

Example: World soybean production was 136.5 million tons in 1980 and 214 million tons in 2005, and has been increasing at an approximately constant rate.

- (a) Determine a linear equation that approximates world soybean production, P , in millions tons, in terms of t , the number of years since 1980.

See Video - <http://www.showme.com/sh/?h=N1HeMwi>

- (b) Graph the linear equation determined above.

- (c) Using units, interpret the slope in terms of soybean production.

- (d) Using units, interpret the vertical intercept in terms of soybean production.

- (e) According to the linear model, what is the predicted world soybean production in 2015?

- (f) According to the linear model, when is soybean production predicted to reach 250 million tons?

SEC. 1.2: LINEAR FUNCTIONS AND APPLICATIONS

Date _____

The topic of a function is one of the major concepts in any College Algebra course. Recall that it is used to represent the dependence of one quantity upon another. A complete understanding of functions, and function notation, is necessary to be successful in any Calculus course.

LINEAR FUNCTION: *When two variables are related by a linear equation, with y in terms of x , we say that y is a linear function of x , and can write*

$$y = f(x) = mx + b.$$

We call x the independent variable and y the dependent variable.

Recall the function notation $f(x)$ is read “ f of x ”, and that f is the name of the function.

Example: Let $f(x) = 3x - 7$. Find the following:

(a) $f(5)$

See Video - <http://www.showme.com/sh/?h=Uz0lExM>

(b) $f(-4)$

(c) $f(c + 3)$

(d) Find x such that $f(x) = 14$.

BREAK-EVEN ANALYSIS: The cost function, $C(q)$, gives the total cost of producing a quantity q of some good. If $C(q)$ is a linear cost function (so $C(q) = mq + b$), then

- the fixed costs are represented by the C -intercept (b),
- and the marginal cost is represented by the slope (m).

The revenue function, $R(q)$, gives the total revenue received by a firm from selling a quantity, q , of some good. The profit, $P(q)$, is revenue minus cost. The number of units for which revenue equals cost is the break-even quantity.

CLICKER QUESTION 4: The Blackbox Calculator Company spends \$7500 to produce 110 calculators, achieving a marginal cost of \$55. Find the linear cost function.

- (A) $C(x) = 110x + 55$ (B) $C(x) = 110x + 7500$ (C) $C(x) = 55x + 7500$
(D) $C(x) = 55x + 1450$ (E) None of these

See Video - <http://www.showme.com/sh/?h=DCPRAQK>

Example: The manager of a restaurant found that the cost to produce 300 cups of coffee is \$52.05, while the cost to produce 500 cups is \$78.45. Assume the cost $C(x)$ is a linear function of x , the number of cups produced.

(a) Find a formula for $C(x)$.

See Video - <http://www.showme.com/sh/?h=sA2CbxY>

(b) What is the fixed cost?

(c) Find the total cost of producing 1100 cups.

(d) Find the marginal cost of a cup of coffee.

See Video - <http://www.showme.com/sh/?h=xkeDzvc>

(e) What does the marginal cost of a cup of coffee mean to the manager?

Example: A company has a cost function $C(q) = 4000 + 2q$ dollars and revenue function $R(q) = 10q$ dollars.

(a) What are the fixed costs for the company?

See Video - <http://www.showme.com/sh/?h=iF9I6iW>

(b) What is the marginal cost?

(c) What price is the company charging for its product?

(d) Graph $C(q)$ and $R(q)$ on the same axes and label the break even quantity, q_0 .

See Video - <http://www.showme.com/sh/?h=5QAjTge>



(e) Explain how you know the company makes a profit if the quantity produced is greater than q_0 .

(f) Find the profit function $P(q)$.

(g) Find the break-even quantity q_0 .

CLICKER QUESTION 5: Erin sells hand-knitted pillow covers on internet. Her marginal cost to produce one pillow cover is \$8.50. Her total cost to produce 30 pillow covers is \$394.65, and she sells them for \$19.95 each. How many pillow covers must she produce and sell in order to break even?

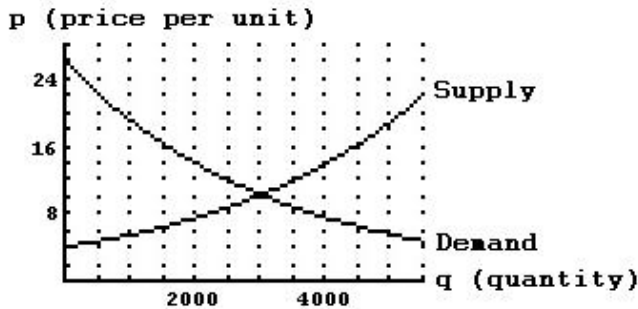
- (A) less than 5 (B) between 5 and 10 (C) between 10 and 15
 (D) between 15 and 20 (E) more than 20

See Video - <http://www.showme.com/sh/?h=XLoLzSi>

SUPPLY AND DEMAND

The *supply curve*, for a given item, relates the quantity, q , of the item that manufacturers are willing to make to the price, p , for which the item can be sold. The *demand curve* relates the quantity, q , of an item demanded by consumers to the price, p , of the item. If we plot the supply and demand curves on the same axes, the graphs cross at the *equilibrium point*.

Example: The graph below shows supply and demand for a product



- (a) At the equilibrium point for this product, $p =$ _____ $q =$ _____
- (b) The price $p = 16$ is _____ the equilibrium price.
 At this price, how many items are suppliers willing to produce? _____
 How many items do consumers want to buy? _____
 Use your answers to these questions to explain why, if prices are _____ the equilibrium price, the market tends to push prices _____ (toward the equilibrium).
- (c) The price $p = 6$ is _____ the equilibrium price.
 At this price, how many items are suppliers willing to produce? _____
 How many items do consumers want to buy? _____
 Use your answers to these questions to explain why, if prices are _____ the equilibrium price, the market tends to push prices _____ (toward the equilibrium).

See Video - <http://www.showme.com/sh/?h=xBuUwD2>

CLICKER QUESTION 6: Suppose the supply and demand functions for a certain model of a wristwatch are given by

$$p = D(q) = 32 - 1.25q \quad \text{and} \quad p = S(q) = 0.75q,$$

where p is the price (in dollars) and q is the quantity in hundreds. Find the equilibrium quantity.

(i) The value of q is

(A) less than 10 (B) between 10 and 15 (C) between 15 and 20

(D) between 20 and 25 (E) more than 25

(ii) The equilibrium quantity is

(A) less than 1000 (B) between 1000 and 1500

(C) between 1500 and 2000 (D) between 2000 and 2500

(E) more than 2500

See Video - <http://www.showme.com/sh/?h=dpeRWMa>

Chapter 2

Nonlinear Functions

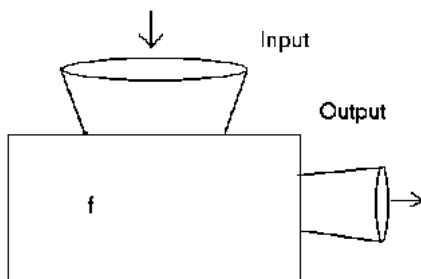
SEC. 2.1: PROPERTIES OF FUNCTIONS

Date _____

Note that this definition of function is slightly different than what is used in the textbook - but the idea is the same.

FUNCTION: *A function is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.*

A function is just an input-output device. In other words, it is some “machine” that when you put something in, it chugs something out.



Often we think of functions represented only by equations, since this is what is stressed in algebra classes. However, we can use equations, graphs, tables, or words to represent functions.

DETERMINING FUNCTIONS

To test your understanding of the concept of a function, you will often be asked to determine if a given relationship is a function. The following fact is useful if given a graph.

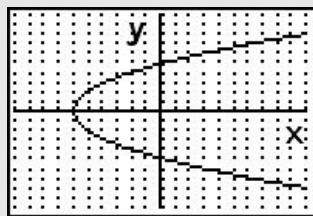
VERTICAL LINE TEST: *If a vertical line intersects a graph in more than one point, the graph is not the graph of a function.*

Example: Draw a graph of a function.

Draw a graph of a non-function.

See Video - <http://www.showme.com/sh/?h=3Z2OG8G>

CLICKER QUESTION 7: Does the following graph represent a function?



(A) Yes (B) No (C) It cannot be determined

See Video - <http://www.showme.com/sh/?h=09VIdXc>

DOMAINS OF FUNCTIONS

In a typical algebra class, we often get into the topic of “implied domain” - when we get a question such as: Find the domain of the function

$$f(x) = \frac{\sqrt{x-2}}{x^2 - x - 12}.$$

In such a question, they want you to understand that certain mathematical operations are forbidden in the real numbers - such as dividing by zero or having a negative under the square root symbol - and numbers that cause such forbidden operations need to be removed from the domain (all other numbers are fine). So we would need

$$x - 2 \geq 0 \text{ and } x^2 - x - 12 \neq 0.$$

Solving each of these, we get

$$x \geq 2 \text{ and } x \neq -3, 4$$

so the domain is all numbers greater than or equal to 2 except 4. We can represent this using interval notation: $[2, 4) \cup (4, \infty)$.

Example: Find the domain of the function $g(x) = \sqrt{36 - x^2}$.

See Video - <http://www.showme.com/sh/?h=J13Lq08>

CLICKER QUESTION 8: Find the domain of the function

$$f(x) = \frac{3}{x^2 - 4}.$$

- (A) $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ (B) $(-\infty, \infty)$ (C) $(-\infty, 0) \cup (0, \infty)$
(D) $(2, \infty)$ (E) None of these

See Video - <http://www.showme.com/sh/?h=i3q7ttg>

For us, domain questions are usually *context questions*.

Example: Suppose the revenue function is $R(q) = -q^2 + 12q$ (where q represents the quantity sold). What is the domain of $R(q)$?

Of course, a mathematician might answer this question and say: *the domain is all real numbers*. However, does it make sense to talk about negative values of q ? Or how about values of q which make $R(q)$ negative? So the domain will be the values of q where the model makes sense. So what is the (reasonable) domain of the revenue function?

See Video - <http://www.showme.com/sh/?h=HhEFGEK>

Ultimately, we will combine the mathematical and contextual approaches.

Example: Suppose the demand function for sugar is given by

$$p = D(q) = 10 + 2\sqrt{100 - 5q},$$

where p is the price per pound and q is the quantity in thousands of pounds. What is the domain of this function?

See Video - <http://www.showme.com/sh/?h=wxurN2m>

Example: For the function defined as

$$f(x) = x^2 - 3x,$$

find:

(a) $f(x + h)$

See Video - <http://www.showme.com/sh/?h=ojecpOq>

(b) $f(x + h) - f(x)$

(c) $\frac{f(x + h) - f(x)}{h}$

PIECEWISE DEFINED FUNCTIONS

Example: Below is the 2010 U.S. Rate Schedule for single persons, showing tax owed for given income levels:

10% of the first \$8,375
15% of the next \$25,625
25% of the next \$48,400
28% of the next \$89,450
33% of the next \$201,800
35% of anything over \$373,650

Let $T(x)$ represent the tax owed on an income of x dollars.

(a) Find $T(50,000)$.

See Video - <http://www.showme.com/sh/?h=GBSz0ZE>

(b) Find $T(500,000)$.

See Video - <http://www.showme.com/sh/?h=gJSEbrc>

(c) Graph $T(x)$.

See Video - <http://www.showme.com/sh/?h=rUgHNnk>

CLICKER QUESTION 9: Let

$$f(x) = \begin{cases} \frac{2x+3}{x-2} & \text{if } x \neq 2 \\ 2 & \text{if } x = 2 \end{cases}$$

(i) Find the value of $f(-\frac{1}{2})$.

- (A) less than -1 (B) between -1 and 1 (C) between 1 and 3
(D) between 3 and 5 (E) more than 5

(ii) Find the value of x such that $f(x) = 1$.

- (A) less than -20 (B) between -20 and -10 (C) between -10 and 0
(D) between 0 and 10 (E) more than 10

See Video - <http://www.showme.com/sh/?h=4OYEI08>

SEC. 2.2: QUADRATIC FUNCTIONS;
TRANSLATION AND REFLECTION

Date _____

QUADRATIC FUNCTION: A quadratic function is defined by

$$f(x) = ax^2 + bx + c,$$

where a , b , and c are real numbers, with $a \neq 0$.

Recall that the graph of a quadratic function is a parabola (U-shaped). The minimum point (if it opens up)/maximum point (if it opens down) is of particular interest to us - especially when we model a real-world application with a quadratic function.

GRAPH OF A QUADRATIC FUNCTION: The graph of a quadratic function

$$f(x) = ax^2 + bx + c,$$

is a parabola that opens up when $a > 0$ and opens down when $a < 0$. The value of $|a|$ determines how wide (if $|a| < 1$) or narrow (if $|a| > 1$) the parabola is. The minimum point when $a > 0$ or maximum point when $a < 0$ is called the **vertex** and has coordinates

$$\text{Vertex} = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right) \right).$$

The axis of symmetry is $x = -b/(2a)$, the y -intercept is at $(0, c)$, and the y -intercept(s) (if they exist) occur where $f(x) = 0$.

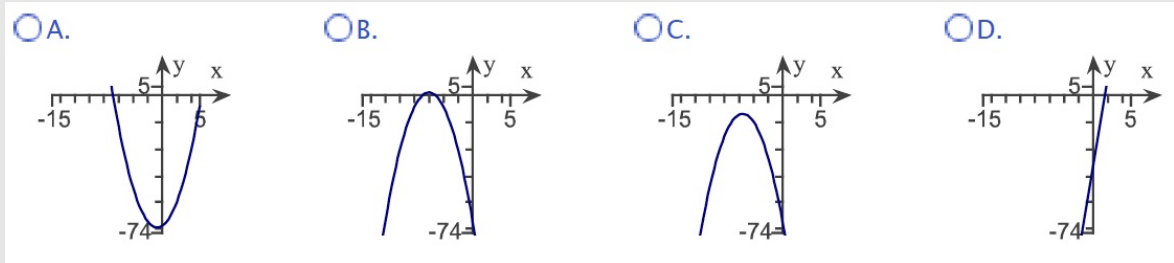
If the function is given in the form

$$f(x) = a(x - h)^2 + k,$$

then the vertex is at (h, k) (and a still determines which way it opens as described above).

CLICKER QUESTION 10: Graph the function $y = -2x^2 - 24x - 70$ and determine its vertex, x -intercepts, and y -intercept.

(i) Choose the correct graph



(ii) The vertex is:

- (A) $(-2, -24)$ (B) $(2, 24)$ (C) $(-6, 2)$ (D) $(6, -2)$ (E) None of these

(iii) The smaller of the x -intercepts is:

- (A) less than -10 (B) between -10 and -8 (C) between -8 and -6
 (D) between -6 and -4 (E) more than -4

(iv) The y -intercept is:

- (A) less than -65 (B) between -65 and -55 (C) between -55 and -45
 (D) between -45 and -35 (E) more than -35

See Video - <http://www.showme.com/sh/?h=ZAzDMNk>

Example: Let $C(x) = \frac{3}{2}x + 3$ be the cost to produce x batches of widgets, and let $R(x) = -\frac{x^2}{2} + 5x$ be the revenue function (each in thousands of dollars).

(a) Graph both functions.

See Video - <http://www.showme.com/sh/?h=9fd9uK0>

(b) Find the minimum break-even quantity.

(c) Find the maximum revenue.

See Video - <http://www.showme.com/sh/?h=9ANP9oO>

(d) Find the maximum profit.

Example: A charter flight charges a fare of \$200 per person plus \$4 per person for each unsold seat on the plane (i.e. the fare changes depending on how many unsold seats there are). The plane holds 100 passengers. Let x represent the number of unsold seats.

- (a) Find an expression for the total revenue received for the flight $R(x)$. (*Hint:* Multiply the number of people flying, $100 - x$, by the price per ticket.)

See Video - <http://www.showme.com/sh/?h=ABS9nyS>

- (b) Graph the expression from above.

- (c) Find the number of unsold seats that will produce the maximum revenue.

See Video - <http://www.showme.com/sh/?h=18cZRCa>

- (d) What is the maximum revenue?

- (e) Some managers might be concerned about empty seats, arguing that it doesn't make economic sense to leave any seats empty. Explain why this is not necessarily so.

Note that if the fare doesn't change - we expect to model revenue with a linear function. However, when the fare does change in such a way as above, we expect to model revenue with a quadratic function.

Example: The manager of an 80-unit apartment complex is trying to decide what rent to charge. Experience has shown that at a rent of \$800, all the units will be full. On the average, one additional unit will remain vacant for each \$25 increase in rent.

(a) Let x represent the number of \$25 increases. Find an expression for the rent of each apartment.

See Video - <http://www.showme.com/sh/?h=TOL2Fsm>

(b) Find an expression for the number of apartments rented.

(c) Find an expression for the total revenue from all rented apartments.

(d) What value of x leads to maximum revenue?

See Video - <http://www.showme.com/sh/?h=BIfrWL2>

(e) What is the maximum revenue?

POLYNOMIALS

POLYNOMIAL FUNCTION: A polynomial function of degree n is defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_1 x + a_0,$$

where $a_n, a_{n-1}, \dots, a_1, a_0$ are real numbers, called **coefficients**, with $a_n \neq 0$. The number a_n is called the **leading coefficient**.

Example: Consider the function $f(x) = -3x^4 + 2x^2 + 17$.

(a) Is it a polynomial function?

See Video - <http://www.showme.com/sh/?h=QMBWHDc>

(b) What is the degree of f ?

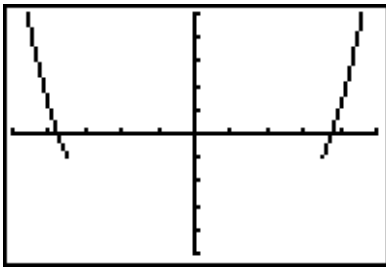
(c) What is the leading coefficient of f ?

(d) What is the basic shape of f ? How many turning points do we expect f to have? What is the end behavior of f (what happens to f as x gets large in the positive and negative directions)?

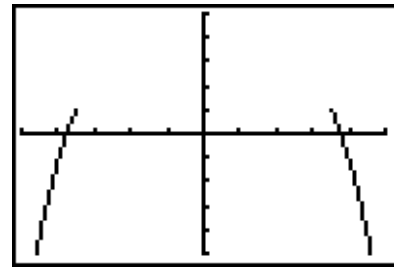
PROPERTIES OF POLYNOMIAL FUNCTIONS:

1. A polynomial function of degree n can have at most $n - 1$ turning points. Conversely, if the graph of a polynomial function has n turning points, it must have degree at least $n + 1$.
2. In the graph of a polynomial function of even degree, both ends go up or both ends go down. For a polynomial function of odd degree, one end goes up and one end goes down.
3. If the graph goes up as x becomes a large positive number, the leading coefficient must be positive. If the graph goes down as x becomes a large positive number, the leading coefficient is negative.

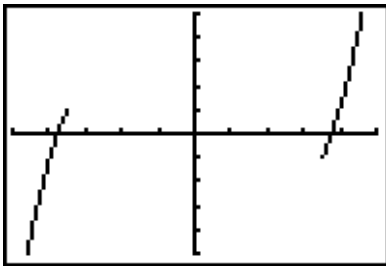
The graphs below illustrate the end behavior of polynomials.



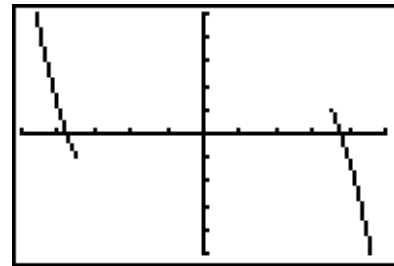
Even Degree, Positive L.C.



Even Degree, Negative L.C.

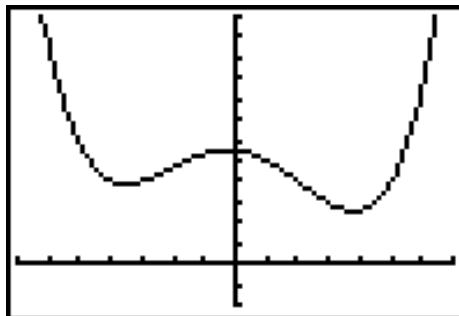


Odd Degree, Positive L.C.



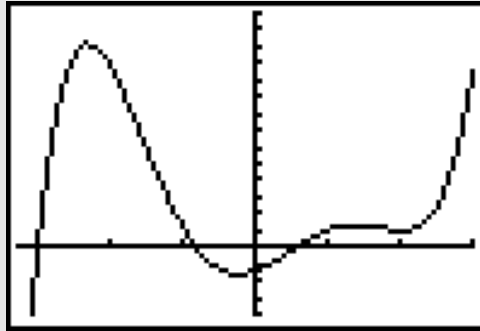
Odd Degree, Negative L.C.

Example: Pictured below is the graph of a polynomial. Give the possible degree of the polynomial, and give the sign (positive or negative) of the leading coefficient.



See Video - <http://www.showme.com/sh/?h=v1TCBrk>

CLICKER QUESTION 11: The graph below is the graph of a polynomial. Give the possible degree of the polynomial, and give the sign (positive or negative) of the leading coefficient.



- (A) degree 4 with negative leading coefficient
- (B) degree 4 with positive leading coefficient
- (C) degree 5 with negative leading coefficient
- (D) degree 5 with positive leading coefficient
- (E) None of the above

See Video - <http://www.showme.com/sh/?h=8fL9648>

Example: This example has to do with the *Laffer curve* (originated by the economist Arthur Laffer). The curve relates the tax rates to the government revenue, and there is much disagreement about the specifics of this curve. One expects that increasing the tax rate would increase the government revenue - however, most agree that it is possible to increase the government revenue by reducing the tax rate. (The idea that reducing the tax rate would stimulate the economy which would yield more government revenue).

A function that might describe the entire Laffer curve is

$$R(x) = x(100 - x)(x^2 + 500),$$

where R is the government revenue in hundreds of thousands of dollars from a tax rate of x percent, with the function valid for $0 \leq x \leq 100$.

(a) Find the revenue from the following tax rates.

10%

40%

80%

See Video - <http://www.showme.com/sh/?h=xvasdg8>

(b) Graph the Laffer curve.

(c) Find the maximum government revenue. (We can do this using our calculator.)

See Video - http://math.arizona.edu/~math116/LafferCurve_pg40.avi

(d) What tax rate gives the maximum government revenue? (We can do this using our calculator.)

Note that finding the maximum value of a function is a very important skill. The calculator has a built in function to do this (which you should know how to use), but it is so important that we will develop another method.

RATIONAL FUNCTIONS

RATIONAL FUNCTION: A rational function is defined by

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Often a distinguishing feature of rational functions is asymptotes.

ASYMPTOTES:

- If a function gets larger and larger in magnitude without bound as x approaches the number k , then the line $x = k$ is a **vertical asymptote**.
- If the values of y approach a number k as $|x|$ gets larger and larger, the line $y = k$ is a **horizontal asymptote**.

In your algebra course, you discussed how to find asymptotes of a rational function.

FINDING ASYMPTOTES AND HOLES: Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function.

Vertical Asymptotes and Holes: Reduce the rational function so that the numerator and denominator have no factors in common. Let $\frac{p_0(x)}{q_0(x)}$ denote the reduced rational function.

- The zeros of $q_0(x)$ are vertical asymptotes of $f(x)$.
- Any zeros of $q(x)$ that are not zeros of $q_0(x)$ are holes in the graph of $f(x)$.

Horizontal Asymptotes: Determine the degree of both the numerator $p(x)$ and the denominator $q(x)$.

- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of $f(x)$.
- If the degree of $p(x)$ is equal to the degree of $q(x)$, then $y = \frac{a}{b}$ is the horizontal asymptotes of $f(x)$ - where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$.
- If the degree of $p(x)$ is more than the degree of $q(x)$, then $f(x)$ does not have a horizontal asymptote.

CLICKER QUESTION 12: Consider the function $g(x) = \frac{5 - 4x}{3x + 11}$.

(i) What is the vertical asymptote of $g(x)$?

- (A) $x = -\frac{4}{3}$ (B) $x = -\frac{11}{3}$ (C) $y = \frac{5}{4}$
(D) $y = -\frac{4}{3}$ (E) There is no vertical asymptote

(ii) What is the horizontal asymptote of $g(x)$?

- (A) $y = 0$ (B) $y = \frac{5}{3}$ (C) $y = -\frac{4}{3}$
(D) $y = -\frac{4}{11}$ (E) There is no horizontal asymptote

See Video - <http://www.showme.com/sh/?h=wAHzkWW>

Example: Graph $f(x) = \frac{3x^2 + 6x - 9}{x^2 - x - 12}$. Identify any holes and asymptotes.

See Video - <http://www.showme.com/sh/?h=71Gan20>

Example: Suppose a cost-benefit model is given by

$$C(x) = \frac{6.5x}{100 - x}$$

where C is the cost in thousands of dollars of removing x percent of a certain pollutant.

(a) Find the cost of removing each percent of pollutants:

50%

80%

95%

See Video -<http://www.showme.com/sh/?h=hyKAJ28>

(b) Graph the function.

(c) According to this model, is it possible to remove all the pollutant?

SEC. 2.4: EXPONENTIAL FUNCTIONS

Date _____

MODELING WITH EXPONENTIAL FUNCTIONS

After the linear function, the exponential function is the most important function for the application of mathematics to real world problems.

Example: Two situations are described below. For each situation, complete the table of values of the sales, S , of the company at the indicated number of years t , and then find a function $S(t)$ for the sales of the company in terms of the years t .

- The company has sales of \$10000 in year 0, and grows at a rate of \$500 per year.

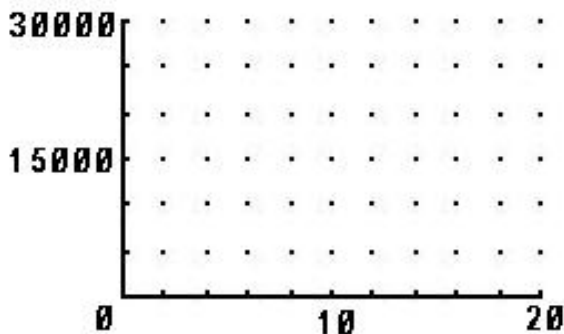
| | | | | | | |
|-------|---|---|---|---|---|----------|
| Years | 0 | 1 | 2 | 3 | 4 | Formula |
| Sales | | | | | | $S(t) =$ |

See Video - <http://www.showme.com/sh/?h=O2nOP2G>

- The company has sales of \$10000 in year 0, and grows at a rate of 5% per year.

| | | | | | | |
|-------|---|---|---|---|---|----------|
| Years | 0 | 1 | 2 | 3 | 4 | Formula |
| Sales | | | | | | $S(t) =$ |

- The function we have discovered in part 1 is a _____ function.
- The function we have discovered in part 2 is an _____ function.
 - The function tells that the initial value of the function (when $t = 0$) was _____.
 - The rate of growth was _____ (or _____ %)
 - The growth factor was _____.
- Graph the two functions on your calculator to see how they compare.



Example: Look at the values in the following table.

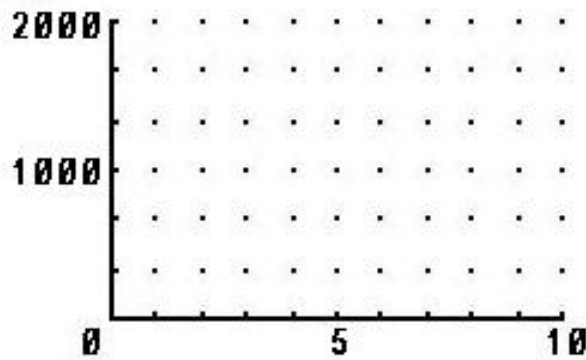
| | | | | | |
|--------|------|------|-----|-----|-----|
| x | 0 | 1 | 2 | 3 | 4 |
| $f(x)$ | 1875 | 1125 | 675 | 405 | 243 |

(a) Do the values in the table represent a linear function? Why or why not?

See Video - <http://www.showme.com/sh/?h=7b2AYAS>

(b) Do the values in the table represent an exponential function? Why or why not?

(c) Find a function for the values in the table. $f(x) =$ _____. Sketch a graph of f on your calculator. Why does this graph look different than the exponential function in the earlier example?



See Video - <http://www.showme.com/sh/?h=itUDh7A>

(d) This is an example of *exponential decay*, not *exponential growth*.

(a) We can tell this is exponential decay because the base of the exponential function is _____.

(b) The rate of decay is _____ (or _____ %)

(c) The decay factor is _____.

THE GENERAL EXPONENTIAL FUNCTION: We say that P is an exponential function of t with base a if:

$$P = P_0 a^t,$$

where P_0 is the initial value (when $t = 0$), and a is the base (or growth factor or decay factor), that is, the factor that P changes when t increases by 1. We have **exponential growth** when $a > 1$, and **exponential decay** when $0 < a < 1$.

The factor a is given by:

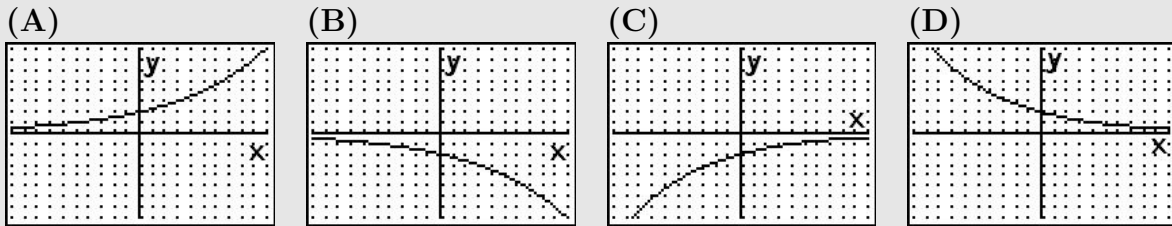
$$a = 1 + r$$

where r is the decimal representation of the percent rate of change. If $r > 0$, then we have exponential growth and if $r < 0$, we have exponential decay.

Example: Joe bought a new car for \$30,000 but it depreciates by 15% per year. Write a function for the value of the car over time. Identify the decay factor.

See Video - <http://www.showme.com/sh/?h=CANRChE>

CLICKER QUESTION 13: Graph the equation $y = -2(1.15)^x$ in the window $[-10, 10] \times [-8, 8]$. Choose the correct graph



See Video - <http://www.showme.com/sh/?h=hZf77D6>

Example: The company that produces Cliff Notes (the abridged versions of classic literature) was started in 1958 with \$4000 and sold in 1998 for \$14,000,000. Find the annual percent increase in the value of this company over the 40 years, and write a function for the value of the Cliff Notes company in terms of the years since 1958.

See Video - <http://www.showme.com/sh/?h=PTzwNX6>

SOLVING EXPONENTIAL EQUATIONS AND e

Now is a good time to review properties of exponents.

PROPERTIES OF EXPONENTS: For any integers m and n , and any real numbers a and b for which the following exist:

$$1) a^m \cdot a^n = a^{m+n}$$

$$2) \frac{a^m}{a^n} = a^{m-n}$$

$$3) (a^m)^n = a^{mn}$$

$$4) (ab)^m = a^m \cdot b^m$$

$$5) \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$6) a^0 = 1$$

$$7) a^{-n} = \frac{1}{a^n}$$

$$8) a^{1/n} = \sqrt[n]{a}$$

$$9) a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Example: Solve for x :

$$3^{2x} = 27$$

See Video - <http://www.showme.com/sh/?h=RcOOmhM>

Example: Solve for x :

$$5^{3-4x} = \frac{1}{625}$$

You may remember, from your previous algebra course, that there is a very important base when studying exponential functions. In fact, this is the most commonly used base when using exponential functions.

This base is the number e , which is approximately equal to 2.7182818.... This number is programmed on your calculator - make sure you use the e^x button or the e button, not the E button.

The base e is used so often that it is called the *natural base*, and the function $y = e^x$ is called the *natural exponential function*. (We will get an indication of “why” when we apply calculus to the exponential function!)

CLICKER QUESTION 14: Solve for x :

$$e^{7x} = (e^{4-x})^3.$$

x is

- (A) less than 2 (B) between 2 and 3 (C) between 3 and 4
(D) between 4 and 5 (E) more than 5

See Video - <http://www.showme.com/sh/?h=myKypHs>

INTEREST

The most important application of exponential function is that of compound interest.

SIMPLE INTEREST: *The product of the principal P , rate r , and time t gives simple interest, I :*

$$I = Prt.$$

COMPOUND INTEREST: *If P dollars is invested at a yearly rate of interest r per year, compounded m times per year for t years, the compound amount is*

$$A = P \left(1 + \frac{r}{m}\right)^{tm} \text{ dollars.}$$

CONTINUOUS COMPOUNDING: *If a deposit of P dollars is invested at a rate of interest r compounded continuously for t years, the compound amount is*

$$A = Pe^{rt} \text{ dollars.}$$

Example: Find the interest earned on \$10,000 invested for 5 years at 4% interest compounded as follows:

(a) Quarterly

See Video - <http://www.showme.com/sh/?h=jW7PqLo>

(b) Monthly

(c) Continuously

Example: Leigh Jacks plans to invest \$500 into a money market account. Find the interest rate that is needed for the money to grow to \$1200 in 14 years if the interest is compounded quarterly.

See Video - <http://www.showme.com/sh/?h=3kB2zA0>

CLICKER QUESTION 15: Assuming continuous compounding, what will it cost to buy a \$50 item in 3 years at an inflation rate of 4%?

It will cost

- (A) less than \$52 (B) between \$52 and \$54 (C) between \$54 and \$56
(D) between \$56 and \$58 (E) more than \$58

See Video - <http://www.showme.com/sh/?h=g3tqrY0>

SEC. 2.5: LOGARITHMIC FUNCTIONS

Date _____

In the last section, we talked about compound interest. What if we encountered the following problem:

Example: May Klingman invests \$15,000 in an account paying 7% per year, compounded quarterly. How many years are required for the compound amount to at least double?

Using the formula from the last section, we get an equation:

How do we solve such an equation algebraically? In this equation the variable t is an exponent, and therefore, it is called an **exponential equation**. In order to solve it, we must be able to “undo” exponential functions - which is what logarithms were invented for.

LOGARITHM: For $a > 0$, $a \neq 1$, and $x > 0$,

$$y = \log_a x \text{ means } a^y = x.$$

In other words, $f(x) = \log_a x$ is the inverse to $g(x) = a^x$.

Two special cases of logarithms:

Common Logarithm This is the base-10 logarithm, usually denoted simply by \log .

$$\log(x) = \log_{10}(x)$$

Natural Logarithm This is the base- e logarithm, usually denoted simply by \ln .

$$\ln(x) = \log_e(x)$$

Using this definition, evaluate the following (without a calculator).

1. $\ln 1 =$ _____ because _____

2. $\log 10 =$ _____ because _____

3. $\ln\left(\frac{1}{e}\right) =$ _____ because _____

4. $\log \sqrt{10} =$ _____ because _____

5. $\ln(-10) =$ _____ because _____

6. $\log 0 =$ _____ because _____

See Video - <http://www.showme.com/sh/?h=4J31o5w>

PROPERTIES OF LOGARITHMS: Let x and y be any positive real numbers and r be any real number. If a and b are positive real numbers, $a \neq 1$, $b \neq 1$, then

1. $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$

2. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

3. $\log_a(x^r) = r \log_a(x)$

4. $\log_a(a) = 1$

5. $\log_a(1) = 0$

6. $\log_a(a^x) = x$

7. $a^{\log_a(x)} = x$

8. $\log_a(x) = \frac{\log(x)}{\log(a)} = \frac{\ln(x)}{\ln(a)}$ (Change of Base Formula)

CLICKER QUESTION 16: Use natural logarithms to evaluate $\log_7 71$.

$\log_7 71$ is

- (A) less than 2 (B) between 2 and 3 (C) between 3 and 4
(D) between 4 and 5 (E) more than 5

See Video - <http://www.showme.com/sh/?h=7LW2WQa>

We will use the properties to solve the equation from this section's introductory example.

Example: Solve

$$30,000 = 15,000 \left(1 + \frac{.07}{4}\right)^{4t}$$

See Video - <http://www.showme.com/sh/?h=vGDRRM8>

CLICKER QUESTION 17: Solve the equation

$$7^{x+1} = 132$$

x is

- (A) less than 1 (B) between 1 and 2 (C) between 2 and 3
(D) between 3 and 4 (E) more than 4

See Video - <http://www.showme.com/sh/?h=O7M6gCW>

Example: In the last section, the function that we developed to solve the Cliff Notes problem was $P(t) = 4000(1.226)^t$, where t represented the number of years after 1958. In what year did the value of the Cliff Notes company reach 5 million dollars?

See Video - <http://www.showme.com/sh/?h=rnzsSQ4>

CLICKER QUESTION 18: A college student invests \$7,000 in an account paying 9% per year compounded annually. In how many years will the amount at least quadruple?

(A) 17.5 years (B) 19.8 years (C) 16.1 years

(D) 18.7 years (E) None of these

See Video - <http://www.showme.com/sh/?h=61gvwWW>

SEC. 2.6: APPLICATIONS: GROWTH AND DECAY;
MATHEMATICS OF FINANCE

Date _____

EXPONENTIAL FUNCTIONS WITH BASE e

In Section 2.4, we saw that an exponential function with base “ a ” has a form:

THE GENERAL EXPONENTIAL FUNCTION: We say that P is an exponential function of t with base a if:

$$P = P_0 a^t,$$

where P_0 is the initial value (when $t = 0$), and a is the base (or growth factor or decay factor), that is, the factor that P changes when t increases by 1. We have **exponential growth** when $a > 1$, and **exponential decay** when $0 < a < 1$.

The factor a is given by:

$$a = 1 + r$$

where r is the decimal representation of the percent rate of change. If $r > 0$, then we have exponential growth and if $r < 0$, we have exponential decay.

This model is generally used when the growth rate is measured “per unit of time”; for example, a population is increasing 5% per year.

If, however, a quantity is increasing (or decreasing) at a **continuous growth rate**, the function is usually written using e as its base.

THE CONTINUOUS EXPONENTIAL FUNCTION: If a quantity is said to grow or decay continuously at a rate of k per unit time, then the amount present at time t is given by

$$y = y_0 e^{kt},$$

where y_0 is the initial value (when $t = 0$), and k is the growth constant (if $k > 0$), or decay constant (if $k < 0$).

There will be times we will want to convert a function of the form $P = P_0 a^t$ to the form $y = y_0 e^{kt}$.

1. Let $a = e^k$. Solve for k . $k =$ _____

See Video - <http://www.showme.com/sh/?h=X5BEH3Y>

2. If $a > 1$, then k _____, and if $0 < a < 1$, then k _____.

Example: Suppose a piece a property was worth \$6000 in 2009 and its value is growing by 2.1% per year.

(a) Write a function that models the value of the property over time.

See Video - <http://www.showme.com/sh/?h=wtqsnjc>

(b) Convert this function to the form $y = y_0e^{kt}$. Determine the continuous growth rate.

Example: Suppose a Treasury Bill (T-Bill) is worth \$500 in 2005 and has a continuous growth rate of 1.7%.

(a) Write a function that models the value of the T-Bill over time.

See Video - <http://www.showme.com/sh/?h=FIpec88>

(b) Convert this function to the form $P = P_0a^t$. Determine the annual growth rate.

EFFECTIVE RATE OF COMPOUND INTEREST

It is often very difficult to know the exact implications of compounding interest. For example, which is a better investment: 6.7% compounded quarterly or 6.6% compounded continuously? To help answer such a question, we use effective rate - which is the equivalent simple interest for a year.

Example: Vanessa Adams invests \$1000 in an account with 6% interest, compounded monthly. After one year, the account has grown to

$$A = 1000 \left(1 + \frac{0.06}{12} \right)^{(12)(1)} = \$1061.68$$

So after one year, Vanessa has made \$61.68 in interest, or

$$\frac{\$61.68}{\$1000} = 6.168\% \text{ of the original investment.}$$

The 6.168% is called the effective rate, and the 6% is called the stated rate.

EFFECTIVE RATE OF COMPOUND INTEREST: *If r is the annual stated rate, the effective rate of interest is*

1. $r_E = \left(1 + \frac{r}{m} \right)^m - 1$ when compounded m times per year.
2. $r_E = e^r - 1$ when compounded continuously.

Example: Christine O'Brien, who is self-employed, wants to invest \$60,000 in a pension plan. One investment offers 8% compounded quarterly. Another offers 7.75% compounded continuously. What is the effective rate in each case?

See Video - <http://www.showme.com/sh/?h=w0iohJA>

CLICKER QUESTION 19: Find the effective rate corresponding to 6% compounded daily.
The effective rate is

- (A) less than 5.9% (B) between 5.9% and 6.1% (C) between 6.1% and 6.3%
(D) between 6.3% and 6.5% (E) more than 6.5%

See Video - <http://www.showme.com/sh/?h=UYuSW6i>

PRESENT VALUE

Example: Frank Steek must make a balloon payment of \$20,000 in 4 years. Find the present value of the payment if it includes annual interest of 6.5% compounded continuously.

See Video - <http://www.showme.com/sh/?h=3QPhPeq>

CLICKER QUESTION 20: *Southwest Dry Cleaners believes that it will need new equipment in 9 years. The equipment will cost \$26,000. What lump sum should be invested today at 8% compounded semiannually, to yield \$26,000? Round to the nearest cent.*

- (A) \$18,172.05 (B) \$12,834.33 (C) \$21,521.24
(D) \$18,353.77 (E) None of these

See Video - <http://www.showme.com/sh/?h=PmrmxOq>

LIMITED GROWTH FUNCTIONS

Example: Sales of a new model of digital camera are approximated by

$$S(x) = 5000 - 4000e^{-x},$$

where x represents the number of years that the digital camera has been on the market, and $S(x)$ represents sales in thousands of dollars.

(a) Find the sales in year 0.

See Video - <http://www.showme.com/sh/?h=6NNVdx2>

(b) When will sales reach \$4,500,000.

(c) Find the limit on sales.

As we leave the first two chapters, we end our short review of pre-calculus material. Make sure you have a good handle on the following classes of functions

- Linear
- Quadratic
- Exponential
- Polynomial
- Rational
- Logarithmic

Make sure you are comfortable with the properties of each - if you see a graph of points, which class of functions would fit it best?

Chapter 3

The Derivative

SEC. 3.1: LIMITS

Date _____

FINDING LIMITS NUMERICALLY AND GRAPHICALLY

Here is where we begin the concepts of calculus. We start with the idea of a limit. It is important to understand that a limit is really a process. It may seem very simple at first - but there is a subtlety to it that takes a bit of time to grasp (especially when we use it to define the operations of calculus).

LIMIT OF A FUNCTION: *Let f be a function and let a and L be real numbers. If*

- 1. as x takes values closer and closer (but not equal) to a on both sides of a , the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to L ; and*
- 2. the value of $f(x)$ can be made as close to L as desired by taking values of x close enough to a ;*

then L is the limit of $f(x)$ as x approaches a , written

$$\lim_{x \rightarrow a} f(x) = L.$$

Note - in the definition above - the limit is about what happens *near* a point, not what happens *at* the point.

Example: Let $f(x) = \frac{x^2 + 6x - 27}{x^2 - 9}$. As x gets closer and closer to 3, does $f(x)$ get closer and closer to some value L ? Lets see.

Evaluate:

$$f(2.9) = \underline{\hspace{2cm}} \quad f(2.99) = \underline{\hspace{2cm}} \quad f(2.999) = \underline{\hspace{2cm}}$$
$$f(3.1) = \underline{\hspace{2cm}} \quad f(3.01) = \underline{\hspace{2cm}} \quad f(3.001) = \underline{\hspace{2cm}}$$

So, the $\lim_{x \rightarrow 3} \frac{x^2 + 6x - 27}{x^2 - 9} =$

See Video - http://math.arizona.edu/~math116/Limit_pg66.avi

CLICKER QUESTION 21: Use the table of values to estimate $\lim_{x \rightarrow 4} f(x)$.

| | | | | | | | | |
|--------|-----|------|-------|--------|----|--------|-------|------|
| x | 3.9 | 3.99 | 3.999 | 3.9999 | 4 | 4.0001 | 4.001 | 4.01 |
| $f(x)$ | 7.9 | 7.99 | 7.999 | 7.9999 | 24 | 8.0001 | 8.001 | 8.01 |

(A) 4 (B) 8 (C) 7.9 (D) 24 (E) None of these

See Video - <http://www.showme.com/sh/?h=VqMaTCa>

CLICKER QUESTION 22: Let

$$f(x) = \begin{cases} e^{2x} + 3 & \text{if } x \neq 0 \\ -5 & \text{if } x = 0 \end{cases}$$

(i) Find $f(0)$.

(A) -5 (B) 0 (C) 3 (D) 4 (E) None of these

(ii) Find $\lim_{x \rightarrow 0} f(x)$.

(A) -5 (B) 0 (C) 3 (D) 4 (E) None of these

See Video - <http://www.showme.com/sh/?h=4hrpMcS>

Example: Find $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 2x}$. Let's look at this problem graphically. Graph the function $Y_1 = \frac{x - 2}{x^2 - 2x}$ on your calculator.

See Video - http://math.arizona.edu/~math116/Limit_pg68.avi

What do you notice about the graph *around* $x = 2$ and *at* $x = 2$?

Evaluate the function close to $x = 2$ to determine the limit.

| | | | | | | |
|----------------|-----|-------|---------|---------|-------|-----|
| x | 1.9 | 1.999 | 1.99999 | 2.00001 | 2.001 | 2.1 |
| Y ₁ | | | | | | |

Conclusion: $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 2x} =$

CLICKER QUESTION 23: Use your calculator to find

$$\lim_{x \rightarrow 0} (1 + x)^{1/x}.$$

Round your answer to 3 decimal places.

- (A) 3.142 (B) 2.718 (C) 1.414 (D) 1.732 (E) None of these

Do you recognize the number in the answer above? Some books use the limit above to define that number.

FINDING LIMITS ALGEBRAICALLY

The key to finding limits algebraically is to be able to manipulate the limit so we can evaluate it by “direct evaluation”. Here are the rules for limits.

RULES FOR LIMITS: Let a , A , and B be real numbers, and let f and g be functions such that

$$\lim_{x \rightarrow a} f(x) = A \text{ and } \lim_{x \rightarrow a} g(x) = B$$

1. If k is a constant, then $\lim_{x \rightarrow a} k = k$ and

$$\lim_{x \rightarrow a} [k \cdot f(x)] = k \cdot \lim_{x \rightarrow a} f(x) = k \cdot A.$$

2. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = A \pm B.$

(The limit of a sum or difference is the sum or difference of the limits.)

3. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = A \cdot B.$

(The limit of a product is the product of the limits.)

4. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{A}{B}.$ (If $B \neq 0$.)

(The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.)

5. If $p(x)$ is a polynomial, then $\lim_{x \rightarrow a} p(x) = p(a).$

6. For any real number k , $\lim_{x \rightarrow a} [f(x)]^k = \left[\lim_{x \rightarrow a} f(x) \right]^k = A^k$, provided this limit exists.

7. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ if $f(x) = g(x)$ for all $x \neq a$.

8. For any real number $b > 0$, $\lim_{x \rightarrow a} b^{f(x)} = b^{\lim_{x \rightarrow a} f(x)} = b^A.$

9. For any real number b such that $0 < b < 1$ or $1 < b$,

$$\lim_{x \rightarrow a} [\log_b f(x)] = \log_b \left[\lim_{x \rightarrow a} f(x) \right] = \log_b A$$

if $A > 0$.

For the most of the functions we deal with, this tells us that limits behave like we would want them to behave - except for “forbidden operations” (such as division by zero, negative under even roots, and logs of non-positive numbers).

Example: Find $\lim_{x \rightarrow -3} (x^2 - 2x + 4)$.

Solution. First ask: Are there any “forbidden operations” for $x^2 - 2x + 4$ around $x = -3$? Since there are no “forbidden operations” there, we can evaluate the limit by just plugging in $x = -3$ into $x^2 - 2x + 4$ and get

$$(-3)^2 - 2(-3) + 4 = 9 + 6 + 4 = 19.$$

Example: Find $\lim_{x \rightarrow 3} (2^{x+2} - 4x)$.

See Video - <http://www.showme.com/sh/?h=fhlWfMu>

Example: Find $\lim_{x \rightarrow 5} \frac{x - 2}{x^2 - 2x}$.

See Video - <http://www.showme.com/sh/?h=uLD8Q6q>

Example: Find $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 2x}$.

See Video - <http://www.showme.com/sh/?h=4KmFqsK>

Let's look back at the graph on this one. Graph the function $Y_1 = \frac{x - 2}{x^2 - 2x}$ using the window $[1, 3, 1] \times [0.3, 1, 1]$ below.

Note that the graph looks pretty normal but it has a hole at $x = 2$. And we were able to find $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 2x}$ using this graph.

Example: Find $\lim_{x \rightarrow 0} \frac{x - 2}{x^2 - 2x}$.

See Video - <http://www.showme.com/sh/?h=VL6piYi>

Example: Find $\lim_{x \rightarrow -4} \frac{x + 4}{x^2 + 3x - 4}$.

See Video - <http://www.showme.com/sh/?h=pZASrGi>

Example: Find $\lim_{x \rightarrow 1} \frac{x + 4}{x^2 + 3x - 4}$.

See Video - <http://www.showme.com/sh/?h=0kOVdJ2>

Example: Find $\lim_{x \rightarrow 0} \frac{x + 4}{x^2 + 3x - 4}$.

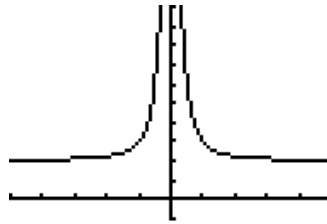
See Video - <http://www.showme.com/sh/?h=0Yrqjaa>

EXISTENCE OF LIMITS: The limit of f as x approaches a may not exist. (The notation $\lim_{x \rightarrow a^-} f(x)$ denotes the limit of $f(x)$ as x approaches a from the left, and $\lim_{x \rightarrow a^+} f(x)$ denotes the limit of $f(x)$ as x approaches a from the right.)

1. If $f(x)$ becomes infinitely large in magnitude (positive or negative) as x approaches the number a from either side, we write $\lim_{x \rightarrow a} f(x) = \infty$ or $\lim_{x \rightarrow a} f(x) = -\infty$. In either case, the limit does not exist.
2. If $f(x)$ becomes infinitely large in magnitude (positive) as x approaches a from one side and infinitely large in magnitude (negative) as x approaches a from the other side, then $\lim_{x \rightarrow a} f(x)$ does not exist.
3. If $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = M$, and $L \neq M$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

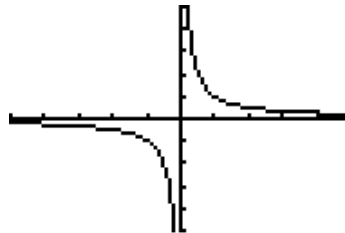
Examples:

1. The graph below illustrates the first case if we consider what happens around $x = 0$.



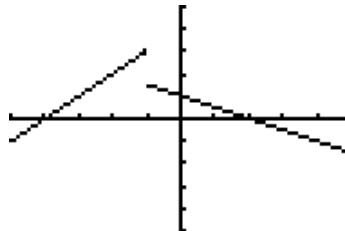
In this case, $\lim_{x \rightarrow 0^-} f(x) = \infty$, and $\lim_{x \rightarrow 0^+} f(x) = \infty$, so we write $\lim_{x \rightarrow 0} f(x) = \infty$, and the limit does not exist.

2. The graph below illustrates the second case if we consider what happens around $x = 0$.



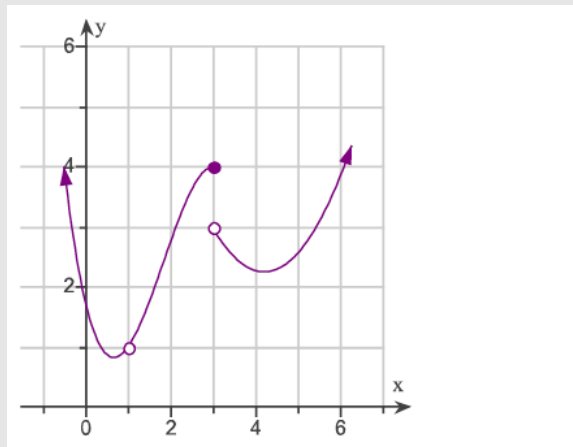
In this case, $\lim_{x \rightarrow 0^-} f(x) = -\infty$, and $\lim_{x \rightarrow 0^+} f(x) = \infty$, so the limit $\lim_{x \rightarrow 0} f(x)$ does not exist.

3. The graph below illustrates the third case if we consider what happens around $x = -1$.



In this case, $\lim_{x \rightarrow -1^-} f(x) = 3$, and $\lim_{x \rightarrow -1^+} f(x) = 2$, so the limit $\lim_{x \rightarrow -1} f(x)$ does not exist.

CLICKER QUESTION 24: The graph of $f(x)$ is given below.



Use the graph to answer parts (i) and (ii).

(i) Find $\lim_{x \rightarrow 3} f(x)$.

- (A) 4 (B) 3 (C) 3.5 (D) 0 (E) the limit does not exist

(ii) Find $\lim_{x \rightarrow 1} f(x)$.

- (A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 0 (E) the limit does not exist

See Video - <http://www.showme.com/sh/?h=6ejcs4G>

CLICKER QUESTION 25: Suppose $\lim_{x \rightarrow 5^-} f(x) = 9$, and $\lim_{x \rightarrow 5^+} f(x) = 9$, but $f(5)$ does not exist.

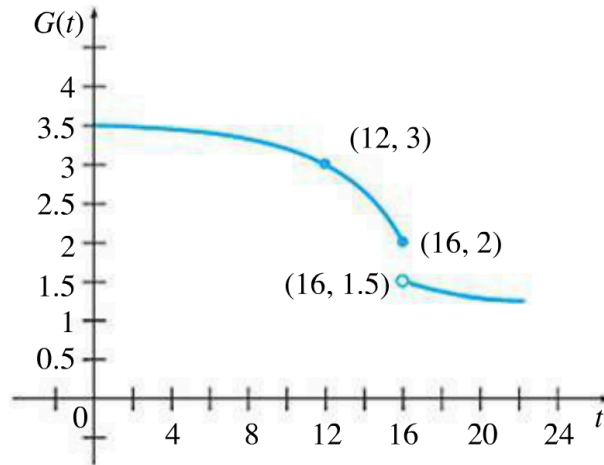
What can you say about $\lim_{x \rightarrow 5} f(x)$?

(A) $\lim_{x \rightarrow 5} f(x) = 9$ (B) $\lim_{x \rightarrow 5} f(x) = -9$ (C) $\lim_{x \rightarrow 5} f(x)$ does not exist

(D) $\lim_{x \rightarrow 5} f(x) = \infty$ (E) None of these

See Video - <http://www.showme.com/sh/?h=ZvcqNgO>

Example: When the price of an essential commodity rises rapidly, consumption drops slowly at first. If the price continues to rise, however, a “tipping” point may be reached, at which consumption takes a sudden substantial drop. Suppose the graph below shows the consumption of gasoline, $G(t)$, in millions of gallons, in a certain area. We assume that the price is rising rapidly. Here t is the time in months after the price began rising. Use the graph to find the following.



(a) $\lim_{t \rightarrow 12} G(t)$

See Video - <http://www.showme.com/sh/?h=rW2GMiW>

(b) $\lim_{t \rightarrow 16} G(t)$

(c) $G(16)$

(d) The tipping point (in months)

LIMITS AT INFINITY

LIMITS AT INFINITY: For any positive real number n

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0 \text{ and } \lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0, \text{ whenever defined.}$$

FINDING LIMITS AT INFINITY: If $f(x) = p(x)/q(x)$, for polynomials $p(x)$ and $q(x)$, $q(x) \neq 0$, $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ can be found as follows.

1. Divide $p(x)$ and $q(x)$ by the highest power of x in $q(x)$.
2. Use the rules for limits, including the rules for limits at infinity,

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0 \text{ and } \lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0, \text{ whenever defined,}$$

to find the limit of the result from step 1.

CLICKER QUESTION 26: Find the following limits:

(i) $\lim_{x \rightarrow \infty} \frac{2x^3 + 4}{x^2 + 3x - 7}$

- (A) $-\frac{4}{7}$ (B) 0 (C) 1 (D) 2 (E) Does not exist

(ii) $\lim_{x \rightarrow \infty} \frac{2x^2 + 4}{x^2 + 3x - 7}$

- (A) $-\frac{4}{7}$ (B) 0 (C) 1 (D) 2 (E) Does not exist

(iii) $\lim_{x \rightarrow \infty} \frac{2x + 4}{x^2 + 3x - 7}$

- (A) $-\frac{4}{7}$ (B) 0 (C) 1 (D) 2 (E) Does not exist

See Video - <http://www.showme.com/sh/?h=GW8Z46K>

Example: A company training program has determined that, on the average, a new employee produces $P(s)$ items per day after s days of on-the-job training, where

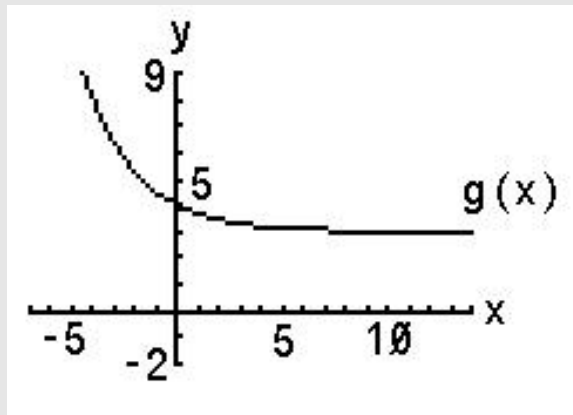
$$P(s) = \frac{63s}{s + 8}.$$

Find and interpret $\lim_{s \rightarrow \infty} P(s)$.

See Video - <http://www.showme.com/sh/?h=KmvTsIa>

See Video - <http://www.showme.com/sh/?h=QNXVGGe>

CLICKER QUESTION 27: The graph of $g(x)$ is given below.



Use the graph to answer parts (i) and (ii).

(i) Find $\lim_{x \rightarrow \infty} f(x)$.

(A) 5 (B) 3 (C) 10 (D) 0 (E) the limit does not exist

(ii) Find $\lim_{x \rightarrow -\infty} f(x)$.

(A) 5 (B) 3 (C) 102 (D) 0 (E) the limit does not exist

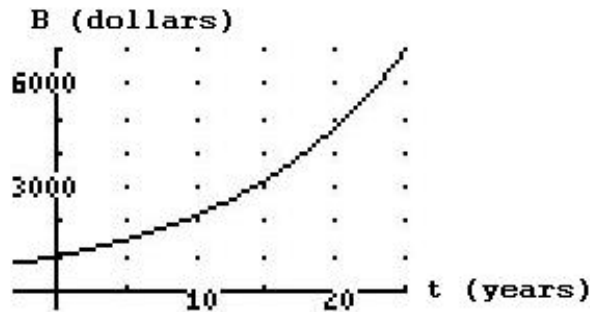
See Video - <http://www.showme.com/sh/?h=7TIyhKS>

SEC. 3.3: RATES OF CHANGE

Date _____

We are often asked to compute the rate of change of a linear function (which is the slope). We know a linear function has a *constant rate of change*. But how do we calculate a rate of change for a function that is not linear?

Example: \$1000 is invested in an interest-bearing account, whose balance B at time t is shown in the graph below.



(a) Estimate the change in the balance over the first 15 years. _____

See Video - <http://www.showme.com/sh/?h=Vy9WeEi>

(b) Estimate the average yearly rate at which balance is increasing over the first 15 years. Note: The word “average” is used because, as we will see, the rate of change can vary within the interval.

(c) Does it make sense that this value is positive? Explain.

(d) Estimate the average rate of change of the balance between $t = 10$ and $t = 15$.

AVERAGE RATE OF CHANGE: *The average rate of change of $f(x)$ with respect to x for a function f as x changes from a to b is:*

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a} = \frac{\Delta f}{\Delta x}$$

A NEW QUESTION: What is the rate of change of the account balance at exactly 15 years? This would be the instantaneous rate of change at $t = 15$.

INSTANTANEOUS RATE OF CHANGE: The instantaneous rate of change for a function f when $x = a$ is

$$\text{Instantaneous rate of change} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ or } \lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a},$$

provided this limit exists.

Example: Given the function $y = f(x) = \sqrt{x+1}$.

- (a) Find the average rate of change of f between $x = 0$ and $x = 3$.

See Video - <http://www.showme.com/sh/?h=bYIY2Cm>

- (b) Sketch the graph of $y = f(x)$, and represent the average rate of change found above as the slope of a line.



- (c) Which is larger, the average rate of change of the function between $x = 0$ and $x = 3$, or $x = 3$ and $x = 6$? What does this tell you about the graph of the function?

See Video - <http://www.showme.com/sh/?h=LbUphQW>

(d) What is the instantaneous rate of change of $f(x) = \sqrt{x+1}$ at $x = 2$?

See Video - <http://www.showme.com/sh/?h=WmisTMe>

Example: Find the instantaneous rate of change of $f(x) = e^{2x} + 4$ at $x = -1$?

CLICKER QUESTION 28: Use the formula for instantaneous rate of change, approximating the limit by using smaller and smaller values of h , to find the instantaneous rate of change for the function $f(x) = 5x^x$ at $x = 2$. The instantaneous rate of change is

- (A) less than 29 (B) between 29 and 32 (C) between 32 and 35
(D) between 35 and 38 (E) more than 38

Example: Suppose customers in a hardware store are willing to buy $N(p)$ boxes of nails at p dollars per box, as given by

$$N(p) = 80 - 5p^2, \quad 1 \leq p \leq 4.$$

- (a) Find the average rate of change of demand for a change in price from \$2 to \$3.

See Video - <http://www.showme.com/sh/?h=hxwvFR2>

- (b) Find and interpret the instantaneous rate of change of demand when the price is \$2.

See Video - <http://www.showme.com/sh/?h=VStltaq>

- (c) Determine how the demand is changing when the price is \$3.

See Video - <http://www.showme.com/sh/?h=DNEb9uq>

- (d) As the price is increased from \$2 to \$3, how is demand changing? Is the change to be expected? Explain.

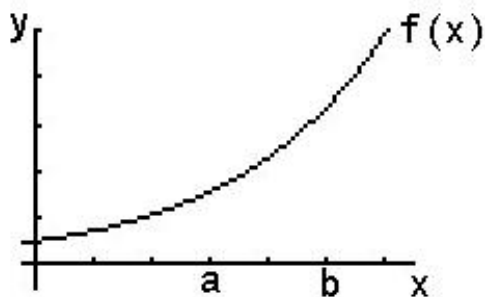
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SEC. 3.4: DEFINITION OF THE DERIVATIVE

Date _____

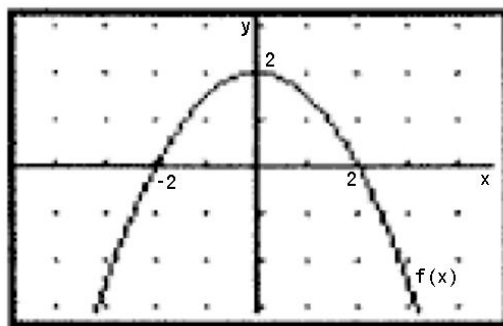
VISUALIZING THE RATES OF CHANGE GRAPHICALLY

Consider the function represented by the graph below.



- The *average rate of change* of a function is represented by the slope of the secant line connecting $(a, f(a))$ and $(b, f(b))$. Draw this line on the graph above.
- The *instantaneous rate of change* of a function is represented by the slope of the tangent line at $x = a$. (We will use the notation $f'(a)$ to represent the instantaneous rate of change of f at $x = a$.) Draw this line on the graph above.

Example: The graph of $f(x) = -\frac{1}{2}x^2 + 2$ is shown below.



- Use the graph to determine whether each of the following quantity is positive (+), negative (-), or zero (0).

(a) $f(1)$ _____ (b) $f(-3)$ _____ (c) $f(0)$ _____ (d) $f'(1)$ _____

(e) $f'(-3)$ _____ (f) $f'(0)$ _____ (g) $\frac{f(1) - f(-1)}{1 - (-1)}$ _____
- Which is larger, $f(-3)$ or $f(-1)$? _____
- Which is larger, $f'(-3)$ or $f'(-1)$? _____

See Video - <http://www.showme.com/sh/?h=VUrFkEC>

SLOPE OF THE TANGENT LINE: The tangent line of the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through this point having the slope

$$\text{Instantaneous rate of change} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ or } \lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a},$$

provided this limit exists. If this limit does not exist, then there is no tangent line at the point. The slope of the tangent line at a point is also called the **slope of the curve** at the point.

CLICKER QUESTION 29: By considering, but not calculating, the slope of the tangent line, give the derivative of the following:

(i) $f(x) = 3$.

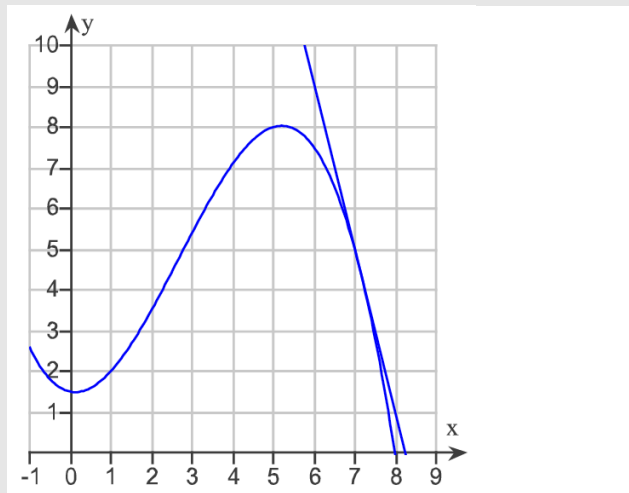
(A) -1 (B) 0 (C) 1 (D) 3 (E) the derivative does not exist

(ii) $f(x) = -x$.

(A) -1 (B) 0 (C) 1 (D) 2 (E) the derivative does not exist

See Video - <http://www.showme.com/sh/?h=DPC50YC>

CLICKER QUESTION 30: Estimate the slope of the tangent line to the curve.



The slope is

(A) less than -7 (B) between -7 and -5 (C) between -5 and -3

(D) between -3 and -1 (E) more than -1

See Video - <http://www.showme.com/sh/?h=4MMUde4>

Using the slope above, and the point-slope form of the equation of the line, we get:

EQUATION OF THE TANGENT LINE: *The tangent line of the graph of $y = f(x)$ at the point $(a, f(a))$ is given by the equation*

$$y - f(a) = f'(a)(x - a)$$

provided $f'(a)$ exists.

Example: Find the equation of the tangent line to $f(x) = 6x^2 - 5x - 1$ at the point $(3, 38)$.

See Video - <http://www.showme.com/sh/?h=0u8vei0>

CLICKER QUESTION 31: For $f(x) = 2x^2 - 3x$, find the equation of the tangent line when $x = -2$.

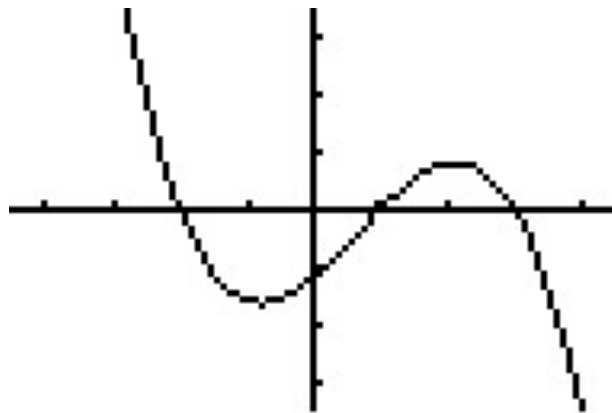
The tangent line is

(A) $y = 14(x + 2) - 11$ (B) $y = -11(x + 2) + 14$ (C) $y = -5(x - 2) - 2$

(D) $y = -2(x - 2) - 5$ (E) None of these

See Video - <http://www.showme.com/sh/?h=qXUEi48>

Example: Let $y = f(x)$ be given by the graph below.



- (a) Estimate any points where the slope of the tangent line is zero.
- (b) Estimate any points where the slope of the tangent line is one.
- (c) Estimate any points where the slope of the tangent line is negative one.

See Video - <http://www.showme.com/sh/?h=RXNw0mW>

The instantaneous rate of change of a function is so important, that it is given another name.

DERIVATIVE OF A FUNCTION AT A POINT: *The derivative of f at a , written $f'(a)$ (read “ f prime of a), is defined to be the instantaneous rate of change of f at the value $x = a$.*

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a}$$

Example: If $f(x) = x^3$, estimate $f'(2)$.

See Video - <http://www.showme.com/sh/?h=n6LPhDc>

If f is a function, then we can define a new function that is the derivative at each point.

THE DERIVATIVE FUNCTION: *The derivative of the function f with respect to x is defined as*

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{b \rightarrow x} \frac{f(b) - f(x)}{b-x}$$

Before we go any further, let's review a few algebraic skills.

Example: Perform the following operations:

(a) $(x+h)^2$

See Video - <http://www.showme.com/sh/?h=X94hMZc>

(b) $\frac{1}{x+h} - \frac{1}{x}$

(c) $\frac{1}{(x+h)^2} - \frac{1}{x^2}$

Example: If $f(x) = x^2 - 3$, find $f'(x)$.

See Video - <http://www.showme.com/sh/?h=OuoSp3w>

FINDING $f'(x)$ FROM THE DEFINITION OF DERIVATIVE: *The four steps to find the derivative $f'(x)$ for a function $y = f(x)$ are summarized here.*

1. Find $f(x + h)$.
2. Find and simplify $f(x + h) - f(x)$.
3. Divide by h to get $\frac{f(x + h) - f(x)}{h}$.
4. Let $h \rightarrow 0$; $\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$, if this limit exists.

Example: If $f(x) = 2x^2 - 3x$, find $f'(x)$.

See Video - <http://www.showme.com/sh/?h=TBbNd7w>

Example: If $f(x) = \frac{1}{x^2}$, use the definition of the derivative to find $f'(x)$.

See Video - <http://www.showme.com/sh/?h=IJgvvo8>

Practice Problem: If $f(x) = \frac{1}{x}$, use the definition of the derivative to find $f'(x)$.

See Video - <http://www.showme.com/sh/?h=cXkZ4eO>

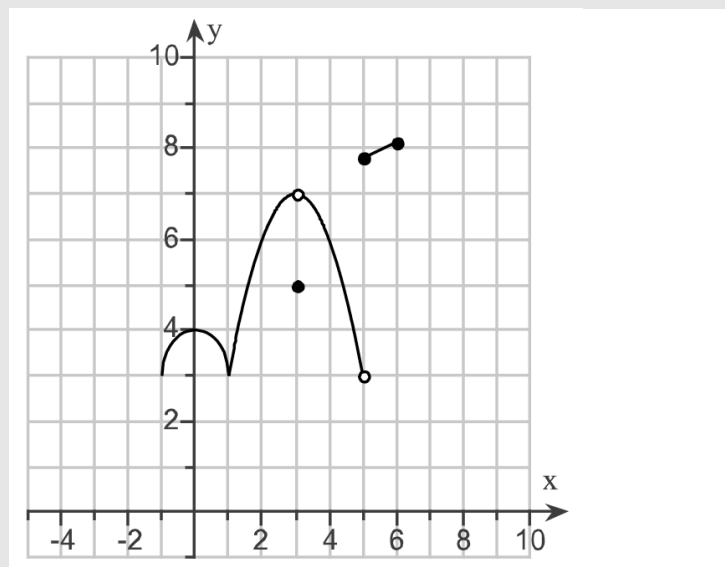
EXISTENCE OF THE DERIVATIVE: The derivative exists when a function f satisfies all of the following conditions at a point.

1. f is continuous,
2. f is smooth, and
3. f does not have a vertical tangent line.

The derivative does not exist when any of the following conditions are true at a point.

1. f is discontinuous,
2. f has a sharp corner, or
3. f has a vertical tangent line.

CLICKER QUESTION 32: Consider the graph of a function below. At how many points in the interval $-1 < x < 6$ is the function not differentiable.



- (A) 0 points (B) 1 point (C) 2 points
(D) 3 points (E) 4 or more points

See Video - <http://www.showme.com/sh/?h=yuCecOO>

MARGINAL COST AND REVENUE

Example: Consider the case of *Dr. X's Bass Guitar Builders*, whose revenues and costs are modeled by the following functions:

$$\text{Revenue: } R(q) = -0.00000000125q^4 + 0.0000211q^3 - 0.000463q^2 + 0.35124q$$

$$\text{Cost: } C(q) = 0.0000156q^3 - 0.2808q^2 + 1684.8q + 630400$$

where R and C are in dollars and q is the number of basses produced per year, with $0 \leq q \leq 12,500$.

We begin by graphing the cost and revenue functions with $X_{\min}=0$, $X_{\max}=12500$, $Y_{\min}=0$, and $Y_{\max}=10700000$.



Currently, *Dr. X's Bass Guitar Builders* is producing 10,795 basses per year, so their profit is

See Video - <http://www.showme.com/sh/?h=hd6phya>

The question for this section is: **Should they build another bass?**

We are going to assume that they make this decision purely on a financial basis, that is, if it increases their profits, they will, but if it reduces their profits, they won't.

Let's consider how this will affect their revenues. Their current revenue is $R(10,795) =$

See Video - <http://www.showme.com/sh/?h=7RmUEca>

If they build another bass, their revenue would be $R(10,796) =$

Their increase in revenue would be $R(10,796) - R(10,795) =$

Let's think about this from a geometric perspective. Graph the revenue function with $X_{\min}=10793$, $X_{\max}=10798$, $Y_{\min}=9515000$, and $Y_{\max}=9525000$.



See Video - <http://www.showme.com/sh/?h=iRgBX6W>

Note that the additional revenue, $R(10,796) - R(10,795)$ could be rewritten as

$$\frac{R(10,796) - R(10,795)}{1} = \frac{R(10,796) - R(10,795)}{10,796 - 10,795}$$

which is the slope of the line connecting the points $(10795, R(10795))$ and $(10796, R(10796))$. We can approximate the slope of this line with the derivative of the revenue function¹. That is, (if we let $Y_1 = R(q)$)

$$\begin{aligned} \text{Deriv}(Y_1, X, 10795) &\approx R'(10,795) \approx \frac{R(10,796) - R(10,795)}{10,796 - 10,795} \\ &\approx \frac{R(10,796) - R(10,795)}{1} = \text{Additional Revenue} \end{aligned}$$

So the derivative of the revenue function is approximately equal to the additional revenue. In general, the derivative of the revenue function is an excellent approximation of this additional revenue, so many economics textbooks use the following definition.

MARGINAL REVENUE: *Marginal Revenue* = $MR = R'(q)$ so *Marginal Revenue* $\approx R(q+1) - R(q)$.

In a similar fashion, the marginal cost can be defined.

MARGINAL COST: *Marginal Cost* = $MC = C'(q)$ so *Marginal Cost* $\approx C(q+1) - C(q)$.

¹In previous sections, we did the reverse. That is, we approximated the derivative with the slope of the secant over a small interval.

So in the case of *Dr. X's Bass Guitar Builders*, the marginal revenue and cost are:

$$\text{Marginal Revenue} = MR = R'(10, 795) = \text{nDeriv}(Y_1, X, 10795) =$$

$$\text{Marginal Cost} = MC = C'(10, 795) = \text{nDeriv}(Y_2, X, 10795) =$$

See Video - <http://www.showme.com/sh/?h=P2BuDmy>

So we conclude:

Suppose *Dr. X's Bass Guitar Builders* is currently producing 10797 basses. Should they build one more? Again, we examine the marginal revenue and cost:

$$\text{Marginal Revenue} = MR = R'(10, 797) = \text{nDeriv}(Y_1, X, 10797) =$$

$$\text{Marginal Cost} = MC = C'(10, 797) = \text{nDeriv}(Y_2, X, 10797) =$$

So we conclude:

Examples:

1. The cost of recycling q tons of paper is given in the following table.

| | | | | | | |
|------------------|------|------|------|------|------|------|
| q (tons) | 1000 | 1500 | 2000 | 2500 | 3000 | 3500 |
| $C(q)$ (dollars) | 2500 | 3200 | 3640 | 3825 | 3900 | 4400 |

- (a) Estimate the marginal cost at $q = 2000$.

See Video - <http://www.showme.com/sh/?h=94vBt8y>

- (b) Give the units of marginal cost and interpret your answer above in terms of cost.
- (c) At approximately what production level does marginal cost appear smallest?

2. A company's cost of producing q liters of a chemical is $C(q)$ dollars; this quantity can be sold for $R(q)$ dollars. Suppose $C(2000) = 5930$ and $R(2000) = 7780$.

(a) What is the profit at a production level of $q = 2000$?

See Video - <http://www.showme.com/sh/?h=5chcuCu>

(b) If $MC(2000) = 2.1$ and $MR(2000) = 2.5$, what is the approximate change in profit if q is increased from 2000 to 2001? Should the company increase or decrease production from $q = 2000$?

(c) If $MC(2000) = 4.77$ and $MR(2000) = 4.32$, should the company increase or decrease production from $q = 2000$?

3. The profit (in thousands of dollars) from the expenditure of x thousand dollars on advertising is given by $P(x) = 1000 + 32x - 2x^2$. Find the marginal profit at the following expenditures. In each case, decide whether the firm should increase the expenditure.

(a) \$8000

See Video - <http://www.showme.com/sh/?h=FU6Z6NE>

(b) \$12,000

Chapter 4

Calculating the Derivative

SEC. 4.1: TECHNIQUES FOR FINDING DERIVATIVES

Date _____

There is an alternative notation for the derivative that we have to be familiar with.

We know that the derivative $f'(x)$ is approximated by the *average rate of change over a small interval*. Therefore, if $y = f(x)$, then the average rate of change is given by $\Delta y/\Delta x$ or $\Delta f/\Delta x$. And, for “small” Δx , we have an approximation:

$$f'(x) \approx \frac{\Delta y}{\Delta x} \text{ or } \frac{\Delta f}{\Delta x}.$$

To remind us of this, if $y = f(x)$, we can write; $f'(x) = \frac{df}{dx} = \frac{dy}{dx}$.

This is known as writing the derivative using “Leibniz notation”.

A disadvantage of this notation is how we need to write the “derivative evaluated at a number”.

Using this alternative notation, in order to write $f'(5)$, we have to write $\left. \frac{dy}{dx} \right|_{x=5}$.

Example: Suppose the demand for a certain item is given by $D(p)$ where p represents the price of the item in dollars. Using Leibniz notation we can write:

$$D'(p) = \text{_____} \text{ and } D'(3) = \text{_____}$$

What are the units for $D'(p)$?

See Video - <http://www.showme.com/sh/?h=xPxspLu>

What is the practical interpretation of $D'(p)$?

NOTATIONS FOR THE DERIVATIVE: *The derivative of $y = f(x)$ may be written in any of the following ways:*

$$f'(x), \quad \frac{dy}{dx}, \quad \frac{df}{dx}, \quad \frac{d}{dx}[f(x)], \quad D_x[f(x)].$$

Example: The cost, C (in dollars) to produce g gallons of ice cream can be expressed as $C = f(g)$. Using units, explain the meaning of the following statements in terms of ice cream.

(a) $f(200) = 350$

See Video - <http://www.showme.com/sh/?h=jrkFDiS>

(b) $f'(200) = 1.4$.

(c) Use the values above to estimate $f(199)$

Example: The table below shows world gold production, $G = f(t)$, as a function of the year, t .

| | | | | | |
|----------------------|------|------|------|------|------|
| t (year) | 1990 | 1993 | 1996 | 1999 | 2002 |
| G (mn troy ounces) | 70.2 | 73.3 | 73.6 | 82.6 | 82.9 |

(a) Does $f'(t)$ appear to be positive or negative? _____

This is because the gold production is _____

See Video - <http://www.showme.com/sh/?h=BooM9Wi>

(b) In which time interval does $f'(t)$ appear to be the greatest? _____

(c) Estimate $f'(2002)$. Give units and interpret your answer in terms of gold production.

(d) Use the estimated value of $f'(2002)$ to estimate $f(2003)$ and $f(2010)$.

$$f(2003) \approx \text{_____} \quad f(2010) \approx \text{_____}$$

Example: The table below shows a function $f(t)$, the total sales of music compact discs (CDs), in millions.

| | | | | | |
|------------|-------|-------|-------|-------|-------|
| t (year) | 1994 | 1996 | 1998 | 2000 | 2002 |
| CD sales | 661.2 | 778.9 | 847.0 | 942.5 | 803.3 |

- (a) On what interval(s) does $f'(t)$ appear to be positive? _____ negative? _____
- (b) Estimate $f'(2002)$. Give units and interpret your answer your answer.

See Video - <http://www.showme.com/sh/?h=N02OvKa>

- (c) Use the estimated value of $f'(2002)$ to estimate $f(2003)$ and $f(2010)$.

$$f(2003) \approx \text{_____} \quad f(2010) \approx \text{_____}$$

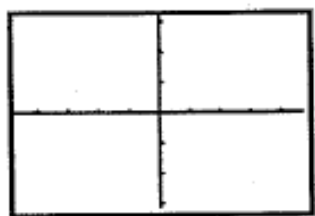
DERIVATIVE PROPERTIES FOR POWERS AND POLYNOMIALS

We know that the derivative of a function at a point represents a slope and a rate of change. In the last chapter we learned how to estimate values of the derivative of a function given by a graph, a table, or a function. Now we learn how to find a formula for the derivative function if we are given a function rule.

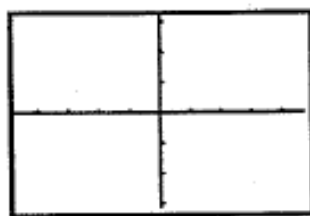
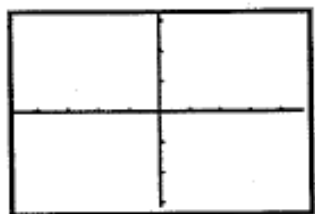
We will discover these properties using a graphical approach. Some with the help of our calculator.

I. The Derivative of a Constant Function

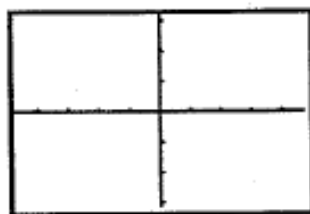
1. Let $f(x) = 2$. Graph $f(x)$ and $f'(x)$.
2. Let $f(x) = -1$. Graph $f(x)$ and $f'(x)$.



$f'(x)$



$f'(x)$



See Video - <http://www.showme.com/sh/?h=DxCoYYi>

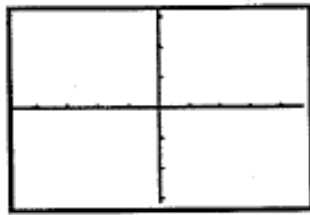
CONSTANT RULE: If $f(x) = k$, where k is any real number, then

$$f'(x) = 0 \quad \text{or} \quad \frac{d}{dx}[k] = 0.$$

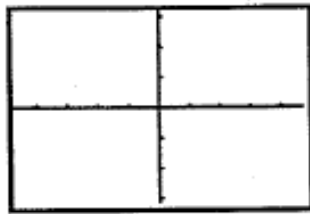
(The derivative of a constant is 0.)

II. The Derivative of a Linear Function

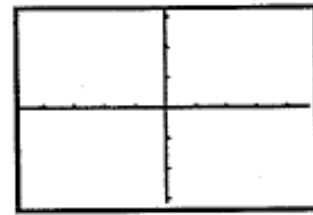
1. Let $f(x) = \frac{1}{2}x - 2$. Graph $f(x)$ and $f'(x)$.



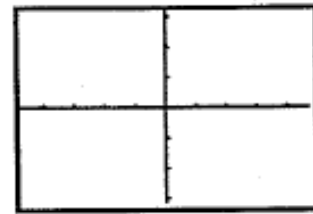
$f'(x)$



2. Let $f(x) = -3x + 3$. Graph $f(x)$ and $f'(x)$.



$f'(x)$



See Video - <http://www.showme.com/sh/?h=hE6242a>

LINEAR FUNCTION RULE: If $f(x) = mx + b$, where m and b are any real numbers, then

$$f'(x) = m \quad \text{or} \quad \frac{d}{dx} [mx + b] = m.$$

(The derivative of a linear function is the slope.)

Example: Find $\frac{d}{dx} [5x + 7]$.

See Video - <http://www.showme.com/sh/?h=JXopwWm>

III. The Derivative of a Power Function ($f(x) = x^n$)

POWER RULE: If $f(x) = x^n$ for any real number n , then

$$f'(x) = nx^{n-1} \quad \text{or} \quad \frac{d}{dx}[x^n] = nx^{n-1}.$$

(The derivative of $f(x) = x^n$ is found by multiplying by the exponent n and decreasing the exponent on x by 1.)

1. $f(x) = x^2$ $f'(x) =$

2. $f(x) = x^{-3}$ $f'(x) =$

3. $f(x) = x^4$ $f'(x) =$

See Video - <http://www.showme.com/sh/?h=YBGRhGi>

QUICK ALGEBRA REVIEW OF EXPONENTS: Rewrite each expression as a power of the base.

1. $\frac{1}{2^3} =$ _____

2. $\frac{1}{x^n} =$ _____

3. $\sqrt[4]{16} =$ _____

4. $\sqrt[n]{x} =$ _____

5. $\sqrt[3]{27^4} =$ _____

6. $\sqrt[n]{x^m} =$ _____

See Video - <http://www.showme.com/sh/?h=AUzFZce>

1. $f(x) = \frac{1}{x} =$ $f'(x) =$

2. $f(x) = \frac{1}{x^2} =$ $f'(x) =$

3. $f(x) = \sqrt{x} =$ $f'(x) =$

4. $f(x) = \sqrt[3]{x^2} =$ $f'(x) =$

See Video - <http://www.showme.com/sh/?h=G5bGxai>

IV. The Derivative of a Constant Times a Function

CONSTANT TIMES A FUNCTION: Let k be a real number. If $g'(x)$ exists, then the derivative of $f(x) = k \cdot g(x)$ is

$$f'(x) = k \cdot g'(x) \quad \text{or} \quad \frac{d}{dx} [k \cdot g(x)] = k \cdot \frac{d}{dx} [g(x)] = k \cdot g'(x).$$

(The derivative of a constant times a function is the constant times the derivative of the function.)

1. $f(x) = -4x^2$ $f'(x) =$

2. $f(x) = -2x^{-3}$ $f'(x) =$

3. $f(x) = 7x$ $f'(x) =$

See Video - <http://www.showme.com/sh/?h=jMUUT4a>

V. The Derivatives of Sums and Differences

SUM OR DIFFERENCE RULE: If $f(x) = u(x) \pm v(x)$ and if $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = u'(x) \pm v'(x) \quad \text{or} \quad \frac{d}{dx} [u(x) \pm v(x)] = \frac{d}{dx} [u(x)] \pm \frac{d}{dx} [v(x)] = u'(x) \pm v'(x).$$

(The derivative of a sum or difference of functions is the sum or difference of the derivatives.)

1. $f(x) = -x^2 - 3x - 6$ $f'(x) =$

2. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6$ $f'(x) =$

See Video - <http://www.showme.com/sh/?h=RGpJjOa>

VI. Practice Problems

Exercises: Find the derivative of each function below. (Hint: Rewrite f first using properties of exponents!)

(a) $y = 2x^4 + 2 + \frac{1}{2x^4}$

See Video - <http://www.showme.com/sh/?h=fuGvU8W>

(b) $y = 4\sqrt{x} + \frac{4}{\sqrt{x}} + \frac{\sqrt[4]{x}}{4}$

See Video - <http://www.showme.com/sh/?h=WrrL7Me>

(c) $y = \sqrt{x} \left(x^2 - \frac{1}{x^2} \right)$

See Video - <http://www.showme.com/sh/?h=ElmANge>

CLICKER QUESTION 33: Which of the following describes the derivative function $f'(x)$ of a quadratic function $f(x)$?

- (A) Cubic (B) Quadratic (C) Linear
(D) Constant (E) None of these

See Video - <http://www.showme.com/sh/?h=yQTOKSO>

VII. Using the Derivative Formulas

Example: Let $f(x) = -2x^3 + 6x + 8$

(a) Find $f'(2)$

See Video - <http://www.showme.com/sh/?h=plfrgIq>

(b) Find the equation of the tangent line at $x = -2$.

(c) Find the x -value(s) where the tangent line to the curve is horizontal.

(d) Check your answers to parts (a) - (c) above graphically.

Example: Assume that a demand equation is give by $q = 5000 - 100p$, and the cost of producing q units is given by $C(q) = 3000 - 20q + 0.03q^2$.

- (a) Find the revenue function $R(q)$. (*Hint:* Solve the demand equation for p and use $R(q) = qp$.)
Remember, normally, in a revenue function the input is quantity, and the output is money.
This may help you use the hint above.

See Video - <http://www.showme.com/sh/?h=0wtuSCu>

- (b) Find the marginal revenue for the following production levels (values of q).

500 units

1000 units

See Video - <http://www.showme.com/sh/?h=ZoP9Leq>

- (c) Find the marginal profit for the following production levels (values of q).

500 units

1000 units

See Video - <http://www.showme.com/sh/?h=Hijybyq>

- (d) When is marginal profit zero? What is the significance of this value?

See Video - <http://www.showme.com/sh/?h=wArEtOq>

SEC. 4.2: DERIVATIVES OF PRODUCTS AND QUOTIENTS Date _____

THE PRODUCT RULE

We already have a property that allows us to find the derivative of a *sum* (or *difference*) of two functions.

Example:

If $f(x) = x^3 + x^5$,

then $f'(x) = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$

Property:

If $f(x) = g(x) + h(x)$,

then $f'(x) = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$

See Video - <http://www.showme.com/sh/?h=lzwhXTE>

The following is the property that can be applied to find the product of two functions. Unfortunately, this product rule is much more complicated than the sum rule.

PRODUCT RULE: If $f(x) = u(x) \cdot v(x)$, and if $u'(x)$ and $v'(x)$ both exist, then

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x).$$

(The derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.)

Example: Let $f(x) = x^2 \cdot (x^3 - 2x)$. Find $f'(x)$ by the using the product rule.

$$f'(x) = \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}} + \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}}$$

$$= \underline{\hspace{4cm}}$$

See Video - <http://www.showme.com/sh/?h=cx77AZ6>

Check your result graphically by graphing $f(x)$ in Y_1 , your derivative in Y_3 , and $Y_2 = \text{nDeriv}(Y_1, X, X)$.

Note: on the newer TI calculators, if you select **nDeriv**(from the **Math** menu, you may see something

like: $\frac{d}{d\square} (\square) \Big|_{\square=\square}$. Fill in the appropriate variables so it looks like: $\frac{d}{dx} (Y_1) \Big|_{x=X}$

Example: Let $f(x) = x^3 \cdot (\sqrt{x} + 1)$. Find $f'(x)$ by using the product rule¹.

$$f'(x) = \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}} + \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}}$$
$$= \underline{\hspace{4cm}}$$

See Video - <http://www.showme.com/sh/?h=FGpv2v2>

Example: Let $f(x) = (3x^2 + 4) \cdot (2x^2 + 3)$. Find $f'(x)$ by using the product rule:

$$f'(x) = \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}} + \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}}$$
$$= \underline{\hspace{4cm}}$$

See Video - <http://www.showme.com/sh/?h=raYivJ2>

¹In many of these examples one could find the derivative without using the product rule. However, there will be problems where you will need the product rule, so practice them now.

Example: Let $f(x) = x^2(x + 3)$. Find all values of x where the graph of f has a horizontal tangent line. (This is also a good problem to check graphically.)

See Video - <http://www.showme.com/sh/?h=6E0Kg9A>

CLICKER QUESTION 34: Suppose that $f(x)$ and $g(x)$ are differentiable functions such that $f(5) = 6$, $f'(5) = 3$, $g(5) = 7$, and $g'(5) = -5$. Find $h'(5)$ when $h(x) = f(x) \cdot g(x)$.

(A) -15 (B) -9 (C) 27 (D) 51 (E) None of these

See Video - <http://www.showme.com/sh/?h=iXj8YXA>

THE QUOTIENT RULE

The following is the property that can be applied to find the quotient of two functions. Unlike the product rule, the order of the operations matter.

QUOTIENT RULE: If $f(x) = \frac{u(x)}{v(x)}$, then

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}.$$

Example: If $f(x) = \frac{5x^2}{3x+5}$, find $f'(x)$.

See Video - <http://www.showme.com/sh/?h=KrRwQt6>

Example: Let $f(x) = \frac{x}{x^2+5}$. Find $f'(x)$ and $f'(2)$.

See Video - <http://www.showme.com/sh/?h=xBAkJH6>

CLICKER QUESTION 35: Suppose that $f(x)$ and $g(x)$ are differentiable functions such that $f(2) = 9$, $f'(2) = 6$, $g(2) = 2$, and $g'(2) = 4$. Find $h'(2)$ when $h(x) = \frac{f(x)}{g(x)}$.

- (A) -6 (B) $\frac{3}{2}$ (C) 6 (D) 12 (E) None of these

See Video - <http://www.showme.com/sh/?h=HPENRx2>

Example: Write the equation of the tangent line to the function $y = \frac{(6x^2 - 7)(5x + 2)}{7x - 5}$ at $x = 0$.

See Video - <http://www.showme.com/sh/?h=G5PGsJU>

Example: A company that manufactures bicycles has determined that a new employee can assemble $M(d)$ bicycles per day after d days of on-the-job training, where

$$M(d) = \frac{100d^2}{3d^2 + 10}.$$

(a) Find the rate of change function for the number of bicycles assembled with respect to time.

See Video - <http://www.showme.com/sh/?h=4NBUC80>

(b) Find and interpret $M'(2)$ and $M'(5)$.

Example: The total cost (in hundreds of dollars) to produce x units of a product is

$$C(x) = \frac{8x - 4}{5x + 3}.$$

Find $\bar{C}'(x)$, the marginal average cost function.

See Video - <http://www.showme.com/sh/?h=ObF7Kq0>

A WEE BIT OF JUSTIFICATION . . .

Example: Suppose $f(x) = u(x) \cdot v(x)$ with $u(x) = 4x + 3$ and $v(x) = 5x - 2$.

Find $u(0) = \underline{\hspace{2cm}}$, $u'(0) = \underline{\hspace{2cm}}$, $v(0) = \underline{\hspace{2cm}}$, and $v'(0) = \underline{\hspace{2cm}}$.

(Of course, the function value at zero is the y -intercept and the derivative is the slope).

Using FOIL, we compute:

$$\begin{aligned} f(x) &= u(x) \cdot v(x) = (4x + 3) \cdot (5x - 2) \\ &= (\underline{\hspace{1cm}} \cdot \underline{\hspace{1cm}}) x^2 + (\underline{\hspace{1cm}} \cdot \underline{\hspace{1cm}} + \underline{\hspace{1cm}} \cdot \underline{\hspace{1cm}}) x + \underline{\hspace{1cm}} \cdot \underline{\hspace{1cm}} \\ &= \underline{\hspace{2cm}} x^2 + \underline{\hspace{2cm}} x + \underline{\hspace{2cm}} \end{aligned}$$

Then $f'(0) = \underline{\hspace{2cm}}$. But notice where this comes from! We see that

$$f'(0) = u(0) \cdot v'(0) + v(0) \cdot u'(0)$$

See Video - <http://www.showme.com/sh/?h=xSkMEQC>

SEC. 4.3: THE CHAIN RULE

Date _____

REVIEW OF COMPOSITION OF FUNCTIONS

1. If $f(x) = |x|$, $g(x) = 1 - x^2$, and $h(x) = 2^x$, what is:

(a) $f(g(5)) =$ _____

(b) $g(h(e^3)) =$ _____

(c) $f(g(x)) =$ _____

(d) $g(h(x)) =$ _____

(e) $h(g(x)) =$ _____

See Video - <http://www.showme.com/sh/?h=YDTfxFA>

2. Look at this problem in reverse.

(a) If $h(x) = f(g(x))$ and $h(x) = (3x^2 + 1)^3$,

what is $f(x) =$ _____

$g(x) =$ _____

(b) If $h(x) = f(g(x))$ and $h(x) = 2e^{5x+1}$,

what is $f(x) =$ _____

$g(x) =$ _____

(c) If $h(x) = f(g(x))$ and $h(x) = \sqrt{\ln x}$

what is $f(x) =$ _____

$g(x) =$ _____

See Video - <http://www.showme.com/sh/?h=do5hLDE>

THE CHAIN RULE

THE CHAIN RULE: If $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Using Leibniz notation, if $y = f(g(x))$, let $u = g(x)$. Then $y = f(u)$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example: Find derivatives of the following functions

(a) $s = (4t - 1)^5$

See Video - <http://www.showme.com/sh/?h=uZvlUzA>

(b) $y = \sqrt{4 - 9x^2}$

See Video - <http://www.showme.com/sh/?h=cUGalJA>

(c) $y = \sqrt[3]{(2x^4 + x)^2}$

See Video - <http://www.showme.com/sh/?h=Z231mN6>

(d) $h(x) = \sqrt{6x^2 + 2x - 1}$

See Video - <http://www.showme.com/sh/?h=GwNr2h6>

(e) $g(x) = \left(3x^9 - 17x + \frac{1}{x}\right)^2$

See Video - <http://www.showme.com/sh/?h=7tYGg3U>

(f) $h(x) = (x^2 - x)^4$

See Video - <http://www.showme.com/sh/?h=pnt5wHI>

(g) $f(x) = 4x(x^2 - x)^4$

See Video - <http://www.showme.com/sh/?h=FcYkSvI>

(h) $g(x) = (3x^4 + 1)^4(x^3 + 4)$

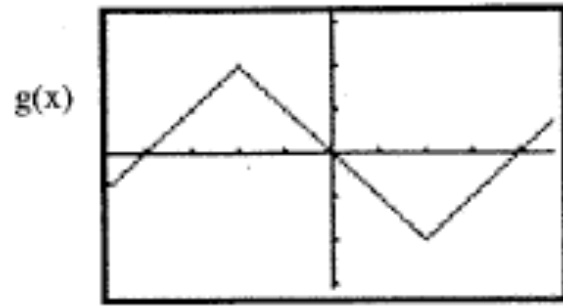
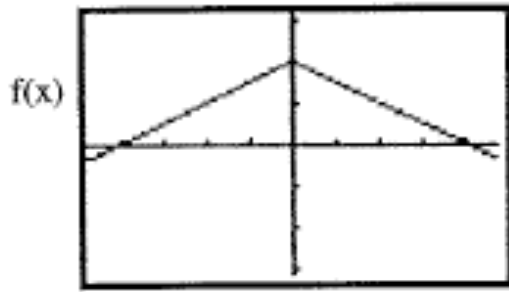
See Video - <http://www.showme.com/sh/?h=oU3zMVU>

$$(i) y = \frac{x^2 + 4x}{(3x^3 + 2)^4}$$

See Video - <http://www.showme.com/sh/?h=SwBFdtQ>

or See Video - <http://www.showme.com/sh/?h=skquAZU>

Example: Use the figures below, to evaluate the expressions.



$$g(3) = \underline{\hspace{2cm}}$$

$$f(-3) = \underline{\hspace{2cm}}$$

$$f(g(2)) = \underline{\hspace{2cm}}$$

$$g(f(1)) = \underline{\hspace{2cm}}$$

See Video - <http://www.showme.com/sh/?h=cnaBq3k>

$$\frac{d}{dx} [f(x)]_{x=-2} = \underline{\hspace{2cm}}$$

$$\frac{d}{dx} [g(x)]_{x=4} = \underline{\hspace{2cm}}$$

$$\frac{d}{dx} [f(g(x))]_{x=1} = \underline{\hspace{2cm}}$$

$$\frac{d}{dx} [g(f(x))]_{x=-1} = \underline{\hspace{2cm}}$$

See Video - <http://www.showme.com/sh/?h=cnaBq3k>

CLICKER QUESTION 36: Suppose that $f(x)$ and $g(x)$ are differentiable functions such that $f(4) = 7$, $f'(4) = -2$, $g(4) = 3$, $g'(4) = 12$, and $g'(7) = -6$. Find $h'(4)$ when $h(x) = g(f(x))$.

- (A) -24 (B) -3 (C) 12 (D) 78 (E) None of these

See Video - <http://www.showme.com/sh/?h=akCCCoK>

Example: Assume that the total revenue (in dollars) from the sale of x television sets is given by

$$R(x) = 24(x^2 + x)^{2/3}.$$

(a) Find the marginal revenue when the following numbers of sets are sold:

100

300

See Video - <http://www.showme.com/sh/?h=x1doylk>

(b) Find and interpret the average revenue from the sale of x sets.

See Video - <http://www.showme.com/sh/?h=nyoEbrc>

(c) Find and interpret the marginal average revenue.

See Video - <http://www.showme.com/sh/?h=fkY04Lw>

Example: Suppose a demand function is given by

$$q = D(p) = 30 \left(5 - \frac{p}{\sqrt{p^2 + 1}} \right)$$

where q is the demand for a product and p is the price per item in dollars. Find the rate of change in the demand for the product per unit change in price (i.e. find dq/dp).

See Video - <http://www.showme.com/sh/?h=R76OJc0>

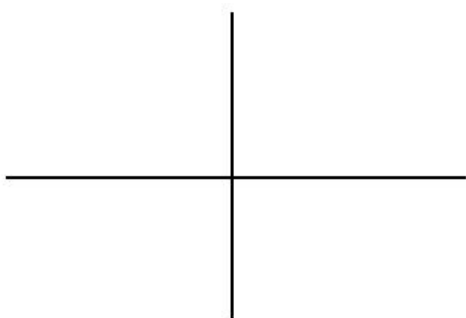
SEC. 4.4: DERIVATIVES OF EXPONENTIAL FUNCTIONS Date _____

Today we want to discover the derivatives of exponential functions. So, what do we expect the derivative of a function in the form $f(x) = a^x$ to look like?

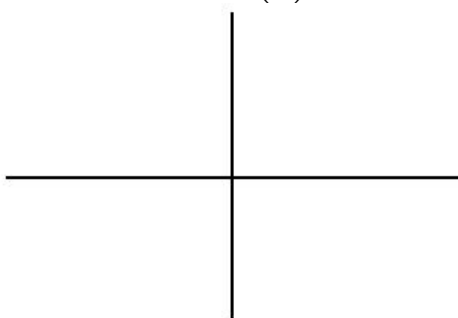
First, we know $f(x) = a^x$ has two different basic shapes. One, if $a > 1$, and the other if $0 < a < 1$.

Sketch the graph two graphs below.

$$f(x) = 2^x$$



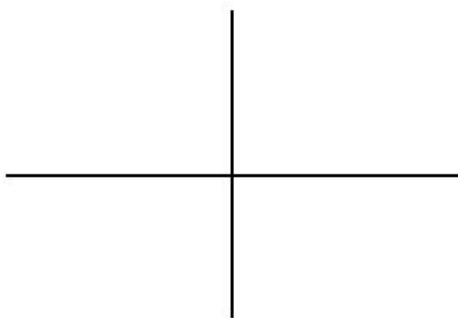
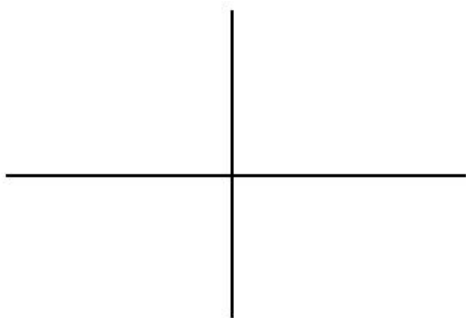
$$g(x) = \left(\frac{1}{3}\right)^x$$



See Video - <http://www.showme.com/sh/?h=u0WnFU8>

Sketch the graph of the derivative function of each of the above.

(On your calculator enter: $Y_1 = 2^X$ and $Y_2 = nDeriv(Y_1, X, X)$)



Note that the derivative of $f(x) = 2^x$ looks like $y = 2^x$ (but it is not identical). And the derivative of $g(x) = \left(\frac{1}{3}\right)^x$ looks like a reflection of $y = \left(\frac{1}{3}\right)^x$.

THE EXPONENTIAL RULE: For any positive constant $a \neq 1$,

$$\frac{d}{dx} (a^x) = (\ln(a))a^x.$$

(The derivative of an exponential function is the original function times the natural logarithm of the base.)

As a special case of the Exponential Rule,

DERIVATIVE OF e^x :

$$\frac{d}{dx}(e^x) = e^x.$$

The above result is one of the reasons the base e is so popular.

Examples: Find the derivative of the following functions:

(a) $f(x) = \frac{5^x}{3} - 9e^x.$

See Video - <http://www.showme.com/sh/?h=NHQ014C>

(b) $g(t) = 12(0.9)^t$

See Video - <http://www.showme.com/sh/?h=EEaQOA4>

Combining the exponential rule and the chain rule, we get the following.

DERIVATIVE OF $a^{g(x)}$ AND $e^{g(x)}$:

$$\frac{d}{dx} (a^{g(x)}) = (\ln a) a^{g(x)} g'(x)$$

and

$$\frac{d}{dx} (e^{g(x)}) = e^{g(x)} g'(x).$$

Examples: Find the derivative of the following functions:

(a) $g(t) = e^{4t}$

See Video - <http://www.showme.com/sh/?h=w8vFeW8>

(b) $g(t) = \frac{1}{e^t}$

CLICKER QUESTION 37: Find the derivative of

$$g(x) = e^{-x^2}.$$

(A) $g'(x) = e^{-x^2}$ (B) $g'(x) = -2xe^{-x^2}$ (C) $g'(x) = -2e^{-2x}$

(D) $g'(x) = e^{-2x}$ (E) None of these

See Video - <http://www.showme.com/sh/?h=ab2Vw2K>

Examples: Find the derivative of the following functions:

(a) $h(x) = 10^{5x^2+3x+10}$

See Video - <http://www.showme.com/sh/?h=up694kK>

(b) $h(x) = x4^{-2x}$

See Video - <http://www.showme.com/sh/?h=wxUbTmK>

(c) $h(x) = (3x^2 - 4x)5^{-6x}$

See Video - <http://www.showme.com/sh/?h=QENozMO>

(d) $f(t) = \frac{e^t - e^{-t}}{2t - 1}$

See Video - <http://www.showme.com/sh/?h=erpQk6K>

(e) $s(t) = 5 \cdot 2^{\sqrt{t-2}}$

See Video - <http://www.showme.com/sh/?h=tVH2UqG>

$$(f) \quad g(x) = \frac{x^2 e^{2x}}{x + e^{3x}}$$

See Video - <http://www.showme.com/sh/?h=AH76eiO>

Example: Find the *equation of the tangent line* to the graph of $f(x) = 4\left(\frac{1}{3}\right)^x$ at $x = 0$. Check your answer by graphing f and the tangent line.

See Video - <http://www.showme.com/sh/?h=dY0KACG>

Example: We found the function that solved the Cliff Notes problem was $P(t) = 4000(1.226)^t$, where t represented the number of years after 1958 and P was the value, in dollars, of the company in that year.

(a) Find $P'(12)$ and interpret your answer.

See Video - <http://www.showme.com/sh/?h=j8cLYIa>

(b) What was the value of the company in 1975?

(c) In 1975, how fast is the value of the company changing?

Example: Suppose \$5000 is deposited in a bank account that pays 4.5% annual interest, compounded continuously,

- (a) Find a formula for $P(t)$, the balance in the account t years after the initial deposit.

$$P(t) =$$

See Video - <http://www.showme.com/sh/?h=ojEMwGe>

- (b) Find $P(10)$ and $P'(10)$ and interpret your answers with units.

Example: The cost in dollars to produce x DVDs can be approximated by

$$C(x) = \sqrt{900 - 800(1.1)^{-x}}.$$

- (a) Find the marginal cost when the following quantities are made
- | | |
|---------|----------|
| $x = 0$ | $x = 20$ |
|---------|----------|

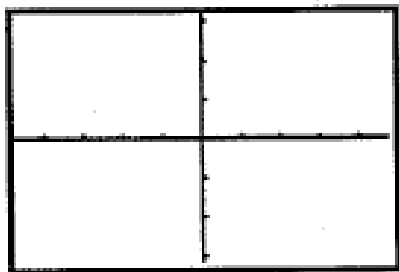
See Video - <http://www.showme.com/sh/?h=IH0nxKa>

- (b) What happens to marginal cost as the number produced becomes larger and larger?

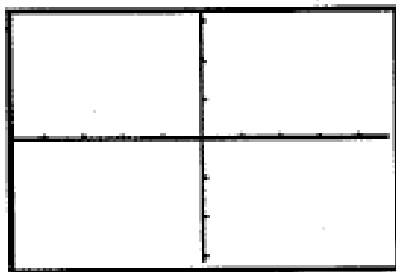
SEC. 4.5: DERIVATIVES OF LOGARITHMIC FUNCTIONS Date _____

Sketch the graph of $f(x) = \ln x$ and its derivative below.

(On your calculator enter: $Y_1 = 2^x$ and $Y_2 = \text{nDeriv}(Y_1, X, X)$)



$f(x)$



$f'(x)$

THE LOGARITHM RULE: For any positive constant a ,

$$\frac{d}{dx} [\log_a(x)] = \frac{1}{(\ln(a))x}.$$

(The derivative of a logarithmic function is the reciprocal of the product of the variable and the natural logarithm of the base.)

And in particular, we get:

THE NATURAL LOGARITHM RULE:

$$\frac{d}{dx} [\ln(x)] = \frac{1}{x}.$$

Example: Find the derivative of the following function:

$$g(x) = 3 \ln(x) + \log(x).$$

(Recall that $\log(x)$ means $\log_{10}(x)$.)

See Video - <http://www.showme.com/sh/?h=gsQwyES>

Using the logarithm rule and the chain rule, we get the following.

DERIVATIVE OF LOGARITHMIC-TYPE FUNCTIONS: For any positive constant a ,

$$\frac{d}{dx} [\log_a |x|] = \frac{1}{(\ln(a))x} \qquad \frac{d}{dx} [\log_a |g(x)|] = \frac{1}{\ln(a)} \cdot \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx} [\ln |x|] = \frac{1}{x} \qquad \frac{d}{dx} [\ln |g(x)|] = \frac{g'(x)}{g(x)}$$

Examples: Find the derivative of the following functions:

(a) $h(x) = \log_5(7x)$.

See Video - <http://www.showme.com/sh/?h=UNNncWW>

(b) $w(x) = \ln(4) \ln(-3x)$.

See Video - <http://www.showme.com/sh/?h=6h6bUsS>

(c) $f(x) = \log(1 + x^3)$.

See Video - <http://www.showme.com/sh/?h=LKYDFcO>

CLICKER QUESTION 38: Suppose $g(x)$ is a differentiable function with $g(3) = 11$ and $g'(3) = -2$. If $h(x) = \ln(g(x))$, find $h'(3)$.

- (A) $\frac{-2}{11}$ (B) -22 (C) $-2\ln(11)$ (D) $\frac{-2}{\ln(11)}$ (E) None of these

See Video - <http://www.showme.com/sh/?h=415PiOO>

Examples: Find the derivative of the following functions:

(a) $s(t) = (3t + 7) \ln(\sqrt{2t + 1})$.

See Video - <http://www.showme.com/sh/?h=l9DrmIS>

(b) $f(x) = (\log(x + 1))^4$.

See Video - <http://www.showme.com/sh/?h=NSwfeeO>

(c) $g(z) = \sqrt{e^{2z} + \ln(z)}$.

See Video - <http://www.showme.com/sh/?h=EQ75HsW>

(d) $f(t) = \frac{\ln(2t^2 + 3)^{5/2}}{\ln(t^2 + 1) + t}$.

See Video - <http://www.showme.com/sh/?h=0bEpMOm>

Let's summarize the derivative properties we now know.

DERIVATIVE RULES FOR TYPES OF FUNCTIONS: Assume f and g are functions and a, b, k and m are constants.

1. $\frac{d}{dx}(k) = \underline{\hspace{2cm}}$

2. $\frac{d}{dx}(m \cdot x + b) = \underline{\hspace{2cm}}$

3. $\frac{d}{dx}(x^n) = \underline{\hspace{2cm}}$

4. $\frac{d}{dx}(e^x) = \underline{\hspace{2cm}}$

5. $\frac{d}{dx}(a^x) = \underline{\hspace{2cm}}$

6. $\frac{d}{dx}(\ln x) = \underline{\hspace{2cm}}$

7. $\frac{d}{dx}(\log_a x) = \underline{\hspace{2cm}}$

DERIVATIVE RULES FOR COMBINATIONS OF FUNCTIONS:

8. $\frac{d}{dx}(f(x) \pm g(x)) = \underline{\hspace{2cm}}$

9. $\frac{d}{dx}(k \cdot g(x)) = \underline{\hspace{2cm}}$

10. $\frac{d}{dx}(f(x) \cdot g(x)) = \underline{\hspace{2cm}}$

11. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \underline{\hspace{2cm}}$

12. $\frac{d}{dx}(f(g(x))) = \underline{\hspace{2cm}}$

See Video - <http://www.showme.com/sh/?h=rYPEzWi>

Don't forget that the point of all of the derivative formulas is to use them to solve problems!

Example: Web TV. The number of homes with access to the Internet by way of cable television between 1998 and 2005 can be modeled by the equation:

$$I(x) = -138.27 + 76.29 \ln x,$$

where $I(x)$ is measured in million homes x years after 1990.

(a) Find $I(12)$ and interpret your answer (with units).

See Video - <http://www.showme.com/sh/?h=iVZeckq>

(b) Find $I'(12)$ and interpret your answer (with units).

See Video - <http://www.showme.com/sh/?h=KpISV6m>

(c) What was the rate of change in the number of homes with access to the Internet by way of cable television in the year 2004?

See Video - <http://www.showme.com/sh/?h=ZSk4Fqi>

Example: If \$1000 is invested in a bank at an annual interest rate of 4.5%, and the interest is compounded continuously, then the amount of money $M(t)$ in the account after t years is given by the equation

$$M(t) = 1000e^{0.045t}$$

(a) Solve this equation for t in terms of M ; *i.e.* find $t(M)$.

See Video - <http://www.showme.com/sh/?h=kdy71mq>

(b) Use the answer to part (a) to determine how long it will take for the investment to triple in value.

See Video - <http://www.showme.com/sh/?h=bb8Wesi>

(c) Find $\left. \frac{dt}{dM} \right|_{M=1000}$ and interpret your answer (with units).

$$\left. \frac{dt}{dM} \right|_{M=1000} =$$

See Video - <http://www.showme.com/sh/?h=vpC9nai>

Example: Suppose the demand function for q units of a certain item is

$$p = D(q) = 100 + \frac{50}{\ln q}, q > 1,$$

where p is in dollars.

(a) Find the revenue function.

See Video - <http://www.showme.com/sh/?h=1PoBBWi>

(b) Find the marginal revenue.

(c) Approximate the revenue from one more unit when 8 units are sold.

Chapter 5

Graphs and the Derivative

SEC. 5.1: INCREASING AND DECREASING FUNCTIONS Date _____

What does the derivative of a function tell us about the function and the graph of the function?

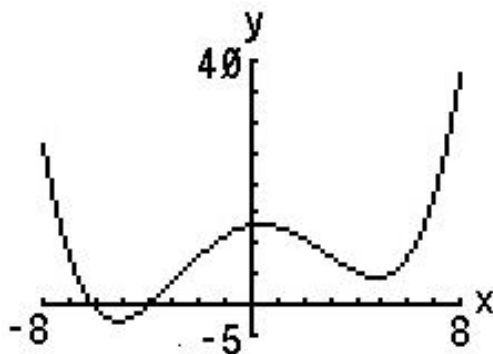
THE DERIVATIVE AND DIRECTION: Suppose a function f has a derivative at each point in an open interval; then

- if $f'(x) > 0$ for each x in the interval, f is increasing on the interval;
- if $f'(x) < 0$ for each x in the interval, f is decreasing on the interval;
- if $f'(x) = 0$ for each x in the interval, f is constant on the interval;

Places where $f'(x) = 0$ are of particular interest to us, and have their own terms.

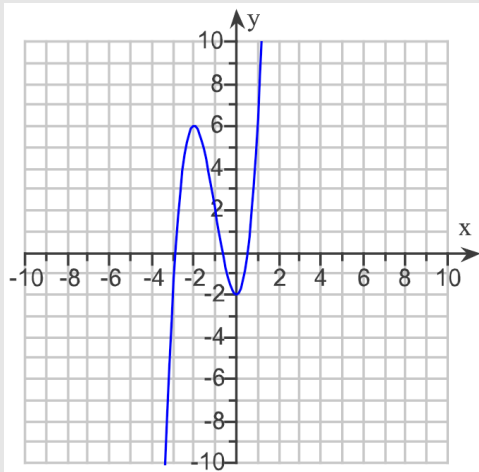
CRITICAL NUMBERS: The critical numbers for a function f are those numbers c in the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist. A critical point is a point whose x -coordinate is the critical number c and whose y -coordinate is $f(c)$.

Example: Determine the critical number(s) of the function graphed below:



See Video - <http://www.showme.com/sh/?h=i0jOeHo>

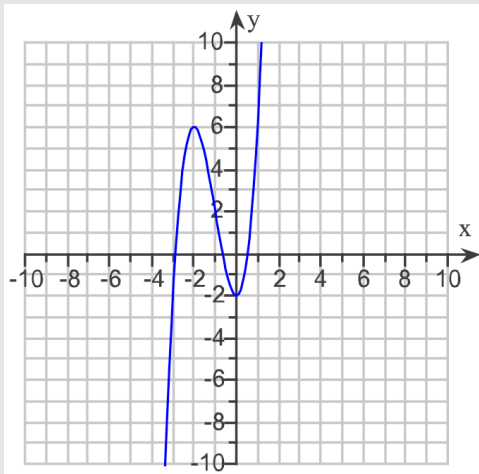
CLICKER QUESTION 39: Determine the critical number(s) of the function graphed below:



- (A) $x = -3$ & $x = -0.7$ & $x = 0.5$ (B) $x = -2$ & $x = 0$ (C) $x = 1$
(D) $x = -2$ & $x = 6$ (E) None of these

See Video - <http://www.showme.com/sh/?h=7pP3B44>

CLICKER QUESTION 40: Suppose that the graph below is the graph below is the graph of $f'(x)$. Find the open intervals where $f(x)$ is increasing.



- (A) $(-\infty, -2), (0, \infty)$ (B) $(-2, 0)$ (C) $(-\infty, -3), (-0.7, 0.5)$
(D) $(-3, -0.7), (0.5, \infty)$ (E) None of these

See Video - <http://www.showme.com/sh/?h=MSqevo0>

Example: Let $f(x) = x^3 - 9x^2 - 48x + 52$.

(a) Find the critical numbers of $f(x)$.

See Video - <http://www.showme.com/sh/?i=1570004>

(b) Determine the intervals where f is increasing, and the intervals where f is decreasing.

(*Process:* Make a chart like a number line. Label the critical numbers. Pick a number on the number line between each critical number and substitute it into f' . Remember, we only care if this value is positive or negative. Label this information on the chart.)

Practice Problems: For the given functions, find the critical points, the intervals where the function increases, and the intervals where the function decreases.

(a) $f(x) = x^4 - 4x^3$

Critical numbers: _____ Increasing: _____ Decreasing: _____

See Video - <http://www.showme.com/sh/?h=vKLtpOC>

(b) $g(x) = \frac{x + 3}{x - 4}$

Critical numbers: _____ Increasing: _____ Decreasing: _____

See Video - <http://www.showme.com/sh/?h=J1hpupE>

(c) $h(x) = x^{18/19} - x^{37/19}$

Critical numbers: _____ Increasing: _____ Decreasing: _____

See Video - <http://www.showme.com/sh/?h=6WegZ7I>

CLICKER QUESTION 41: Find the critical numbers for the function

$$f(x) = 4x^3 - 27x^2 - 210x - 21.$$

Round to two decimal places as needed.

- (A) $x = -4.55$ & $x = -0.10$ & $x = 11.40$
- (B) $x = 4.00$ & $x = -27.00$ & $x = -210.00$ & $x = -21.00$
- (C) $x = -2.50$ & $x = 7.00$
- (D) There are no critical numbers
- (E) None of these

See Video - <http://www.showme.com/sh/?h=WLKL5nM>

Example: A manufacturer sells video games with the following cost and revenue functions (in dollars), where x is the number of games sold, for $0 \leq x \leq 3300$.

$$C(x) = 0.32x^2 - 0.00004x^3$$

$$R(x) = 0.848x^2 - 0.0002x^3$$

Determine the interval(s) on which the profit function is increasing.

See Video - <http://www.showme.com/sh/?h=JqHBk5Q>

Example: The projected year-end assets in the Social Security trust funds, in trillions of dollars, where t represents the number of years since 2000, can be approximated by

$$A(t) = 0.0000329t^3 - 0.00450t^2 + 0.0613t + 2.34,$$

where $0 \leq t \leq 50$. Determine where $A(t)$ is increasing, and where $A(t)$ is decreasing.

See Video - <http://www.showme.com/sh/?h=uqAt2hc>

RELATIVE EXTREMA: *Let c be a number in the domain of the function f . Then $f(c)$ is a relative (or local) maximum for f if there exists an open interval (a, b) containing c such that*

$$f(x) \leq f(c)$$

for all x in (a, b) .

Likewise, $f(c)$ is a relative (or local) minimum for f if there exists an open interval (a, b) containing c such that

$$f(x) \geq f(c)$$

for all x in (a, b) .

A function has a relative (or local) extremum (plural: extrema) at c if it has either a relative maximum or a relative minimum there.

If c is an endpoint of the domain of f , we only consider x in the half-open interval that is in the domain.

What we called “turning points” in an algebra class, we now call “relative extrema”.

Example: In the last section, we looked at the function:

$$f(x) = x^3 - 9x^2 - 48x + 52.$$

First, we found $f'(x)$. $f'(x) =$ _____.

Then we found the critical numbers of $f(x)$ are $x =$ _____ and $x =$ _____.

Finally, we determined that $f(x)$ was increasing on _____

and decreasing on _____.

We are going to take this problem one step further. From the chart (number line) we constructed in the previous section, we can find the local maximum and local minimum points on the graph of the function.

See Video - <http://www.showme.com/sh/?h=eMZfzai>

Local Maxima: _____

Local Minima: _____

Example: Given $f(x) = x^4 - 4x^3$, find the critical numbers, and local maxima and minima points.

Critical numbers: _____ Local Maxima: _____ Local Minima: _____

See Video - <http://www.showme.com/sh/?h=yadJ8AS>

Let's summarize what we have learned from the examples:

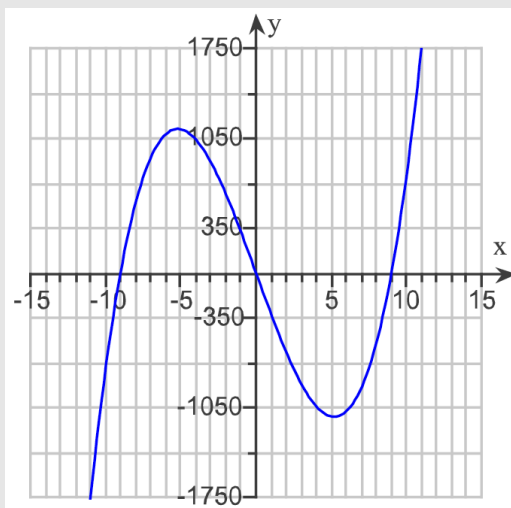
FIRST DERIVATIVE TEST: Let c be a critical number for a function f . Suppose that f is continuous on (a, b) and differentiable on (a, b) except possibly at c .

1. $f(c)$ is a local maximum of f if $f'(x)$ changes from positive to negative at c .
2. $f(c)$ is a local minimum of f if $f'(x)$ changes from negative to positive at c .

Note: Although it is a true statement that “If a function f has a local maximum or a local minimum at a number p , then p is critical number of f ”, it is *not* true that “If a function f has a critical number at a point p , then f is a local maximum or local minimum of f ”.

Example: Consider the function $f(x) = x^3$. Note that f has a critical point at $x = 0$, but it is neither a local maximum nor a local minimum.

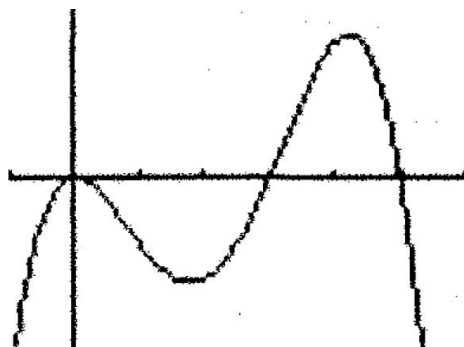
CLICKER QUESTION 42: Find the locations of all relative maximums of $f(x)$.



- (A) $x = 5$ (B) $x = -9$ & $x = 9$ (C) $x = -5$
(D) $x = 0$ (D) None of these

See Video - <http://www.showme.com/sh/?h=v8Pk9Me>

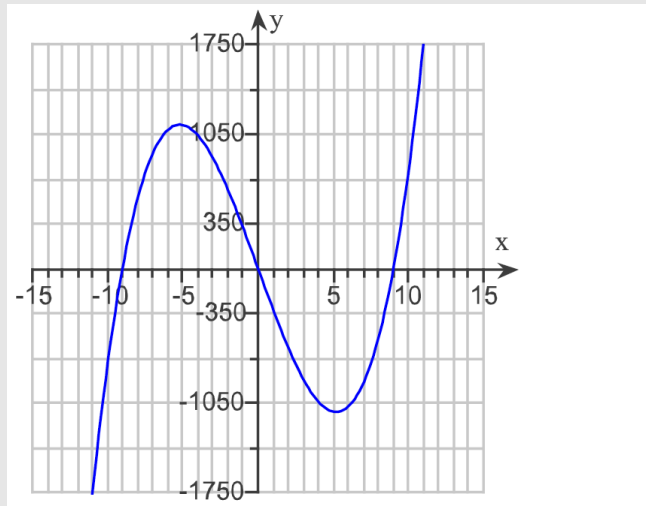
Example: Below is a graph of f' , the derivative of a function f . What are the critical points of the function f ? Over what intervals is the function f increasing and decreasing? For what values of x does f have a local maximum and a local minimum?



- Critical numbers: _____
- Increasing: _____
- Decreasing: _____
- Local Maximum: _____
- Local Minimum: _____

See Video - <http://www.showme.com/sh/?h=Tzuz2um>

CLICKER QUESTION 43: Suppose that the graph below is the graph of $f'(x)$, the derivative of a function $f(x)$. Find the locations of all relative extrema of $f(x)$, and tell whether each is a relative maximum or minimum.



- (A) $x = -5$ is a relative maximum
& $x = 5$ is a relative minimum
- (B) $x = -9$ & $x = 9$ are relative minimums
& $x = 0$ is a relative maximum
- (C) $x = -5$ is a relative minimum
& $x = 5$ is a relative maximum
- (D) $x = -9$ & $x = 9$ are relative maximums
& $x = 0$ is a relative minimum
- (E) None of these

See Video - <http://www.showme.com/sh/?h=xGoCYWu>

Example: Find the x -value of all the points where the functions defined as follows have any relative extrema. Find the value(s) of any relative extrema.

(a) $f(x) = x^3 + 6x^2 + 9x - 8$

See Video - <http://www.showme.com/sh/?h=oDycBcm>

(b) $f(x) = x^4 - 8x^2 + 9$

See Video - <http://www.showme.com/sh/?h=HUrphCq>

$$(c) f(x) = \frac{x^2 - 6x + 9}{x + 2}$$

See Video - <http://www.showme.com/sh/?h=rIJromG>

(d) $f(x) = 3xe^x + 2$

See Video - <http://www.showme.com/sh/?h=Tc2fh8C>

Example: A small company manufactures and sells lamps. The production manager has determined that the cost and demand functions for q lamps per week are

$$\text{Cost: } C(q) = 100 + 20qe^{-0.01q}; \text{ Demand: } p = 40e^{-0.01q}$$

where p is the price per lamp.

- (a) Find the number, q , of units that produces maximum profit. Why does this value of q produce maximum profit?

See Video - <http://www.showme.com/sh/?h=KZD5KE4>

- (b) Find the price, p , per unit that produces maximum profit.

- (c) Find the maximum profit, P .

Example: Suppose that the cost function for a product is given by

$$C(x) = 0.002x^3 + 9x + 6912.$$

Find the production level (i.e., value of x) that will produce the minimum average cost per unit $\bar{C}(x)$.

See Video - <http://www.showme.com/sh/?h=2TXuaY4>

Example: The demand equation for one type of computer networking system is

$$p = D(q) = 500qe^{-0.0016q^2},$$

where p is the price (in dollars) and q is the quantity of servers sold per month. Find the values of q and p that maximize revenue.

See Video - <http://www.showme.com/sh/?h=tQiKDoG>

Chapter 6

Applications of the Derivative

SEC. 6.1: ABSOLUTE EXTREMA

Date _____

WHAT IS AN ABSOLUTE/GLOBAL EXTREMA?

Let's begin with an example.

Example: Consider the function $f(x) = x^4 - 2x^2 + 3$.

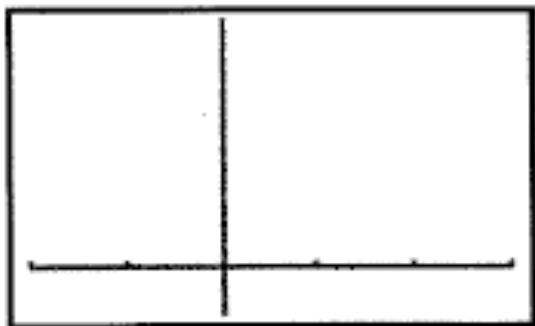
- (a) Find all local minima and local maxima for $f(x)$.

See Video - <http://www.showme.com/sh/?h=C2hzhfv6>

Local minimum: _____ Local maximum: _____

Located at: _____ Located at: _____

- (b) On your calculator, sketch the graph of f on the interval $[-2, 3]$. Show the graph below. Label the local maximum and local minimum.



We are often interested in where a function is larger or smaller than *all other points* on an interval. These points are called global (or absolute) maxima and minima.

ABSOLUTE EXTREMA: Let f be a function defined on some interval. Let c be a number in that interval. Then $f(c)$ is the absolute (or global) maximum of f on the interval if

$$f(x) \leq f(c)$$

for every x in the interval, and $f(c)$ is the absolute (or global) minimum of f on the interval if

$$f(x) \geq f(c)$$

for every x in the interval.

A function has an absolute (or global) extremum (plural: extrema) at c if it either has an absolute maximum or an absolute minimum there.

Example: Let's return to the example on the previous page, with $f(x) = x^4 - 2x^2 + 3$. We found local minimums at $x = -1$ and $x = 1$, and we found a local maximum at $x = 0$. Our new question is:

On the interval $[-2, 3]$, what are the global minimum and global maximum values of f ?

See Video - <http://www.showme.com/sh/?h=dzm6bdA>

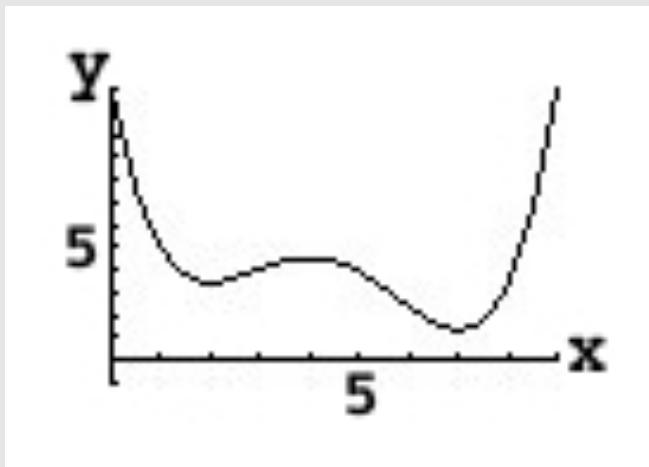
Global minimum: _____ Global maximum: _____

Located at: _____ Located at: _____

It turns out on a closed interval (a closed interval is an interval that includes its endpoints), continuous functions always have global extrema.

EXTREME VALUE THEOREM: A function f that is continuous on a closed interval $[a, b]$ will have both an absolute maximum and an absolute minimum on the interval.

CLICKER QUESTION 44: The graph of $f(x)$ is given below. Find the location of the absolute maximum of $f(x)$ on the following intervals.



(i) The location of the absolute maximum of $f(x)$ on the interval $[2, 6]$ is at

- (A) $x = 2$ (B) $x = 4$ (C) There is no absolute maximum
(D) $x = 9$ (E) $x = 6$

(ii) The location of the absolute maximum of $f(x)$ on the interval $[2, 9]$ is at

- (A) $x = 2$ (B) $x = 4$ (C) There is no absolute maximum
(D) $x = 9$ (E) $x = 6$

(iii) The location of the absolute maximum of $f(x)$ on the interval $(-\infty, \infty)$ is at

- (A) $x = 2$ (B) $x = 4$ (C) There is no absolute maximum
(D) $x = 9$ (E) $x = 6$

See Video - <http://www.showme.com/sh/?h=jaO7zbE>

FINDING ABSOLUTE/GLOBAL EXTREMA ON CLOSED INTERVALS

How do we find global maxima and global minima on a closed interval?

FINDING ABSOLUTE EXTREMA ON A CLOSED INTERVAL: *To find absolute extrema for a function f continuous on a closed interval $[a, b]$:*

1. *Find all critical numbers for f in (a, b) .*
2. *Evaluate f for all critical numbers in (a, b) .*
3. *Evaluate f for the endpoints a and b of the interval $[a, b]$.*
4. *The largest value found in Step 2 or 3 is the absolute maximum for f on $[a, b]$, and the smallest value found is the absolute minimum for f on $[a, b]$.*

Example: Find the absolute extrema of

$$f(x) = \frac{x - 1}{x^2 + 1}$$

on the interval $[1, 5]$, and indicate the x -values at which they occur.

See Video - <http://www.showme.com/sh/?h=RUixFmy>

Example: Find the absolute extrema of

$$f(x) = (x^2 - 4)^{1/3}$$

on the interval $[-2, 3]$, and indicate the x -values at which they occur.

See Video - <http://www.showme.com/sh/?h=g8AZ0fA>

Example: Find the absolute extrema of

$$f(x) = 3x - e^{0.2x}$$

on the interval $[-5, 20]$, and indicate the x -values at which they occur.

See Video - <http://www.showme.com/sh/?h=X5Kyd12>

CLICKER QUESTION 45: Find the absolute minimum value of the function

$$f(x) = 10x^3 + 39x^2 - 18x - 31$$

over the interval $[-1, 2]$. (Round to two decimal place as needed.)

The absolute minimum occurs at

- (A) $x = -1.00$ (B) $x = 0.40$ (C) $x = 1.00$ (D) $x = 2.00$ (E) None of these

See Video - <http://www.showme.com/sh/?h=iGZ1PpQ>

Example: A company has found that its weekly profit from the sale of x hundred units of an auto part is given by

$$P(x) = -0.02x^3 + 600x - 20,000$$

Production bottlenecks limit the number of units that can be made per week to no more than 15,000, while a long-term contract requires that at least 5,000 units be made each week. Find the number of units of the auto part that must be sold to maximize the weekly profit. Find the maximum possible weekly profit.

See Video - <http://www.showme.com/sh/?h=nrB2nnU>

FINDING ABSOLUTE/GLOBAL EXTREMA ON INTERVALS THAT ARE NOT CLOSED

We can now find global extrema on a closed interval. But what if we are interested in finding global extrema on an interval that is not closed; *i.e.*, an interval that does not contain an endpoint? Or an interval that includes ∞ ?

FINDING ABSOLUTE EXTREMA ON AN OPEN INTERVAL: *To find absolute extrema for a function f continuous on a open interval (a, b) :*

1. Find all critical numbers for f in (a, b) .
2. Evaluate f for all critical numbers in (a, b) .
3. Evaluate $\lim_{x \rightarrow a} f$ and $\lim_{x \rightarrow b} f$.
4. If either limit above is the larger than the largest value found in step 2, there is no absolute maximum. Otherwise, the largest value found in Step 2 is the absolute maximum for f on (a, b) .
5. If either limit above is smaller than the smallest value found in step 2, there is no absolute minimum. Otherwise, the smallest value found in Step 2 is the absolute minimum for f on (a, b) .

Example: Find the absolute extrema of

$$f(x) = 2x + \frac{8}{x^2} + 1$$

on the interval $(0, \infty)$.

See Video - <http://www.showme.com/sh/?h=tRn4BdI>

CRITICAL POINT THEOREM: Suppose a function f is continuous on a open interval I and that f has exactly one critical number in the interval I , located at $x = c$.

If f has a relative maximum at $x = c$, then this relative maximum is the absolute maximum of f on the interval I .

If f has a relative minimum at $x = c$, then this relative minimum is the absolute minimum of f on the interval I .

Example: Find the absolute extrema of

$$f(x) = 12 - x + \frac{9}{x}$$

on the interval $(0, \infty)$.

See Video - <http://www.showme.com/sh/?h=VIVs3zE>

Example: Find the absolute extrema of

$$f(x) = \frac{x}{x^2 + 1}.$$

See Video - <http://www.showme.com/sh/?h=MigHhDM>

CLICKER QUESTION 46: Find the minimum value of the average cost for the cost function

$$C(x) = x^3 + 17x + 432$$

The absolute minimum is

- (A) less than 100 (B) between 100 and 110 (C) between 110 and 120
(D) between 120 and 130 (E) more than 130

See Video - <http://www.showme.com/sh/?h=DfqhKRU>

Example: Suppose the cost function is given by

$$C(x) = 81x^2 + 17x + 324.$$

(a) Minimize the average cost on the interval $[1, 10]$.

See Video - <http://www.showme.com/sh/?h=JGSiiHI>

(b) Minimize the average cost on the interval $[10, 20]$.

SEC. 6.2: APPLICATIONS OF EXTREMA

Date _____

From the previous few sections, we see that it is important to be able to find a local max/min or a global max/min of a function. But, when solving problems in the real world, it is very seldom that we are given the specific function to work with. Therefore, we need to build the function to which we will then apply our calculus concepts.

Example: A fence must be built to enclose a rectangular area of 20,000 ft². Fencing material costs \$2.50 per foot for the two sides facing north and south and \$3.20 per foot for the other two sides. Find the cost of the least expensive fence.

Illustrate: Draw a figure that illustrates the problem.

See Video - <http://www.showme.com/sh/?h=CKwqma8>

Try Some Numbers: Set up a table and try some numbers.

| North & South Faces | | East & West Faces | | Total |
|---------------------|------|-------------------|------|-------|
| Length | Cost | Length | Cost | Cost |
| | | | | |
| | | | | |
| | | | | |

Plug in a Variable in the Table: Once you have a feel how the numbers work - plug in a variable in for one of the quantities. It will allow you to develop a formula to optimize.

Analyze: Optimize the function developed above.

Check: Check your answer graphically. Read the problem again, do your numbers make sense?

Example: You are planning to make an open rectangular box with a square base that will hold a volume of 50 cubic feet. Determine the dimensions that minimize the materials needed. (Minimize the surface area of the box.)

Illustrate :

See Video - <http://www.showme.com/sh/?h=uFHg2wC>

Try Some Numbers:

Plug in a Variable in the Table:

Analyze:

Check:

Example: A rectangular storage container with an open top is to have a volume of 32 ft^3 . The length of the base is equal to the width. Material for the base costs \$6 per square foot and material for the sides cost \$4 per square foot. Find the cost of the cheapest such container.

See Video - <http://www.showme.com/sh/?h=ETLJBc8>

Example: At a price of \$6 per ticket, a musical theater group can fill every seat in the theater, which has a capacity of 1700. For every additional dollar charged, the number of people buying tickets decreases by 85. What ticket price maximizes revenue and how many people will attend?

See Video - <http://www.showme.com/sh/?h=wNg8S6S>

CLICKER QUESTION 47: A hotel has 250 units. All rooms are occupied when the hotel charges \$110 per day for a room. For every increase of x dollars in the daily room rate, there are x rooms vacant. Each occupied room costs \$36 per day to service and maintain. What should the hotel charge per day in order to maximize daily profit?

- (A) \$88 (B) \$162 (C) \$190 (D) \$198 (E) None of these

See Video - <http://www.showme.com/sh/?h=B17kCoK>

Example: A baseball team is trying to determine what price to charge for tickets. At a price of \$10 per ticket, it averages 50,000 people per game. For every increase of \$1, it loses 5000 people. Every person at the game spends an average of \$5 on concessions. What price per ticket should be charged in order to maximize revenue?

See Video - <http://www.showme.com/sh/?h=PeZLxgW>

ELASTICITY OF DEMAND

Example: In 2008, the Valve Corporation, a software entertainment company, ran a holiday sale on its popular Steam software program. Using data collected from the sale, it is possible to estimate the demand corresponding to various discounts in the price of the software. Assuming that the original price was \$40, the demand for the software can be estimated by the function

$$q = 3,751,000p^{-2.826},$$

where p is the price and q is the demand.

We are interested in knowing how changing the price will impact the demand.

- When the price is \$40, the demand is
- What happens if we increase the price by \$2? (Note that a \$2 increase is a $\frac{2}{40} = 0.05 = 5\%$ increase). We expect the demand to decrease - but by how much? 5%? Less than 5%? More than 5%? Compute the percentage decrease of demand.

- Compute the ratio:

$$\frac{\text{percentage change in demand}}{\text{percentage change in price}}$$

- Use the ratio above to estimate the percentage change in demand if we decrease the price by 2%.

The ratio

$$\frac{\text{percentage change in demand}}{\text{percentage change in price}}$$

is of particular importance for us - in many cases, it helps us set the price to maximize revenue. It should be noted that most economics textbooks keep this ratio negative, while this textbook multiplies the ratio by -1 to make it positive.

ELASTICITY OF DEMAND: Let $q = f(p)$, where q is the demand at a price p . The elasticity of demand is

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}.$$

Demand is inelastic if $E < 1$.

Demand is elastic if $E > 1$.

Demand has unit elasticity if $E = 1$.

Example: Compute the elasticity of demand for the previous example, where the original price was \$40, the demand is estimated by the function

$$q = 3,751,000p^{-2.826}.$$

The importance of the elasticity of demand is that it measures the impact of pricing decisions on revenue.

REVENUE AND ELASTICITY:

1. *If the demand is inelastic ($E < 1$), total revenue increases as price increases.*
2. *If the demand is elastic ($E > 1$), total revenue decreases as price increases.*
3. *Total revenue is maximized at the price where demand has unit elasticity ($E = 1$).*

Example: Suppose the demand function is given by

$$q = 48,000 - 10p^2.$$

(a) Find the elasticity, E .

(b) Find values of q (if any) at which total revenue is maximized.

CLICKER QUESTION 48: The short-term demand for graphing calculators in the United States in 2008 can be approximated by

$$q = f(p) = 326,412p^{-1.15},$$

where p represents the price of graphing calculators (in dollars per calculator) and q represents the quantity of graphing calculators purchased. Answer parts (i) and (ii)

(i) What is the elasticity of demand for graphing calculators when the price is \$110 per calculator?

- (A) less than 0.25 (B) between 0.25 and 0.75 (C) between 0.75 and 1.25
(D) between 1.25 and 1.75 (E) more than 1.75

(ii) Interpret the elasticity of demand. Choose the correct answer below.

- (A) The demand is elastic, so as price increases, revenue decreases.
(B) The demand is elastic, so as price increases, revenue increases.
(C) The demand is inelastic, so as price increases, revenue increases.
(D) The demand is inelastic, so as price increases, revenue decreases.

A bit of justification: Note that the percentage change of demand is $\frac{\Delta q}{q}$ and the percentage change of price is $\frac{\Delta p}{p}$.

$$\frac{\text{percentage change in demand}}{\text{percentage change in price}} = \frac{\Delta q/q}{\Delta p/p} = \frac{p}{q} \frac{\Delta q}{\Delta p} \approx \frac{p}{q} \frac{dq}{dp}.$$

This book multiplies the ratio by negative one in the definition of elasticity simply to make elasticity a positive number. (Many other books define it without the negative sign.)

Chapter 7

Integration

SEC. 7.1: ANTIDERIVATIVES

Date _____

ANTIDERIVATIVES AND INDEFINITE INTEGRALS

Up to this point in the course, one of our major problems has been

Given a function f , find the derivative f' .

We needed to be able to find the derivative of a function, before we could use it!

Today, we are going to look at the opposite problem, which also occurs in many applications.

Given the derivative f' , find the function f .

or we could state this problem as:

Given a function f , find a function F , such that $F' = f$.

Examples:

(a) If $f'(x) = 20x^4$, then $f(x) =$ _____

A question about this answer. Is this a unique answer? _____

So, we say the antiderivative to $f'(x) = 20x^4$ is $f(x) =$ _____.

This solution represents a “family of functions” for the antiderivative and is called a “general antiderivative”. The C is called the “constant of integration” and should be part of every answer to an antiderivative problem. (We will talk about finding “C” later.)

See Video - <http://www.showme.com/sh/?h=W35Tz8K>

(b) If $f(x) = x^4$, then an antiderivative $F(x) =$ _____.

(c) If $f(x) = x^{15}$, then an antiderivative $F(x) =$ _____.

(d) If $f(x) = \frac{1}{x^2} =$ _____, then $F(x) =$ _____ = _____

See Video - <http://www.showme.com/sh/?h=N0FtcEC>

(e) If $f(x) = \sqrt{x} =$ _____, then $F(x) =$ _____ = _____

Before we write this as a property, let's discuss some new notation.

Derivative Notation

1. If $f(x) = x^2$, then $f'(x) =$ _____

2. $\frac{d}{dx}(x^2) =$ _____

Antiderivative Notation

1. If $f(x) = x^2$, then $F(x) =$ _____

2. $\int x^2 dx =$ _____

This is called an *indefinite integral*.

INDEFINITE INTEGRAL: If $F'(x) = f(x)$, then

$$\int f(x)dx = F(x) + C,$$

for any real number C .

In the indefinite integral notation, do not forget the “ dx ” at the end!

ANTIDERIVATIVES AND INDEFINITE INTEGRALS

Since the antiderivative is just the undoing of the derivative, we will undo the derivative rules.

POWER RULE: For any real number $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

(The antiderivative of $f(x) = x^n$ for $n \neq -1$ is found by increasing the exponent n by 1 and dividing x raised to the new power by the new value of the exponent.)

Examples: Find the following indefinite integrals.

(a) $\int x^5 dx$

See Video - <http://www.showme.com/sh/?h=bdhVN6O>

(b) $\int \frac{1}{\sqrt[3]{x}} dx$

See Video - <http://www.showme.com/sh/?h=qH977qK>

CLICKER QUESTION 49: Evaluate the following indefinite integral:

$$\int \frac{1}{x^4} dx$$

- (A) $-\frac{4}{x^5} + C$ (B) $-\frac{1}{3x^3} + C$ (C) $-\frac{4}{x^3} + C$ (D) $-\frac{1}{3x^5} + C$ (E) None of these

See Video - <http://www.showme.com/sh/?h=4uaisYC>

Let's find some more antiderivatives, and, from the results, perhaps obtain some properties of indefinite integrals.

Examples and Properties: (Assume k is a constant)

1. $\int 7dx = \underline{\hspace{2cm}}$ **Property:** $\int kdx = \underline{\hspace{2cm}}$

See Video - <http://www.showme.com/sh/?h=YBTwOAK>

2. $\int 5x^7 dx = \underline{\hspace{2cm}}$ **Property:** $\int (k \cdot f(x)) dx = \underline{\hspace{2cm}}$

3. $\int (2 - x^3) dx = \underline{\hspace{2cm}}$ **Property:** $\int (f(x) \pm g(x)) dx = \underline{\hspace{2cm}}$
 $\hspace{30em} = \underline{\hspace{2cm}}$

CONSTANT MULTIPLE AND SUM OR DIFFERENCE RULE: *If all indicated integrals exist,*

$$\int k \cdot f(x) dx = k \int f(x) dx,$$

for any real number k , and

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$$

(The antiderivative of a constant times a function is the constant times the antiderivative of the function. The antiderivative of a sum or difference of functions is the sum or difference of the antiderivatives.)

Examples: Find the following indefinite integrals.

$$(a) \int \left(17x^{5/2} - \frac{4}{x^3} + 5 \right) dx$$

See Video - <http://www.showme.com/sh/?h=dm5xm8O>

$$(b) \int 4x(x^2 + 2x) dx$$

See Video - <http://www.showme.com/sh/?h=72zBHAc>

INDEFINITE INTEGRALS FOR EXPONENTIAL FUNCTIONS

Let's recall the derivative rule for exponential functions, and use it to find antiderivatives for exponential functions.

Examples:

(a) Since $\frac{d}{dx} [e^x] = \underline{\hspace{2cm}}$, then $\int e^x dx = \underline{\hspace{2cm}}$

(b) Since $\frac{d}{dx} [e^{kx}] = \underline{\hspace{2cm}}$, then $\int e^{kx} dx = \underline{\hspace{2cm}}$

(c) Since $\frac{d}{dx} [2^x] = \underline{\hspace{2cm}}$, then $\int 2^x dx = \underline{\hspace{2cm}}$

(d) Since $\frac{d}{dx} [3^{kx}] = \underline{\hspace{2cm}}$, then $\int 3^{kx} dx = \underline{\hspace{2cm}}$

See Video - <http://www.showme.com/sh/?h=CdbCfYG>

INDEFINITE INTEGRALS OF EXPONENTIAL FUNCTIONS:

$$\int e^x dx = e^x + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C, k \neq 0$$

For $a > 0, a \neq 1$:

$$\int a^x dx = \frac{a^x}{\ln(a)} + C$$

$$\int a^{kx} dx = \frac{a^{kx}}{k \ln(a)} + C, k \neq 0$$

(The antiderivative of the exponential function e^x is itself. If x has a coefficient of k , we must divide by k in the antiderivative. If the base is not e , we must divide by the natural logarithm of the base.)

Examples: Find the following indefinite integrals:

(a) $\int 3e^x dx$

See Video - <http://www.showme.com/sh/?h=RH2oQIC>

(b) $\int 4e^{-0.2x} dx$

See Video - <http://www.showme.com/sh/?h=IEDE3WK>

(c) $\int (e^{2u} + 4u) du$

See Video - <http://www.showme.com/sh/?h=9BNdgcC>

CLICKER QUESTION 50: Evaluate the following indefinite integral:

$$\int (5x^4 - 2e^{-0.1x}) dx$$

- (A) $x^5 + 20e^{-0.1x} + C$ (B) $x^5 + 20e^{0.9x} + C$ (C) $20x^3 + 0.2e^{-0.1x} + C$
(D) $20x^3 + 0.2e^{-1.1x} + C$ (E) None of these

See Video - <http://www.showme.com/sh/?h=Elzf4aG>

THE INDEFINITE INTEGRAL OF $\frac{1}{x}$

There is one more antiderivative property we need. What is $\int \frac{1}{x} dx = \int x^{-1} dx$? As we have seen,

we can't apply the Power Rule because _____

See Video - <http://www.showme.com/sh/?h=KMbgSga>

INDEFINITE INTEGRAL OF $\frac{1}{x}$:

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln |x| + C$$

(The antiderivative of $f(x) = x^n$ for $n = -1$ is the natural logarithm of the absolute value of x .)

Examples: Find the following indefinite integrals

(a) $\int \frac{3}{x} dx$

See Video - <http://www.showme.com/sh/?h=PxDhqWO>

(b) $\int \left(\frac{9}{x} - 3e^{-0.4x} \right) dx$

See Video - <http://www.showme.com/sh/?h=2GwVj0a>

$$(c) \int (10u^{-3.5} + \pi u^{-1}) du$$

See Video - <http://www.showme.com/sh/?h=GuO7TkW>

$$(d) \int \frac{\sqrt{x} + 1}{\sqrt[3]{x}} dx$$

See Video - <http://www.showme.com/sh/?h=VXpjEUS>

$$(e) \int \frac{e + 2t^3}{4t} dt$$

See Video - <http://www.showme.com/sh/?h=b8RkcSW>

$$(f) \int (x + \sqrt{2})^2 dx$$

See Video - <http://www.showme.com/sh/?h=S5cAFYO>

SPECIFIC ANTIDERIVATIVES (OR SOLVING FOR C)

One final topic. We know that the *general antiderivative* of

$$f(x) = \frac{3}{x} - 4x \text{ is } F(x) = \underline{\hspace{10cm}}$$

What would we need to find the *specific antiderivative*; that is, what do we need to find the value of C ?

So, if we were given that $f(x) = \frac{3}{x} - 4x$, and $F(1) = -10$, where F is the antiderivative of f , find the *specific antiderivative* of $f(x)$.

See Video - <http://www.showme.com/sh/?h=mJfnOOe>

Therefore, if $f(x) = \frac{3}{x} - 4x$ and $F(1) = -10$, then $F(x) = \underline{\hspace{10cm}}$

Example: If $f(x) = e^{2x} - 9x^2$ and $F(0) = 4$, find $F(x)$, the specific antiderivative of $f(x)$.

See Video - <http://www.showme.com/sh/?h=u2gHBGS>

Example: Find the cost function if the marginal cost is

$$C'(x) = x + \frac{1}{x^2}$$

and 2 units cost \$5.50.

See Video - <http://www.showme.com/sh/?h=WMP53cO>

Example: Find the demand function if the marginal revenue function is

$$R'(x) = 500 - 0.15\sqrt{x}.$$

See Video - <http://www.showme.com/sh/?h=zDIIZEW>

Example: The marginal profit in dollars on Brie cheese sold at a cheese store is given by

$$P'(x) = x(50x^2 + 30x),$$

where x is the amount of cheese sold, in hundreds of pounds. The “profit” is -\$40 when no cheese is sold.

(a) Find the profit function.

See Video - <http://www.showme.com/sh/?h=e5PYqki>

(b) Find the profit from selling 200 lb of Brie cheese.

In the last class we learned how to find antiderivatives of some functions. We learned that:

$$(a) \int \frac{1}{\sqrt{x}} dx =$$

See Video - <http://www.showme.com/sh/?h=QpH1z7o>

$$(b) \int 4e^{5x} dx =$$

$$(c) \int \left(5 + \frac{7}{x}\right) dx =$$

Our goal today is to “undo” the chain rule. Let’s recall the chain rule:

Example: Find the derivative of $h(x) = e^{x^2}$.

Solution. Note that this function is the composition of two functions - the “outer function” $f(x) = e^x$ and the “inner function” $g(x) = x^2$ (so that $h(x) = f(g(x))$). Now we can apply the chain rule:

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x).$$

Since $f'(x) = e^x$ and $g'(x) = 2x$, we get

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x) = f(x^2) \cdot 2x = e^{x^2} \cdot 2x.$$

Today, we want to undo this process, so we want to find $\int 2xe^{x^2} dx$. The key will be indentifying the “inner function” (in this case, the $g(x) = x^2$).

Example: Find the indefinite integral $\int 2xe^{x^2} dx$.

Solution. The inner function of a composition is $g(x) = x^2$. We will introduce a new variable u , and make a change to the integral - changing it from an integral of x , to an integral in u . However, when we do this, we must change the entire integral to u , including changing dx to a du .

So we let $u = x^2$. Now we will determine how dx and du are related. Note that $\frac{du}{dx} = 2x$ (recall - this is just the Leibniz notation of the derivative). This means that¹

$$du = 2x dx.$$

This is the equation that relates du and dx , and allows us to change the integral from one of x to one of u .

$$\int 2xe^{x^2} dx = \int e^{\overbrace{x^2}^u} \underbrace{(2x dx)}_{du} = \int e^u du.$$

Note that the last integral has been completely converted to an integral of u , and is much easier to solve than the original one.

$$\int e^u du = e^u + C.$$

We back-substitute, and get

$$e^u + C = e^{x^2} + C.$$

Finally, this process looks like

$$\int 2xe^{x^2} dx = \int e^u du = e^u + C = e^{x^2} + C.$$

This method of finding indefinite integrals is called substitution.

¹It looks like we are multiplying both sides of the equation $\frac{du}{dx} = 2x$ by dx . That is not really what is going on - however, it may help you remember it. It is one of the beautiful things about the Leibniz notation - it is not really a fraction, but if you treat it like one, it works out fine.

Example: Find $\int 4x^3\sqrt{x^4+5}dx$.

Solution. The inner function of a composition is $g(x) = x^4 + 5$, so we set $u = x^4 + 5$. Now we will determine how dx and du are related. Since $\frac{du}{dx} = 4x^3$,

$$du = 4x^3 dx.$$

Now we can change the integral from one of x to one of u .

$$\int 4x^3\sqrt{x^4+5}dx = \int \sqrt{\underbrace{x^4+5}_u} \underbrace{(4x^3 dx)}_{du} = \int \sqrt{u} du.$$

Now we can integrate the resulting integral of u

$$\int \sqrt{u} du = \frac{2u^{3/2}}{3} + C = \frac{2\sqrt{u^3}}{3} + C.$$

We back-substitute, and get

$$\frac{2\sqrt{u^3}}{3} + C = \frac{2\sqrt{(x^4+5)^3}}{3} + C.$$

Finally, our process looks like

$$\int 4x^3\sqrt{x^4+5}dx = \int \sqrt{u} du = \frac{2\sqrt{u^3}}{3} + C = \frac{2\sqrt{(x^4+5)^3}}{3} + C.$$

Examples: Find the following indefinite integrals.

(a) $\int 10x\sqrt{5x^2 + 3}dx$

See Video - <http://www.showme.com/sh/?h=M3EMQHg>

(b) $\int (t + 2)^8 dt$

See Video - <http://www.showme.com/sh/?h=gHHzYzg>

$$(c) \int \frac{6x^2}{2x^3 - 3} dx$$

See Video - <http://www.showme.com/sh/?h=iPgRy1g>

$$(d) \int -4e^{2-4x} dx$$

See Video - <http://www.showme.com/sh/?h=kY4uNBw>

(e) $\int \frac{2x + 2}{(x^2 + 2x + 1)^3} dx$

See Video - <http://www.showme.com/sh/?h=cJofpY0>

CLICKER QUESTION 51: Which of the following substitutions would be the most useful in determining the indefinite integral:

$$\int \frac{4x}{2x^2 + 4} dx$$

(A) $u = 4x$ (B) $u = x^2$ (C) $u = 2x^2 + 4$ (D) $u = x$ (E) None of these

See Video - <http://www.showme.com/sh/?h=huQhDW4>

Let's make a small change to an example we did earlier.

Example: Find $\int x^3\sqrt{x^4+5}dx$.

Solution. The inner function of a composition is $g(x) = x^4 + 5$, so we set $u = x^4 + 5$. Now we will determine how dx and du are related. Since $\frac{du}{dx} = 4x^3$,

$$du = 4x^3dx.$$

We rewrite the original integral to prepare for substitution

$$\int x^3\sqrt{x^4+5}dx = \int \sqrt{x^4+5}(x^3dx),$$

and we see there is only x^3dx - and we need $4x^3dx$ to make the change to du . However, since $du = 4x^3dx$, then $\frac{du}{4} = x^3dx$, and

$$\int x^3\sqrt{x^4+5}dx = \int \sqrt{\underbrace{x^4+5}_u} \underbrace{(x^3dx)}_{du/4} = \int \sqrt{u} \left(\frac{du}{4} \right).$$

Now we can integrate the resulting integral of u

$$\int \frac{\sqrt{u}}{4} du = \frac{u^{3/2}}{6} + C = \frac{\sqrt{u^3}}{6} + C.$$

We back-substitute, and get

$$\frac{\sqrt{u^3}}{6} + C = \frac{\sqrt{(x^4+5)^3}}{6} + C.$$

Finally, our process looks like

$$\int x^3\sqrt{x^4+5}dx = \int \frac{\sqrt{u}}{4} du = \frac{\sqrt{u^3}}{6} + C = \frac{\sqrt{(x^4+5)^3}}{6} + C.$$

Example: Find $\int \frac{x}{4x^2 + 1} dx$.

Solution. The inner function of a composition is $g(x) = 4x^2 + 1$, so we set $u = 4x^2 + 1$. Now we will determine how dx and du are related. Since $\frac{du}{dx} = 8x$,

$$du = 8x dx, \text{ or } \frac{du}{8} = x dx.$$

Now we can change the integral from one of x to one of u .

$$\int \frac{x}{4x^2 + 1} dx = \int \frac{\overbrace{x}^{du/8}}{\underbrace{4x^2 + 1}_u} dx = \int \frac{du/8}{u} = \int \frac{du}{8u}.$$

Now we can integrate the resulting integral of u

$$\int \frac{du}{8u} = \frac{\log(u)}{8} + C.$$

We back-substitute, and get

$$\frac{\log(u)}{8} + C = \frac{\log(4x^2 + 1)}{8} + C.$$

Finally, our process looks like

$$\int \frac{x}{4x^2 + 1} dx = \int \frac{du}{8u} = \frac{\log(u)}{8} + C = \frac{\log(4x^2 + 1)}{8} + C.$$

Examples: Find the following indefinite integrals.

(a) $\int \frac{1}{\sqrt{3-4x}} dx$

See Video - <http://www.showme.com/sh/?h=PolWTq4>

(b) $\int x^3 e^{-5x^4} dx.$

See Video - <http://www.showme.com/sh/?h=k2p9cgK>

(c) $\int x^2\sqrt{x^3 - 10}dx.$

See Video - <http://www.showme.com/sh/?h=tuE5oiO>

(d) $\int \frac{20x}{4 + 5x^2}dx.$

See Video - <http://www.showme.com/sh/?h=1dEZbY8>

CLICKER QUESTION 52: If we use the substitution $u = 3x^4 + 6x$, then which of the following is equivalent to

$$\int \frac{2x^3 + 1}{(3x^4 + 6x)^4} dx.$$

- (A) $\int \frac{u^4}{6} du$ (B) $\int \frac{6du}{u^4}$ (C) $\int \frac{du}{6u^4}$ (D) $\int \frac{du}{u^4}$ (E) none of these

See Video - <http://www.showme.com/sh/?h=Nzgf9Ye>

SUBSTITUTION METHOD:

1. We make a variable substitution and let u equal an “inside function”.
2. We then find du , and see if the Substitution Method applies.
3. If it does, we then express the integrand in terms of u .
4. We then find this antiderivative of the function expressed in terms of u .
5. Finally, we rewrite the answer back in terms of x .

Examples: Find the following indefinite integrals.

(a) $\int \frac{e^{\sqrt{y}}}{2\sqrt{y}} dy$

See Video - <http://www.showme.com/sh/?h=hiUXXcm>

(b) $\int \frac{1}{x(\ln(x))} dx$

See Video - <http://www.showme.com/sh/?h=wLw9IMi>

Example: The rate of growth of profit (in millions of dollars per year) from a new technology is approximated by

$$P'(x) = xe^{-x^2},$$

where x represents time measured in years. The total profit in the third year that the new technology is in operation is \$10,000.

(a) Find the total profit function.

See Video - <http://www.showme.com/sh/?h=MAbnp0i>

(b) What happens to the total amount of profit in the long run?

DISTANCE AND ACCUMULATED CHANGE

So far in chapter 7, we have looked at undoing derivative rules (finding anti-derivatives).

Example: $\int 3x^2 dx = x^3 + C$.

Remember that $\int f(x)dx$ is called the *indefinite integral* and the symbols $\int \cdots dx$ can be thought of as “find the anti-derivative of what is in between”.

In this section, we introduce the *definite integral*. The notation for the definite integral is $\int_a^b f(x)dx$. The terminology (definite vs. indefinite) and the notation are very close, but they are different things! While the indefinite integral is a class of functions, the definite integral is a number.

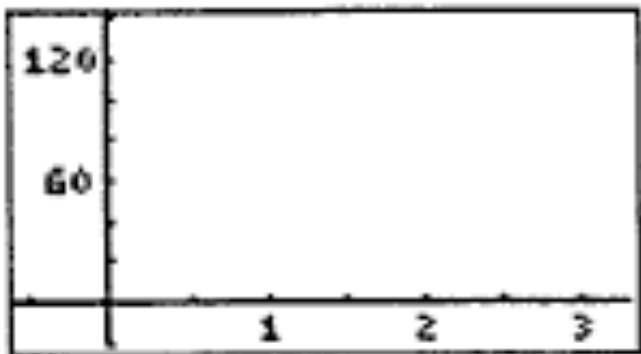
By the end of this section, make sure you

- understand what $\int_a^b f(x)dx$ means;
- can compute the value of $\int_a^b f(x)dx$ using
 - left-hand sums (LHS);
 - right-hand sums (RHS);
 - the average of LHS and RHS;
 - midpoint sums
 - the built-in feature on your calculator;
- understand two interpretations of $\int_a^b f(x)dx$
 - the (signed) area under the curve $f(x)$ from $x = a$ to $x = b$;
 - if $f(x)$ is the rate of change, then $\int_a^b f(x)dx$ gives the total change from $x = a$ to $x = b$.

AN INTRODUCTORY EXAMPLE

The distance from Tucson to Mesa, Arizona, is approximately 120 miles.

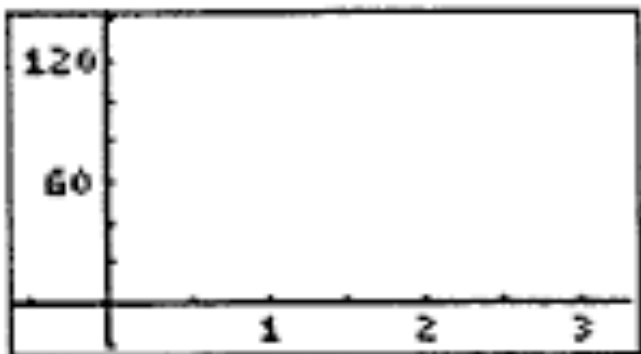
1. If I make this trip at a *constant velocity*, and it takes me 2 hours, how fast am I traveling? What would the graph of the velocity vs. time graph look like? Sketch it below.



An interesting question is, how could the velocity vs. time graph tell the distance I traveled on the trip?

See Video - <http://www.showme.com/sh/?h=qtmJH4y>

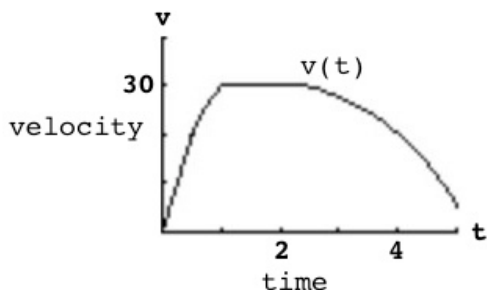
2. Suppose that instead of driving 60 mph for the entire trip, I drive 40 mph for $\frac{3}{4}$ hour. Then I get on a highway and I drive 70 mph for 1 hour, then get off the highway and drive 40 mph again for $\frac{1}{2}$ hour. Sketch the velocity vs. time graph below.



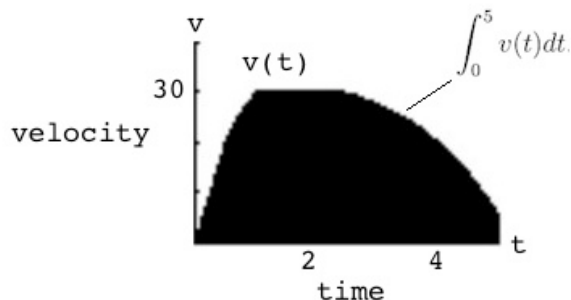
Will I get to my destination (120 miles from Tucson) in that amount of time? Explain!

There are a couple of ideas we want to take from these examples.

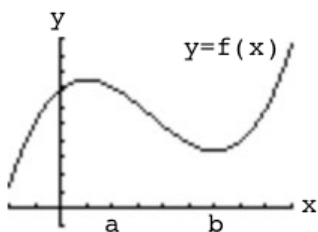
1. (a) If we have a velocity function (the rate of change of position) plotted on a set of axes, such as:



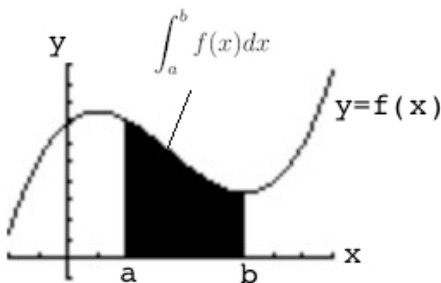
then finding the area under the velocity function is a meaningful calculation, and will give us distance traveled (the total change in position). The process of finding the area under the velocity function from time $t = 0$ to time $t = 5$ is called the *definite integral* of the velocity function and is denoted by $\int_0^5 v(t)dt$.



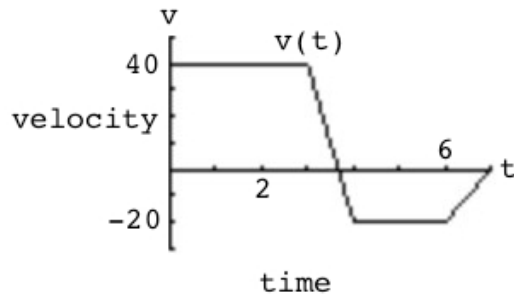
- (b) We can find the area under any function, not just velocity functions.



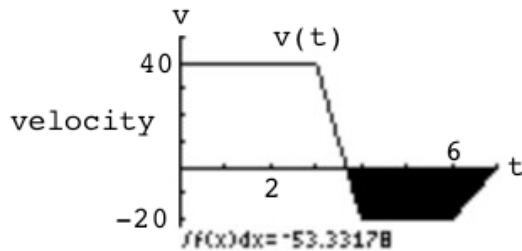
If $y = f(x)$ is a function, then the (signed) area under the function $f(x)$ from $x = a$ to $x = b$ is given by $\int_a^b f(x)dx$.



2. (a) If we have a velocity function, what does negative velocity mean? For example, if the velocity function looked like

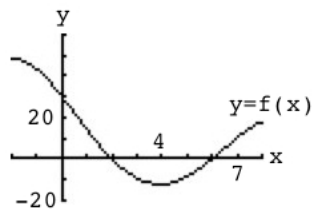


what does the part under the t -axis represent? How should the area under the t -axis be counted?

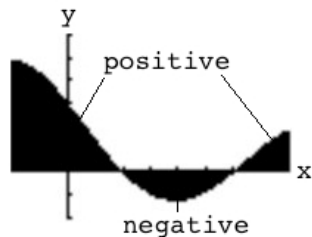


Since the negative velocity means traveling in the opposite direction of a positive velocity, any area below the t -axis should be counted negatively. (If you travel east from Tucson on I-10 at 70 mph for 2 hours, then turn around and travel west on I-10 at 60 mph for 1 hour, how far will you be from Tucson?) So area above the t -axis is counted as positive distance traveled, and area below the t -axis is counted as negative distance traveled.

- (b) For a general function $y = f(x)$, the same principle applies.



When we compute the definite integral $\int_a^b f(x) dx$, the part above the x -axis is counted positively, and the part below the x -axis is counted negatively. Because of this, we say the definite integral gives a *signed area*.



In the textbook, they define the definite integral as follows.

THE DEFINITE INTEGRAL: If $f(x)$ is defined on the interval $[a, b]$, the definite integral of f from a to b is given by^a

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x,$$

provided the limit exists, where $\Delta x = \frac{b-a}{n}$ and x_i is any value of x in the i th interval.

a is called the lower limit of integration and b is called the upper limit of integration.

^aWe are using summation notation (the \sum) which is a convenient way of writing the sum of a bunch of numbers.

$$\sum_{i=1}^n f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

Don't worry, you won't be tested on this notation.

- 3. (a)** We see that if $v(t)$ is the velocity at time t , then $\int_0^5 v(t)dt$ is the area under the velocity function from $t = 0$ to $t = 5$. In addition, the integral would also give the change in position (i.e. distance traveled) from time $t = 0$ to time $t = 5$. In other words, if we integrate the rate of change of position (which is velocity) from 0 to 5, then the result is the total change in position from $t = 0$ to $t = 5$.

$$\begin{aligned} \int_0^5 (\text{Rate of Change of Position}) dt &= \text{Total Change of Position from } t = 0 \text{ to } t = 5 \\ &= (\text{Position at } t = 5) - (\text{Position at } t = 0) \end{aligned}$$

- (b)** Again, we can apply this idea to any rate of change function. If $f(x)$ represents a rate of change function, then $\int_a^b f(x)dx$ gives the total change from $x = a$ to $x = b$.

Again, the textbook gives the following definition.

TOTAL CHANGE IN $F(x)$: If $f(x)$ gives the rate of change of $F(x)$ for x in $[a, b]$, then the total change in $F(x)$ as x goes from a to b is given by

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x = \int_a^b f(x)dx.$$

COMPUTING DEFINITE INTEGRALS USING FORMULAS FROM GEOMETRY

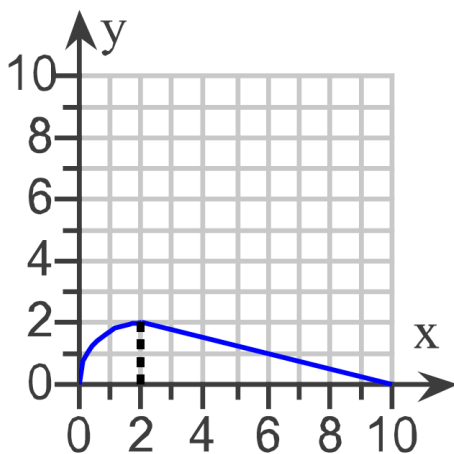
Since we are computing areas, a few formulas from geometry will be helpful. The main ones we will use are:

$$\text{Area of a Rectangle} = (\text{Base})(\text{Height})$$

$$\text{Area of a Triangle} = \frac{1}{2}(\text{Base})(\text{Height})$$

$$\text{Area of a Circle} = \pi (\text{Radius})^2$$

Example: Find $\int_0^{10} f(x)dx$ for the graph of $y = f(x)$ below, where $f(x)$ consists of line segments and circular arcs.



See Video - <http://www.showme.com/sh/?h=aRFqBvM>

Example: Find the exact value of the integral using formulas from geometry.

$$\int_0^4 \sqrt{16 - x^2} dx$$

See Video - <http://www.showme.com/sh/?h=ROQFp1E>

CLICKER QUESTION 53: Find the exact value of the integral using formulas from geometry.

$$\int_1^5 (2 + 3x) dx$$

The exact value of the integral is

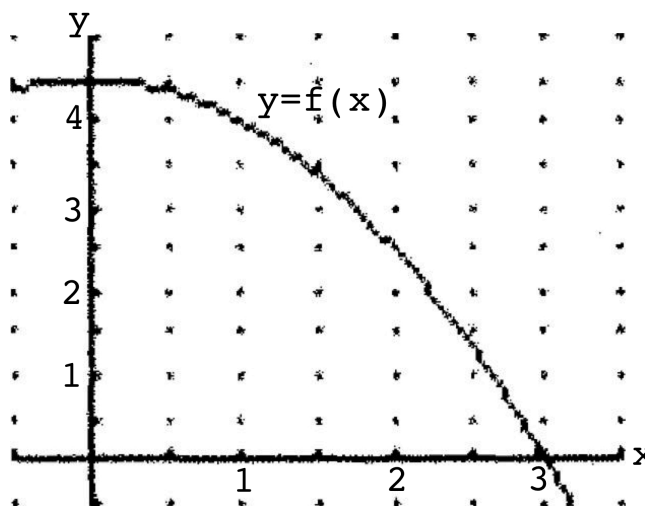
- (A) less than 35 (B) between 35 and 40 (C) between 40 and 45
(D) between 45 and 50 (E) more than 50

See Video - <http://www.showme.com/sh/?h=EtN6TRY>

APPROXIMATING DEFINITE INTEGRALS USING RECTANGLES

Example: The graph of a function f is given below. Approximate the area under the function from $x = 0$ to $x = 3$ with rectangles, using the following methods with $n = 3$ (i.e. use 3 rectangles).

- (a) Use left endpoints (LHS).
- (b) Use right endpoints (RHS).
- (c) Average the LHS and RHS.
- (d) Use midpoints.



Left-hand sum = _____

See Video - <http://www.showme.com/sh/?h=H1lYsTY>

Right-hand sum = _____

Average of LHS and RHS = _____

Midpoint sum = _____

CLICKER QUESTION 54: Approximate the area under the graph of $f(x) = 2^x$ from $x = -2$ and $x = 6$ and above the x -axis with rectangles, using the following methods with $n = 4$.

(i) The approximation using left endpoints is

- (A) less than 50 (B) between 50 and 75 (C) between 75 and 100
(D) between 100 and 125 (E) more than 125

(ii) The approximation using right endpoints is

- (A) less than 50 (B) between 50 and 75 (C) between 75 and 100
(D) between 100 and 125 (E) more than 125

(iii) The approximation using the average of LHS and RHS is

- (A) less than 50 (B) between 50 and 75 (C) between 75 and 100
(D) between 100 and 125 (E) more than 125

(iv) The approximation using midpoints is

- (A) less than 50 (B) between 50 and 75 (C) between 75 and 100
(D) between 100 and 125 (E) more than 125

See Video - <http://www.showme.com/sh/?h=SCzbeHQ>

APPROXIMATING DEFINITE INTEGRALS USING A GRAPHING CALCULATOR

We can approximate definite integrals with a graphing calculator. If you have a TI calculator, this can be done in two ways:

1. From the Home Screen, press the **MATH** key and select **9:fnInt**. (This stands for a “function numerical integral”.)

Then to evaluate $\int_{-2}^6 2^x dx$, enter: **fnInt(2^X,X,-2,6)**

2. If the function has been graphed on your calculator with the interval included, from the graph select **CALC**, and **7:∫ f(x)dx**, and enter the limits of integration.

Example: Approximate the definite integral $\int_{-1}^4 e^{-x^2} dx$

See Video - <http://www.showme.com/sh/?h=JAA1HVY>

SEC. 7.4: THE FUNDAMENTAL THEOREM OF CALCULUS Date _____

In the first two sections of this chapter we computed antiderivatives (or indefinite integrals), and in the third section we computed areas under curves (the definite integral). The goal of this section is to show how those two computations are related.

THE FUNDAMENTAL THEOREM OF CALCULUS (FTC): *Let $f(x)$ be continuous (no breaks) on the interval $[a, b]$, and let F be any antiderivative of f . Then*

$$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$$

We can use the FTC to evaluate definite integrals, as long as we can find an antiderivative of the integrand.

Example: Evaluate the definite integral $\int_{-3}^3 (9 - x^2) dx$.

See Video - <http://www.showme.com/sh/?h=xAVV7cu>

Examples: Evaluate the following definite integrals with the FTC.

(a) $\int_4^9 \sqrt{x} dx$

See Video - <http://www.showme.com/sh/?h=QROidCy>

(b) $\int_1^2 \frac{3}{x^4} dx$

See Video - <http://www.showme.com/sh/?h=2l7WVhA>

$$(c) \int_0^1 (5e^x + 6x) dx$$

See Video - <http://www.showme.com/sh/?h=HOZ8GR6>

$$(d) \int_1^3 \frac{1}{t} dt$$

See Video - <http://www.showme.com/sh/?h=kfSLluy>

THE FTC WITH THE SUBSTITUTION METHOD

How do we use the FTC to evaluate a definite integral if it is necessary to use the Substitution Method to find the antiderivative?

There are two methods that we can use to do this. Let's look at an example.

Example: Evaluate: $\int_0^1 x(x^2 + 1)^3 dx$

Method #1: To evaluate this definite integral using the FTC, we first need to find an antiderivative of the function $f(x) = x(x^2 + 1)^3$, which requires the Substitution Method. So, let's first treat the problem as an indefinite integral (antiderivative). and find:

$$\int x(x^2 + 1)^3 dx =$$

See Video - <http://www.showme.com/sh/?h=SZnB2Ey>

Now, that we have the antiderivative, we can find the definite integral.

$$\int_0^1 x(x^2 + 1)^3 dx =$$

Note: Check the answer with your calculator.

Method #2: Once we have established the “change of variable” in the definite integral, and we have written the antiderivative in terms of the new variable, we can also rewrite the limits of integration in terms of the new variable.

$$\int_0^1 x(x^2 + 1)^3 dx =$$

See Video - <http://www.showme.com/sh/?h=JWxafT6>

With this method, we do not convert the antiderivative back to the original variable.

CLICKER QUESTION 55: *If we use the substitution $u = x^2$, then which of the following is equivalent to*

$$\int_1^3 xe^{x^2} dx.$$

(A) $\int_1^3 e^u du$

(B) $\int_1^3 \frac{e^u}{2} du$

(C) $\int_1^9 \frac{e^u}{2} du$

(D) $\int_1^9 e^u du$

(E) none of these

See Video - <http://www.showme.com/sh/?h=P7Zc3Iu>

Let's try some more examples to get some practice.

Examples: Analytically, evaluate the following definite integrals. You can then check answers with your calculator.

(a) $\int_0^2 x^2 \sqrt{1+x^3} dx$

See Video - <http://www.showme.com/sh/?h=71uRJIA>

$$(b) \int_0^4 \frac{dx}{\sqrt{2x+1}}$$

See Video - <http://www.showme.com/sh/?h=uWrHxx2>

$$(c) \int_1^2 \frac{dx}{(3-5x)^2}$$

See Video - <http://www.showme.com/sh/?h=KLWwUjI>

$$(d) \int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

See Video - <http://www.showme.com/sh/?h=Pw8xshM>

$$(e) \int_0^3 \frac{6x}{x^2 + 1} dx$$

See Video - <http://www.showme.com/sh/?h=VWkzGXA>

Example: A small company of science writers found that its rate of profit (in thousands of dollars) after t years of operation is given by the function below.

$$P'(t) = (6t + 12)(t^2 + 4t + 4)^{\frac{1}{4}}.$$

(a) Find the total profit in the first four years.

See Video - <http://www.showme.com/sh/?h=dFIT3XE>

(b) Find the profit in the fifth year of operation.

(c) What is happening to the annual profit in the long run?

PROPERTIES OF THE DEFINITE INTEGRAL

PROPERTIES OF DEFINITE INTEGRALS: *If all indicated integrals exist,*

$$1. \int_a^a f(x)dx = 0;$$

$$2. \int_a^b k \cdot f(x)dx = k \int_a^b f(x)dx \text{ for any real constant } k,$$

$$3. \int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

$$4. \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \text{ for any real number } c,$$

$$5. \int_a^b f(x)dx = - \int_b^a f(x)dx.$$

Note that the 5th property above is how we reverse the limits of integration.

Example: Find $\int_7^{-2} 6x^2 dx$

See Video - <http://www.showme.com/sh/?h=n7APFhY>

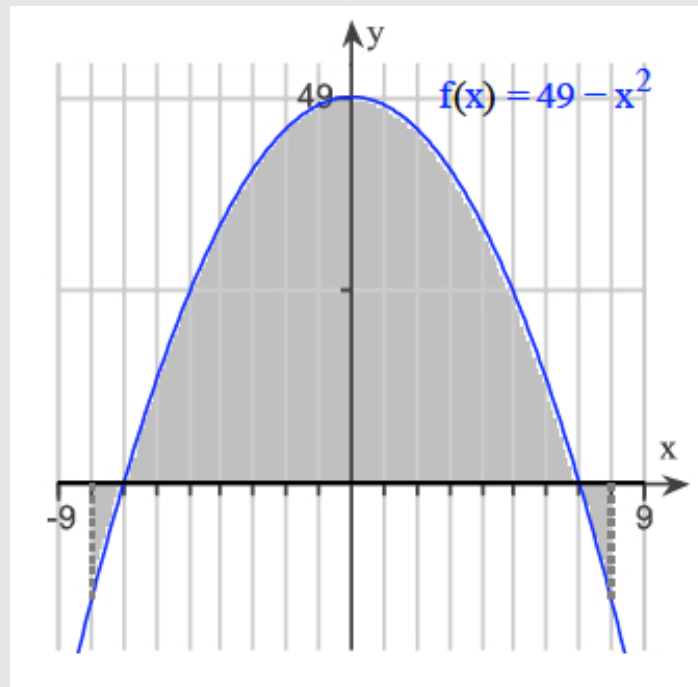
Example: Use the properties of definite integrals to find $\int_4^8 f(x)dx$ for the following function

$$f(x) = \begin{cases} 1.5x - 1 & \text{if } x \leq 6 \\ -2x + 20 & \text{if } x > 6 \end{cases}$$

See Video - <http://www.showme.com/sh/?h=RZHfX5U>

Example: Use the definite integral to find the area between the x -axis and $f(x) = 16 - x^2$ over the interval $[0, 10]$. Check first to see if the graph crosses the x -axis in the given interval. (Note: In this question, they want all areas to be counted positively, not just areas above the x -axis.)

CLICKER QUESTION 56: Find the total area of the shaded regions.



The total area of the shaded regions is

- (A) between 440 and 450 (B) between 450 and 460 (C) between 460 and 470
(D) between 470 and 480 (E) between 480 and 490

See Video - <http://www.showme.com/sh/?h=F4EWBNY>

AREA BETWEEN TWO CURVES: If f and g are continuous and $f(x) \geq g(x)$ on the interval $[a, b]$, then the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b (f(x) - g(x)) dx.$$

Example: Find the area between the curves $y = 10x$ and $y = x^2 - 12$ on the interval $[-1, 3]$.

See Video - <http://www.showme.com/sh/?h=FhwSIZU>

Finding the area between curves can become a bit more difficult when the curves cross.

CLICKER QUESTION 57: Find the area between the curves:

$$x = -2, \quad x = 2, \quad y = 3e^{3x}, \quad y = 2e^{3x} + 3$$

The area is

- (A) less than 110 (B) between 110 and 120 (C) between 120 and 130
(D) between 130 and 140 (E) more than 140

See Video - <http://www.showme.com/sh/?h=QtAV4Vc>

Example: A construction company has an expenditure rate of $E'(x) = e^{0.14x}$ dollars per day on a particular paving job and an income rate of $I'(x) = 133.9 - e^{0.14x}$ dollars per day on the same job, where x is the number of days from the start of the job. The company's profit on the job will equal total income less total expenditures. Profit will be maximized if the job ends at the optimum time, which is the point where the two curves meet.

(a) Find the optimum number of days for the job to last.

See Video - <http://www.showme.com/sh/?h=8nVKKpc>

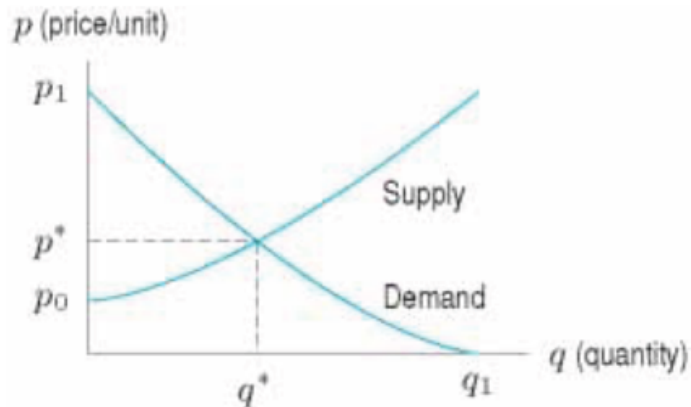
(b) Find the total income for the optimum number of days.

(c) Find the total expenditures for the optimum number of days.

(d) Find the maximum profit for the job.

CONSUMER AND PRODUCER SURPLUS

Consider the typical supply and demand curves.



- (a) What does the demand curve mean? (What does each point on the curve mean?)

- (b) What does the supply curve mean? (What does each point on the curve mean?)

- (c) What is the significance of the point (q^*, p^*) ?

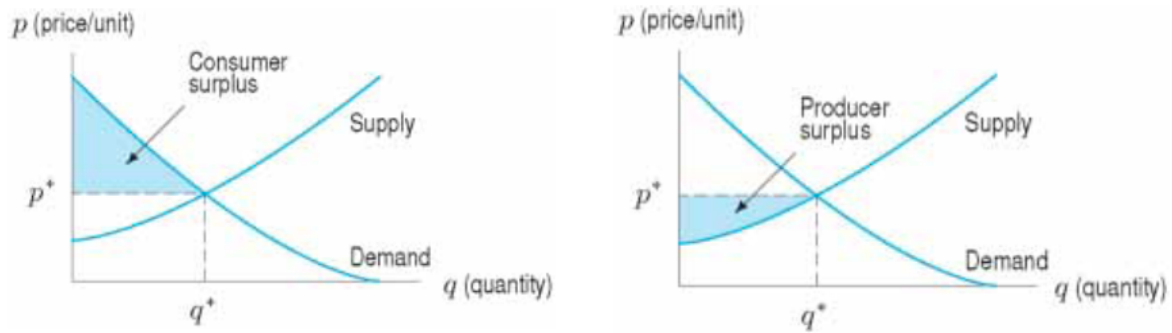
In the typical scenario, we assume that the market has settled to its equilibrium - that is the price is p^* and the quantity sold is q^* .

Note that in this scenario, there are many customers who end up paying less than they were willing to pay - and these consumers benefit from the current price (in this case, the equilibrium price). The *consumer surplus* measures the consumers' gain from trade. It is the total amount gained by consumers by buying the item at the current price rather than at the price they would have been willing to pay.

In addition, there are many producers who would have supplied the product at lower price - and these producers benefit from the current price (in this case, the equilibrium price). The producer surplus measures the suppliers' gain from trade. It is the total amount gained by producers by selling at the current price, rather than at the price they would have been willing to accept.

Both the consumer and the producer benefit from trade - and the *consumer surplus* and the *producer surplus* quantify that benefit.

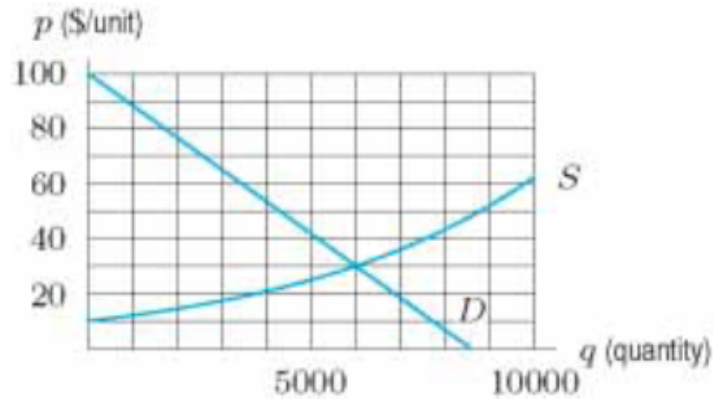
Visualizing and Computing the Consumer and Producer Surplus



From these graphs, we can determine the integrals we want to set up.

Examples:

1. Consider the following supply and demand curves.

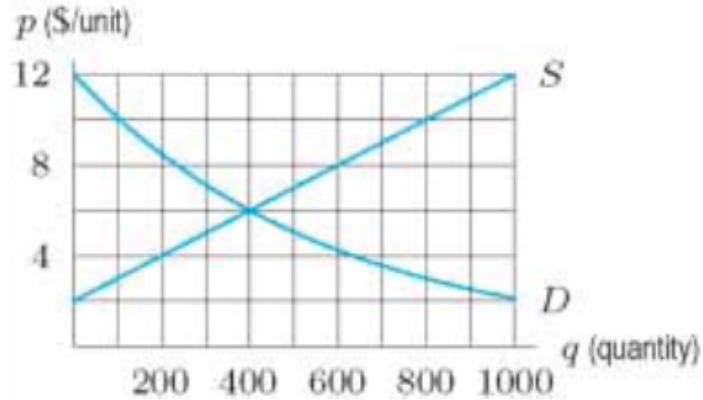


- (a) What are the equilibrium price and quantity for the supply and demand curves above?

See Video - <http://www.showme.com/sh/?h=c4OXqRk>

- (b) Shade the areas representing the consumer and producer surplus and estimate them.

2. Consider the following supply and demand curves.



(a) What are the equilibrium price and quantity for the supply and demand curves above?

See Video - <http://www.showme.com/sh/?h=5LHIM1o>

(b) Shade the areas representing the consumer and producer surplus and estimate them. The total gains from trade is the sum of consumer surplus and producer surplus. Estimate the total gains from trade.

(c) A price of \$4 is artificially imposed. Estimate the consumer and producer surplus and the total gains from trade.²

²This part is added to demonstrate why somebody would want to calculate consumer or producer surplus - it gives us a way to quantify the impact of prices that are imposed by regulation or monopolies.

3. Supply and demand data are in given below.

| | | | | | | | |
|----------------|----|-----|-----|-----|-----|-----|-----|
| q (quantity) | 0 | 100 | 200 | 300 | 400 | 500 | 600 |
| p (\$/unit) | 60 | 50 | 41 | 32 | 25 | 20 | 17 |

| | | | | | | | |
|----------------|----|-----|-----|-----|-----|-----|-----|
| q (quantity) | 0 | 100 | 200 | 300 | 400 | 500 | 600 |
| p (\$/unit) | 10 | 14 | 18 | 22 | 25 | 28 | 34 |

(a) Which table shows supply and which shows demand? Why?

See Video - <http://www.showme.com/sh/?h=jnP1dRo>

(b) Estimate the equilibrium price and quantity.

(c) Estimate the consumer and producer surplus.

4. The demand curve for a product has equation $p = 20e^{-0.002q}$ and the supply curve has equation $p = 0.02q + 1$ for $0 \leq q \leq 1000$, where q is quantity and p is price in \$/unit.

(a) Which is higher, the price at which 300 units are supplied or the price at which 300 units are demanded? Find both prices.

See Video - <http://www.showme.com/sh/?h=41Sem7k>

(b) Sketch the supply and demand curves. Find the equilibrium price and quantity.

(c) Using the equilibrium price and quantity, calculate the consumer and producer surplus.

5. The demand curve for a product has equation $p = 35 - q^2$ and the supply curve has equation $p = 3 + q^2$ for $0 \leq q \leq 6$, where q is quantity and p is price in \$/unit.

(a) Sketch the supply and demand curves. Find the equilibrium price and quantity.

See Video - <http://www.showme.com/sh/?h=OFWHupk>

(b) Using the equilibrium price and quantity, calculate the consumer and producer surplus.

Chapter 8

Further Techniques and Applications of Integration

SEC. 8.3: CONTINUOUS MONEY FLOW

Date _____

In most businesses, revenue is generated on an almost continuous basis. For example, if you own a small shop that generates \$100,000 of revenue a year, you are making a fraction of that every day you are open, rather than one lump sum of \$100,000. So we will treat this kind of money flow (such as revenue) as if it were continuous.

If we have a rate of continuous money flow, we would like to be able to compute:

1. The total money flow
2. The present value of the money flow
3. The accumulated amount of the money flow

Keep in mind, there are two issues that complicate these computations. First, if your business is successful, the money flow will increase over time. The second is that because of inflation, the value of money is always decreasing. If you generate \$10,000 of revenue in January, it is not worth \$10,000 at the end of the year. The good news is that the integral helps us overcome these complications.

TOTAL MONEY FLOW

Recall one of the major interpretations of the definite integral: if $f(t)$ is the rate of change, then $\int_a^b f(t)dt$ gives the total change from $t = a$ to $t = b$. Applying this to money flow, we get:

TOTAL MONEY FLOW: *If $f(t)$ is the rate of money flow, then the total money flow over the time interval $t = 0$ to $t = T$ is given by*

$$\int_0^T f(t)dt.$$

(Note the difference between t - the variable, and T - a fixed amount of time.)

Example: The function $f(t) = 0.06t + 700$ represents the rate of flow of money in dollars per year. Determine the total money flow for the first 5 years.

See Video - <http://www.showme.com/sh/?h=rWPVQRs>

Example: The rate of continuous money flow starts at \$1000 and increases exponentially at 5% per year for 4 years. Determine the total money flow.

Solution. First we would like to determine an equation for $f(t)$, the rate of money flow. Since it starts at \$1000 and increases exponentially at 5% per year, then

$$f(t) = 1000e^{0.05t}.$$

Using the expression for total money flow in the box above, we get the integral representing the total money flow for the first 4 years

$$\int_0^4 1000e^{0.05t} dt.$$

Note that we have two methods to compute this integral - the Fundamental Theorem of Calculus, or numerically using technology¹. Using the Fundamental Theorem of Calculus, we get

$$\int_0^4 1000e^{0.05t} dt = \left[1000 \frac{e^{0.05t}}{0.05} \right]_0^4 = 20000e^{0.05(4)} - 20000e^{0.05(0)} = 20000e^{0.2} - 20000 \approx \$4428.06.$$

Using technology, we also get \$4428.06 (rounded to the nearest cent).

CLICKER QUESTION 58: The function $f(x) = 1000x - 100x^2$ represents the rate of flow of money in dollars per year. Determine the total money flow for the first 10 years.

The total money flow is

- (A) less than \$15,000 (B) between \$15,000 and \$15,500
(C) between \$15,500 and \$16,000 (D) between \$16,000 and \$16,500
(E) more than \$16,500

See Video - <http://www.showme.com/sh/?h=Tq8JIno>

¹Note that we will encounter examples that we cannot do using the Fundamental Theorem of Calculus, and we will be forced to compute those with technology.

PRESENT VALUE OF MONEY FLOW

The total money flow does not take into account inflation. As we know, \$1000 today is worth more than \$1000 will be in 10 years. Another way to look at this is that if we were paid \$1000 today, we could put it in an account and earn interest on it. If someone pays us \$1000 in 10 years - we have missed out on 10 years of interest on that money. We introduced the idea of “present value” to determine the value of money paid to us in the future.

Example: Richard Hamm must make a balloon payment of \$1000 in 10 years. Find the present value of the payment if it includes 2.5% interest, compounded continuously.

Solution. We use the continuous compounding formula

$$A = Pe^{rt}$$

with $A = \$1000$, $r = 0.025$, and $t = 10$,

$$\$1000 = Pe^{(0.025)(10)}.$$

Now we solve for P and get

$$P = \frac{\$1000}{e^{(0.025)(10)}} = \$1000e^{-(0.025)(10)} \approx \$778.80.$$

In this case, we see that \$1000 in 10 years is only worth \$778.80 in today's money.

Note in the solution above, we multiplied the future value (the \$1000) by e^{-rt} to find the present value. If we apply this logic to the rate of continuous money flow, we get the following.

PRESENT VALUE OF MONEY FLOW: *If $f(t)$ is the rate of continuous money flow at an interest rate r for T years, then the present value is*

$$P = \int_0^T f(t)e^{-rt} dt.$$

Example: The function $f(t) = 0.06t + 700$ represents the rate of flow of money in dollars per year. Determine the present value of the first five years of money flow if interest earned is 1.2% compounded continuously. (You will probably need to use technology to calculate the resulting integral.)

See Video - <http://www.showme.com/sh/?h=Knliw1w>

Example: The rate of continuous money flow starts at \$1000 and increases exponentially at 5% per year for 4 years. Find the present value if interest earned is 3.5% compounded continuously.

Solution. First we would like to determine an equation for $f(t)$, the rate of money flow. Since it starts at \$1000 and increases exponentially at 5% per year, then

$$f(t) = 1000e^{0.05t}.$$

Using the expression for total money flow in the box above, we get the integral representing the total money flow for the first 4 years

$$\int_0^4 (1000e^{0.05t})(e^{-0.035t})dt = \int_0^4 1000e^{0.015t}dt.$$

Again, we have two methods to compute this integral - the Fundamental Theorem of Calculus, or numerically using technology. Using the Fundamental Theorem of Calculus, we get

$$\begin{aligned} \int_0^4 1000e^{0.015t}dt &= \left[1000 \frac{e^{0.015t}}{0.015} \right]_0^4 = \frac{200000}{3}e^{0.015(4)} - \frac{200000}{3}e^{0.015(0)} \\ &= \frac{200000}{3}e^{0.06} - 200000 \approx \$4122.44. \end{aligned}$$

Using technology, we also get \$4122.44 (rounded to the nearest cent).

CLICKER QUESTION 59: The function $f(x) = 1000x - 100x^2$ represents the rate of flow of money in dollars per year. Determine the present value of the total money flow for the first 10 years if interest earned is 2.8% compounded continuously.

The present value of the total money flow is

- (A) less than \$15,000 (B) between \$15,000 and \$16,000
(C) between \$16,000 and \$17,000 (D) between \$17,000 and \$18,000
(E) more than \$18,000

See Video - <http://www.showme.com/sh/?h=QNukJrk>

ACCUMULATED AMOUNT OF MONEY FLOW AT TIME T

In the previous example, we considered the present value of money flow - so we took into account that money is worth less in the future than it is now. But what if we put the money into an account (earning interest) as soon as we received it? How much total money would we have at the end of T years?

ACCUMULATED AMOUNT OF MONEY FLOW AT TIME T : If $f(t)$ is the rate of money flow at an interest rate r at time t , the accumulated amount of money flow at time T is

$$A = e^{rT} \int_0^T f(t)e^{-rt} dt.$$

Example: The rate of continuous money flow starts at \$1000 and increases exponentially at 5% per year for 4 years. Find the accumulated amount if interest earned is 3.5% compounded continuously.

See Video - <http://www.showme.com/sh/?h=MvhBL3w>

Example: An investment is expected to yield a uniform continuous rate of money flow of \$20,000 per year for 3 years. Find the total money flow, the present value, and the accumulated amount at an interest rate of 4% compounded continuously.

See Video - <http://www.showme.com/sh/?h=h9koTlw>

Example: A money market fund has a continuous flow of money at a rate of $f(t) = 1500 - 60t^2$, reaching 0 in 5 years. Find the present value and the accumulated amount if interest is 5% compounded continuously.

See Video - <http://www.showme.com/sh/?h=Y6vE704>