A very brief introduction to the world of determinants of matrices.

A matrix is simply a rectangular array of numbers, expressions, or other entries arranged in rows and columns. For example, $\begin{bmatrix} 4 & 8 & -2 \\ -1 & 0 & 5 \end{bmatrix}$ is a 2×3 matrix (2 rows and 3 columns). $\begin{bmatrix} \heartsuit & \spadesuit \\ \text{☯} & \text{☢} \\ \clubsuit & \text{☠} \\ \text{☽} & \text{☕} \end{bmatrix}$ is a 4×2 matrix. For our purposes, we will only be concerned with square matrices (having the same number of rows and columns), and we will specifically focus on 3×3 square matrices.

However, we begin our discussion of determinants with 2×2 matrices. The determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is given by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Looking at a few quick examples:

$$\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} = (2)(8) - (4)(3) = 4$$

$$\begin{vmatrix} 5 & 1 \\ 6 & -2 \end{vmatrix} = (5)(-2) - (6)(1) = -16$$

$$\begin{aligned} \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} &= \cos^2(\theta) - (-\sin^2(\theta)) \\ &= \cos^2(\theta) + \sin^2(\theta) \\ &= 1 \end{aligned}$$

Now that we're pros at taking determinants of 2×2 matrices, let's move on to the ones we really care about– 3×3 matrices. The equivalent formula (to what we have for 2×2 matrices) for 3×3 matrices is:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi \qquad (1)$$

This is all well and fine, **if** we really want to try and remember this string of 23 characters, with each a through i appearing twice. Since most of us probably don't want to do this, we can hope that there is a better way to compute the determinant of a 3×3 matrix. Thankfully, there are actually 2 "better" ways. The first is to augment our matrix with copies of the first two rows added to the right-hand side, to make it a 3×5 matrix like so:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightsquigarrow \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

and concern ourselves with each of the 6 diagonals in this new matrix. We take the 3 diagonals that begin along the top row (at the first a, first b, and c) and proceed down and to the right, take the product of the entries along each diagonal and add them. We then take the 3 diagonals which begin in the top row (at the second a, second b, and c) and proceed down and to the left, take the product of the entries in each diagonal and subtract them from the previous result. This idea is shown below, with the "positive" diagonals color-coded:

$$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} & a & b \\ d & \boldsymbol{e} & \boldsymbol{f} & \boldsymbol{d} & e \\ g & h & \boldsymbol{i} & \boldsymbol{g} & \boldsymbol{h} \end{bmatrix} \rightsquigarrow aei + bfg + cdh,$$

the "negative" diagonals denoted with underlines:

$$\begin{bmatrix} a & b & \underline{\boldsymbol{c}} & \underline{\underline{\boldsymbol{a}}} & \underline{\underline{\underline{\boldsymbol{b}}}} \\ d & \underline{\boldsymbol{e}} & \underline{\underline{\boldsymbol{f}}} & \underline{\underline{\underline{\boldsymbol{d}}}} & e \\ \underline{\boldsymbol{g}} & \underline{\underline{\boldsymbol{h}}} & \underline{\underline{\underline{\boldsymbol{i}}}} & g & h \end{bmatrix} \rightsquigarrow -ceg - afh - bdi,$$

and finally, both shown together:

$$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \underline{c} & \underline{\underline{\boldsymbol{a}}} & \underline{\underline{\underline{\boldsymbol{b}}}} \\ d & \underline{\boldsymbol{e}} & \underline{\underline{\boldsymbol{f}}} & \underline{\underline{\underline{\boldsymbol{d}}}} & e \\ \underline{\boldsymbol{g}} & \underline{\underline{\boldsymbol{h}}} & \underline{\underline{\underline{\boldsymbol{i}}}} & \boldsymbol{g} & \boldsymbol{h} \end{bmatrix} \rightsquigarrow aei + bfg + cdh - ceg - afh - bdi \qquad (2)$$

Which, lo and behold, $aei + bfg + cdh - ceg - afh - bdi$ is exactly the expression that was stated above for the determinant of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. So, the good news is that we have a (hopefully) easier way to remember how to compute our determinant than rote memorization; The bad news is that this method is still not the most useful one for what we will need in this class.

The method that is of best/simplest use to us is the one that uses the ease of computing determinants of 2×2 matrices to compute the determinant of a 3×3 matrix. In order to do this, we must first discuss the concept of a *minor* of a matrix. The ij^{th} minor of an $n \times n$ matrix, $\boldsymbol{A}$, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the i^{th} row and the j^{th} column from $\boldsymbol{A}$. Since that's probably not crystal clear, let's look at a couple examples.

Let M_{ij} denote the ij^{th} minor of a matrix, and let $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Thus, M_{11} is the minor obtained by removing the first column and the first row from $\boldsymbol{A}$ and computing the determinant. That is,

$$M_{11} = \begin{vmatrix} a & b & c \\ d & \boldsymbol{e} & \boldsymbol{f} \\ g & \boldsymbol{h} & \boldsymbol{i} \end{vmatrix} = ei - fh$$

While M_{32} is obtained by removing the third row and second column:

$$M_{32} = \begin{vmatrix} \boldsymbol{a} & b & \boldsymbol{c} \\ \boldsymbol{d} & e & \boldsymbol{f} \\ g & h & i \end{vmatrix} = af - cd$$

Now that we know what minors are, we can compute $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ by *expanding* along any row or column and using the minors. To expand along a row or column, we must first mention that there is an implicit sign associated with each position in a matrix. If we define a_{ij} to be the ij^{th} entry (that is, the entry in the i^{th} row and the j^{th} column) in the matrix $\boldsymbol{A}$ then the sign associated with a_{ij} is given by $(-1)^{i+j}$. So, the sign is positive if the row number and column number have the same parity, and negative if they have opposite parities. From a simpler viewpoint, the first entry in the first column (the upper left entry) a_{11} of $\boldsymbol{A}$ has an associated "+" ($(-1)^{1+1} = 1$) and signs of consecutive entries alternate along both rows and columns. Here's the illustration: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$. To compute the determinant of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ by expanding across a row or column, we compute the sum of the product of each entry in a row or column, the associated sign with the entry position, and the corresponding minor. That is, multiply the entry (along with the appropriate sign) and the corresponding minor, for all three entries in a row or column, and add them all together. The example that we will be most concerned with is expanding along the first (top) row, which is illustrated below.

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= (-1)^{1+1}a_{11}M_{11} + (-1)^{1+2}a_{12}M_{12} + (-1)^{1+3}a_{13}M_{13} \\ &= (-1)^2 a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^3 b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + (-1)^4 c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \end{aligned}$$

So we see that computing the determinant of a 3×3 matrix is really no more difficult than computing the determinant of three 2×2 matrices. The remaining portion of the computation is now shown:

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei - afh - bdi + bfg + cdh - ceg \\ &= aei + bfg + cdh - ceg - afh - bdi \end{aligned} \tag{3}$$

Hey, look at that, that's the same result that we got from (1) and (2)! Now to see an example that this third method (the cofactor expansion method) is independent of our choice of which row or column we choose to expand across, let's expand across, say the second column instead:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -bM_{12} + eM_{22} - hM_{32}$$
$$= -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix}$$
$$= -b(di - fg) + e(ai - cg) - h(af - cd)$$
$$= -bdi + bfg + aei - ceg - afh + cdh$$
$$= aei + bfg + cdh - ceg - afh - bdi$$

Alright, let's not even act surprised (maybe still be impressed) that it's the same result as (1), (2), and (3). With all of this now under our belts, let actually do a few concrete examples.

Ex 1: Compute the determinant of $\begin{bmatrix} 5 & 2 & 3 \\ 4 & -1 & 6 \\ 2 & 8 & 3 \end{bmatrix}$.

First let's use the method given by equation (1) which seems computationally simple, but is not easy to remember (or necessarily implement, since it requires keeping track of 9 entries):

$$\begin{vmatrix} 5 & 2 & 3 \\ 4 & -1 & 6 \\ 2 & 8 & 3 \end{vmatrix} = (5)(-1)(3) + (2)(6)(2) + (3)(4)(8) - (3)(-1)(2) - (5)(6)(8) - (2)(4)(3)$$
$$= -15 + 24 + 96 + 6 - 240 - 24$$
$$= -153$$

Now let's try using the cofactor expansion method by expanding along the first row (equation (3)) which may seem computationally more difficult (it takes more lines to write out here), but is much easier to implement:

$$\begin{vmatrix} 5 & 2 & 3 \\ 4 & -1 & 6 \\ 2 & 8 & 3 \end{vmatrix} = 5\begin{vmatrix} -1 & 6 \\ 8 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 4 & -1 \\ 2 & 8 \end{vmatrix}$$
$$= 5\big((-1)(3) - (6)(8)\big) - 2\big((4)(3) - (2)(6)\big) + 3\big((4)(8) - (2)(-1)\big)$$
$$= 5(-3 - 48) - 2(12 - 12) + 3(32 + 2)$$
$$= 5(-51) - 2(0) + 3(34)$$
$$= -255 - 0 + 102$$
$$= -153$$

Ex 2: Compute the determinant of $\begin{bmatrix} 8 & 0 & -6 \\ -4 & 2 & 3 \\ 7 & -5 & 1 \end{bmatrix}$.

$$\begin{aligned}
\begin{vmatrix} 8 & 0 & -6 \\ -4 & 2 & 3 \\ 7 & -5 & 1 \end{vmatrix} &= 8\begin{vmatrix} 2 & 3 \\ -5 & 1 \end{vmatrix} - 0\begin{vmatrix} -4 & 3 \\ 7 & 1 \end{vmatrix} + (-6)\begin{vmatrix} -4 & 2 \\ 7 & -5 \end{vmatrix} \\
&= 8\big((2)(1) - (-5)(3)\big) - 0\big((-4)(1) - (7)(3)\big) - 6\big((-4)(-5) - (7)(2)\big) \\
&= 8(2 + 15) - 0(-4 - 21) - 6(20 - 14) \\
&= 8(17) - 0 - 6(6) \\
&= 136 - 36 \\
&= 100
\end{aligned}$$

Ex 3: Compute the determinant of $\begin{bmatrix} 2 & 4 & 3 \\ 1 & 6 & 0 \\ -5 & 2 & -4 \end{bmatrix}$.

$$\begin{aligned}
\begin{vmatrix} 2 & 4 & 3 \\ 1 & 6 & 0 \\ -5 & 2 & -4 \end{vmatrix} &= 2\big((6)(-4) - (2)(0)\big) - 4\big((1)(-4) - (-5)(0)\big) + 3\big((1)(2) - (-5)(6)\big) \\
&= 2(-24 - 0) - 4(-4 - 0) + 3(2 + 30) \\
&= -48 + 16 + 96 \\
&= 64
\end{aligned}$$

Ex 4: Compute the determinant of $\begin{bmatrix} x & y & z \\ 3 & 2 & 5 \\ 6 & -4 & 1 \end{bmatrix}$.

$$\begin{aligned}
\begin{vmatrix} x & y & z \\ 3 & 2 & 5 \\ 6 & -4 & 1 \end{vmatrix} &= x\big((2)(1) - (-4)(5)\big) - y\big((3)(1) - (6)(5)\big) + z\big((3)(-4) - (6)(2)\big) \\
&= x(2 + 20) - y(3 - 30) + z(-12 - 12) \\
&= 22x + 27y - 24z
\end{aligned}$$

Ex 5: Compute the determinant of $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 5 \\ 4 & 1 & -1 \end{bmatrix}$.

$$\begin{aligned}
\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 5 \\ 4 & 1 & -1 \end{vmatrix} &= \big((3)(-1) - (1)(5)\big)\vec{i} - \big((-2)(-1) - (4)(5)\big)\vec{j} + \big((-2)(1) - (4)(3)\big)\vec{k} \\
&= (-3 - 5)\vec{i} - (2 - 20)\vec{j} + (-2 - 12)\vec{k} \\
&= -8\vec{i} + 18\vec{j} - 14\vec{k}
\end{aligned}$$