1. Let $\vec{F} = 3y\vec{i} - 2\vec{j} + 5\vec{k}$ and C be the line from $(-2, -3, 2)$ to $(1, -4, 11)$. Find $\int_C \vec{F} \cdot d\vec{r}$.

2. Let C be the line from $(-2, -3, 2)$ to $(1, -4, 11)$ and
$\vec{G} = 3xe^{3x^2-2y^2+5z^2}\vec{i} - 2ye^{3x^2-2y^2+5z^2}\vec{j} + 5ze^{3x^2-2y^2+5z^2}\vec{k}$. Compute $\int_C \vec{G} \cdot d\vec{r}$.

3. Let $\vec{F} = (y + \sin x)\,\vec{i} + (z^2 + \cos y)\,\vec{j} + x^3\vec{k}$. Let C be the (closed) curve $\vec{r}(t) = \sin(t)\vec{i} + \cos(t)\vec{j} + \sin(2t)\vec{k}$ for $0 \le t \le 2\pi$. Compute $\int_C \vec{F} \cdot d\vec{r}$. [*Hint:* First show that the curve C lies on the graph of the function $z = 2xy$]

4. Let C be the circumference of a circle of radius 3 in the plane $2x + y + 2z = 8$, oriented counterclockwise when viewed from the origin.
Compute $\displaystyle\int_C \left((5x + 6y)\vec{i} + (4y - 2z)\vec{j} + (3z + x)\vec{k}\right) \cdot d\vec{r}$.

5. Let $F = 5x\vec{i} + 5y\vec{j}$.

(a) Find the flux of $\vec{F}$ out of a closed cylinder of radius b, and height $2b$, centered along the z-axis whose base is in the $z = -b$ plane.

(b) Find the flux density of $\vec{F}$ at the origin.

6. Let S be the ellipsoid $\dfrac{x^2}{(a+1)^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, where a, b, and c are positive constants, and let $\vec{F} = \dfrac{\vec{r} - a\vec{i}}{\|\vec{r} - a\vec{i}\|^3}$.

(a) Where is $\vec{F}$ undefined?

(b) Find $\text{div}\vec{F}$ at points where $\text{div}\vec{F}$ is defined.

(c) Find $\int_S \vec{F} \cdot d\vec{A}$

7. Let S be the part of a sphere of radius 4, centered at the origin, oriented outward, for $x \ge 3$, and let C be the boundary of S.

(a) Compute the integral of $\vec{F} = 5z\vec{j} - 5y\vec{k}$ along C from $(3, -\sqrt{7}, 0)$ to $(3, \sqrt{7}, 0)$.

(b) Compute the integral of $\vec{G} = (4x - y)\vec{i} + (2y - x)\vec{j} + 24z^2\vec{k}$ along C from $(3, -\sqrt{7}, 0)$ to $(3, \sqrt{7}, 0)$.

8. Let S be the part of a sphere of radius 4, centered at the origin, oriented outward, for $x \le 3$, and let C be the boundary of S.

(a) Repeat part (a) from the previous problem.

(b) Repeat part (b) from the previous problem.

9. Determine $\int_S \left(2\vec{i}+3\vec{j}+4\vec{k}\right)\cdot d\vec{A}$ where S is a sphere of radius 5 centered at the origin, oriented outward.

10. Compute $\int_S \left(x\vec{i}+y\vec{j}+z\vec{k}\right)\cdot d\vec{A}$ where S is a

(a) Sphere of radius 3 centered at the origin

(b) Cube of side length 6 centered at the origin and with sides parallel to the axes

(c) Torus (donut) of volume 67

11. Compute $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=y\vec{i}-3x\vec{j}$ and C is a

(a) Circle of radius 5 in the xy-plane, centered at the origin and oriented counterclockwise when viewed from above.

(b) Triangle with vertices at $(0,0,0)$, $(3,5,0)$, and $(3,0,0)$, traversed in that order.

(c) Star of area 23 lying in the plane $z=5$, traversed counterclockwise when viewed from above.

(d) Star of area 27 lying in the plane $y=7$, traversed counterclockwise when viewed from larger y-values.

12. Let $\vec{F}=e^{yz}\vec{i}+\cos(xz)\vec{j}+e^{x^2+y^2}\vec{k}$.

(a) Find div $\vec{F}$.

(b) Let S be the paraboloid $z=x^2+y^2$, for $0\le z\le 9$, oriented downward. Find the flux of $\vec{F}$ through S.

13. An open box of side length 2 lies in the first octant with one corner at the origin. The open face is in the plane $x=2$. Find the flux of $\vec{F}=\text{curl}\left(-z^2\vec{j}+3yz\vec{k}\right)$ through the surface of the box, oriented outward.

14. Find the flux of $\vec{F}=xz^2\vec{i}+x^2z^2\vec{j}+zy^2\vec{k}$ out of the closed cylinder $y^2+z^2=25$, $0\le x\le 10$.

15. Let C be a square of side length 3 whose center is at $(2,3,6)$ in the plane $3x+2y+z=18$, oriented in the clockwise direction when viewed from the origin. Compute $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=(z+x^2)\,\vec{i}+(y-4x)\,\vec{j}+(y-5x)\,\vec{k}$.

16. Consider the vector field $\vec{F}=((2-12x^2)\,z-y^2)\,\vec{i}+(4x-2xy+5y)\,\vec{j}+(y-4x^3)\,\vec{k}$. Determine the circulation around the closed curve C, which is the circle of radius 6 centered at $(2,\sqrt{2},10)$, in the plane $z=5x+\sqrt{10}y-2\sqrt{5}$, oriented clockwise when viewed from the point $(0,0,12)$.

17. Consider the closed path consisting of three line segments from $(0, 3, 0)$ to $(6, 0, 0)$ to $(9, 6, 15)$, traversed in that order. Compute $\oint_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F} = 3y^2\vec{\imath} + \left(8x + \frac{1}{2}x^2\right)\vec{\jmath} + (4xy - yz).\vec{k}$$

[*Hint(?):* all three points lie in the plane $z = x + 2y - 6$.]

18. Let C_1, C_2, and C_3 be the curves described below:

C_1 is parameterized by $\vec{r}_1(t) = 4\cos(t)\vec{\imath} + 4\sin(t)\vec{\jmath}$, for $0 \le t \le 2\pi$

C_2 is the straight line segment from $(4, 0, 0)$ to $(4, 0, 6)$

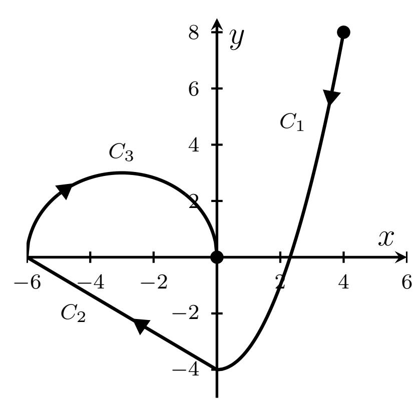
C_3 is parameterized by $\vec{r}_3(t) = 4\cos(t)\vec{\imath} - 4\sin(t)\vec{\jmath} + 6\vec{k}$, for $0 \le t \le 2\pi$

Determine $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \frac{1}{2}yz^2\vec{\imath} + z^2\vec{\jmath} + (y^2 + x^2 + xyz)\,\vec{k}$, and $C = C_1 + C_2 + C_3$.

19. Determine the flux of $\vec{G} = (2xz - 4x - 3x^2)\,\vec{\imath} + (yz^2 + 9y + 6xy)\,\vec{\jmath} + (3 - z^2)\,\vec{k}$ out of the surface, H, given by $z = \sqrt{6 - x^2 - y^2}$, oriented upward.

20. Let C be the path parameterized by $\vec{r}(t) = (4\cos(\pi t))\,\vec{\imath} + (2\sin(\pi t))\,\vec{\jmath} + (5 - t)\vec{k}$ for $0 \le t \le 5$. Compute $\int_C \vec{F} \cdot d\vec{r}$, where
$\vec{F} = (3x^2y - 10)\,\vec{\imath} + (x^3 - 2yz)\,\vec{\jmath} + (2z - y^2)\,\vec{k}$.

21. Given the vector field $\vec{F} = (2xy - y^2 - 3x^2y^3)\,\vec{\imath} + (x^2 + 3 - 3x^3y^2)\,\vec{\jmath}$ and the curve $C = C_1 + C_2 + C_3$ shown below, compute $\int_C \vec{F} \cdot d\vec{r}$.



C_1 is a portion of the graph of $y = \frac{3}{4}x^2 - 4$

C_2 is a line segment

C_3 is a semicircle

22. Let $\vec{G} = (x^2 + 3y - 2yz^2 + 5)\,\vec{\imath} + (y + 3x - y^2 - 2xz^2)\,\vec{\jmath} + (\sin(z) - 4xyz)\,\vec{k}$, and let $C = C_1 + C_2$ where C_1 and C_2 are the curves described below.

C_1 is the parabolic curve parameterized by $\vec{r}_1(t) = t\vec{\imath} + (t^2 - 9)\vec{k}$, for $-3 \le t \le 3$

C_2 is the semicircle parameterized by $\vec{r}_2(t) = 3\cos(t)\vec{\imath} - 3\sin(t)\vec{\jmath}$ for $0 \le t \le \pi$

Determine the value of the line integral of $\vec{G}$ along C.

23. Let S be the closed surface created by intersecting the graphs of $z = \sqrt{2 - x^2 - y^2}$ and $z = \sqrt{x^2 + y^2}$. Compute the flux out of S of

$$\vec{G} = \left(2x^3 + 4x\left(z - z^2\right)\right)\vec{i} + \left(8z + y^3 - yx^2\right)\vec{j} + \left(3z^3 + 2y^2 z\right).$$

24. Consider two vector fields $\vec{F}$ and $\vec{G}$, where

$$\vec{F} = \left(x^2 - 2y + 5\right)\vec{i} + \left(xy + 3x - xz^2\right)\vec{j} + \left(x^2 + 3xyz\right)\vec{k}$$

and $\vec{G}$ =curl$\vec{F}$. Compute $\int_S \vec{G} \cdot d\vec{A}$ for the following surfaces:

(a) S_1 which is the portion of the paraboloid $z = 100 - x^2 - y^2$, for $z \geq 0$, oriented upward.

(b) S_2 which is the hemisphere $z = \sqrt{100 - x^2 - y^2}$, oriented away from the origin.

(c) S_3 which is the disk of radius 10, in the xy-plane, centered at the origin, oriented upward.

(d) S_4 which is the portion of the cone $z = \sqrt{x^2 + y^2} - 10$, for $z < 0$, oriented downward.

(e) S_5 which is the graph of $z = \sqrt{10 - \sqrt{x^2 + y^2}}$, oriented upward.

25. Consider a two-dimensional vector field $\vec{F}(x, y) = F_1(x, y)\vec{i} + F_2(x, y)\vec{j}$, and a simple closed curve C which lies entirely in the xy-plane, oriented in the counterclockwise direction when viewed from above. Use Stokes' Theorem to rewrite the line integral $\oint_C \vec{F} \cdot d\vec{r}$ as a flux integral through a flat surface, and simplify the flux integral as much as possible. [*I know these directions are not completely clear, and well, a little poor, but there's a cute punch line*].

Answers

1 $\frac{31}{2}$

2 $\frac{1}{2}\left(e^{576}-e^{14}\right)$

3 π

4 27π

5a $20\pi b^3$

5b 10

6a $(a,0,0)$

6b $\text{div}\vec{F}=0$, for $(x,y,z)\neq(a,0,0)$

7a -35π

7b $-6\sqrt{7}$

8a 35π

8b $-6\sqrt{7}$

9 0

10a 108π

10b 648

10c 201

11a -100π

11b 30

11c -92

11d 0

12a $\text{div }\vec{F}=0$

12b $\pi(1-e^9)$

13 -20

14 3125π

15 $\frac{99}{\sqrt{14}}$

16 $\left(6+12\sqrt{10}\right)\pi$

17 225

18 288π

19 $\frac{124\sqrt{6}-90}{5}\pi$

20 55

21 $\frac{491092}{15}$

22 0

23 $\left(8\sqrt{2}-6\right)\pi$

24a 500π

24b 500π

24c 500π

24d -500π

24e 500π

25 It's Green's Theorem!

Solutions

2

$$\begin{aligned}
\text{curl}\vec{G} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xe^{3x^2-2y^2+5z^2} & -2ye^{3x^2-2y^2+5z^2} & 5ze^{3x^2-2y^2+5z^2} \end{vmatrix} \\
&= \left(-20yze^{3x^2-2y^2+5z^2} - (-20yze^{3x^2-2y^2+5z^2})\right)\vec{i} - \left(30xze^{3x^2-2y^2+5z^2} - 30xze^{3x^2-2y^2+5z^2}\right)\vec{j} + \\
&\quad \left(-12xye^{3x^2-2y^2+5z^2} - (-12xye^{3x^2-2y^2+5z^2})\right)\vec{k} \\
&= \vec{0}
\end{aligned}$$

Since $\vec{G}$ is defined on all of 3-space, and $\text{curl}\vec{G} = \vec{0}$, then by the curl test, $\vec{G}$ is a gradient field. In fact, $\vec{G}$ is the gradient of the scalar function $g(x,y,z) = \frac{1}{2}e^{3x^2-2y^2+5z^2}$. So, by the Fundamental Theorem of Calculus for Line Integrals:

$$\begin{aligned}
\int_C \vec{G} \cdot d\vec{r} &= g(1,-4,11) - g(-2,-3,2) \\
&= \frac{1}{2}e^{3(1)^2-2(-4)^2+5(11)^2} - \frac{1}{2}e^{3(-2)^2-2(-3)^2+5(2)^2} \\
&= \frac{1}{2}e^{576} - \frac{1}{2}e^{14}
\end{aligned}$$

6b $\vec{F} = \dfrac{\vec{r} - a\vec{i}}{\|\vec{r} - a\vec{i}\|^3} = \dfrac{(x-a)\vec{i} + y\vec{j} + z\vec{k}}{\left(\sqrt{(x-a)^2 + y^2 + z^2}\right)^3}$

$$\begin{aligned}
\text{div}\vec{F} &= \frac{\partial}{\partial x}\left(\frac{(x-a)}{((x-a)^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{((x-a)^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{((x-a)^2+y^2+z^2)^{3/2}}\right) \\
&= \frac{((x-a)^2+y^2+z^2)^{3/2} - (x-a)(\frac{3}{2})\left((x-a)^2+y^2+z^2\right)^{1/2}(2(x-a))}{\left(((x-a)^2+y^2+z^2)^{3/2}\right)^2} \\
&\quad + \frac{((x-a)^2+y^2+z^2)^{3/2} - y(\frac{3}{2})\left((x-a)^2+y^2+z^2\right)^{1/2}(2y)}{\left(((x-a)^2+y^2+z^2)^{3/2}\right)^2} \\
&\quad + \frac{((x-a)^2+y^2+z^2)^{3/2} - z(\frac{3}{2})\left((x-a)^2+y^2+z^2\right)^{1/2}(2z)}{\left(((x-a)^2+y^2+z^2)^{3/2}\right)^2} \\
&= \frac{((x-a)^2+y^2+z^2)^{3/2} - 3(x-a)^2\left((x-a)^2+y^2+z^2\right)^{1/2}}{((x-a)^2+y^2+z^2)^3} \\
&\quad + \frac{((x-a)^2+y^2+z^2)^{3/2} - 3y^2\left((x-a)^2+y^2+z^2\right)^{1/2}}{((x-a)^2+y^2+z^2)^3} \\
&\quad + \frac{((x-a)^2+y^2+z^2)^{3/2} - 3z^2\left((x-a)^2+y^2+z^2\right)^{1/2}}{((x-a)^2+y^2+z^2)^3} \\
&= \frac{((x-a)^2+y^2+z^2) - 3(x-a)^2}{((x-a)^2+y^2+z^2)^{5/2}} + \frac{((x-a)^2+y^2+z^2) - 3y^2}{((x-a)^2+y^2+z^2)^{5/2}} + \frac{((x-a)^2+y^2+z^2) - 3z^2}{((x-a)^2+y^2+z^2)^{5/2}} \\
&= \frac{((x-a)^2+y^2+z^2) - 3(x-a)^2 + ((x-a)^2+y^2+z^2) - 3y^2 + ((x-a)^2+y^2+z^2) - 3z^2}{((x-a)^2+y^2+z^2)^{5/2}} \\
&= \frac{3\left((x-a)^2+y^2+z^2\right) - 3(x-a)^2 - 3y^2 - 3z^2}{((x-a)^2+y^2+z^2)^{5/2}} \\
&= 0
\end{aligned}$$

25

$$\begin{aligned}
\text{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(F_2)\right)\vec{i} - \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(F_1)\right)\vec{j} + \left(\frac{\partial}{\partial x}(F_2) - \frac{\partial}{\partial y}(F_1)\right)\vec{k} \\
&= (0-0)\vec{i} - (0-0)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} \\
&= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}
\end{aligned}$$

Let S be the flat surface in the xy plane, which has C as it's boundary. S is oriented in the upward direction by the orientation on C.

$$\text{by Stokes' Theorem: } \oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\vec{F} \cdot d\vec{A}$$
$$= \int_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k} \cdot \vec{k}\, dA$$
$$= \int_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

Which is Green's Theorem exactly, if we simply call the region in the xy-plane R instead of S, as we have all semester long.

So, we've shown that Green's Theorem is one special case of Stokes' Theorem!