24. pg 557 An aquarium pool has volume $2 \cdot 10^6$ liters. The pool initially contains pure fresh water. At $t = 0$ minutes, water containing 10 grams/liter of salt is poured into the pool at a rate of 60 liters/minute. The salt water instantly mixes with the fresh water, and the excess mixture is drained out of the pool at the same rate (60 liters/minute).

(a) Write a differential equation for $S(t)$, the mass of salt in the pool at time t.

(b) Solve the differential equation to find $S(t)$.

(c) What happens to $S(t)$ as $t \to \infty$?

Solution (a) Since there was no salt in the aquarium to start with, at any time t we have,

$$\text{Quantity of salt present} = \text{Quantity of salt entered} - \text{Quantity of salt exited}$$

So we also have,

$$\begin{pmatrix}\text{Rate of change of}\\ \text{quantity of salt}\end{pmatrix} = \begin{pmatrix}\text{Rate of salt}\\ \text{entering}\end{pmatrix} - \begin{pmatrix}\text{Rate of salt}\\ \text{exiting}\end{pmatrix}$$

Now looking at each of these pieces separately, we first see

$$\begin{pmatrix}\text{Rate of salt}\\ \text{entering}\end{pmatrix} = \begin{pmatrix}\text{Concentration}\\ \text{of salt}\end{pmatrix}\begin{pmatrix}\text{Volume}\\ \text{entering}\end{pmatrix}$$

For the water entering the aquarium, both the concentration of salt and the volume are constant: 60 liters of of water enter the aquarium each minute, and each liter contains 10 grams of salt, so numerically:

$$\begin{pmatrix}\text{Rate of salt}\\ \text{entering}\end{pmatrix} = \begin{pmatrix}\text{Concentration}\\ \text{of salt}\end{pmatrix}\begin{pmatrix}\text{Volume}\\ \text{entering}\end{pmatrix} = (10)(60) = 600$$

For the water exiting the aquarium, things are a bit more complicated because the concentration of salt is changing. The volume of water exiting is still a constant 60 liters of water per minute. The concentration though, is a function of the quantity of salt. The concentration of salt in the aquarium is the ratio of salt in the aquarium to the volume of the aquarium. Thus, $C = \frac{S}{20{,}000{,}000}$. So,

$$\begin{pmatrix}\text{Rate of salt}\\ \text{exiting}\end{pmatrix} = \begin{pmatrix}\text{Concentration}\\ \text{of salt}\end{pmatrix}\begin{pmatrix}\text{Volume}\\ \text{exiting}\end{pmatrix} = \left(\frac{S}{20,000,000}\right)(60) = \frac{3S}{100,000}$$

So, putting everything together, we can write an equation for the rate of change of quantity of salt:

$$\frac{dS}{dt} = 600 - \frac{3S}{100,000}$$

We should check that the units of are differential equation are correct (this is a good way to see if the equation even makes sense). Since $\frac{dS}{dt}$ is the rate of change in quantity of salt, S is measured in grams, and t is measured in minutes, the units of $\frac{dS}{dt}$ are grams/min. Checking the other side of the equation:

$$\left(10\frac{\text{grams}/}{\text{liter}}\right)\left(60\frac{\text{liters}/}{\text{minute}}\right) - \left(\frac{S\text{ grams}}{20000000\text{ liters}}\right)\left(60\frac{\text{liters}/}{\text{minute}}\right)$$

The liters cancel, and we do in fact get grams/minute.

(b)

$$\frac{dS}{dt} = \frac{3}{100,000}(20,000,000 - S)$$

$$\frac{dS}{20,000,000 - S} = \frac{3}{100,000}dt$$

$$\int \frac{dS}{20,000,000 - S} = \int \frac{3}{100,000}dt$$

$$-\ln(20,000,000 - S) = \frac{3}{100,000}t + C$$

$$\ln(20,000,000 - S) = D - \frac{3}{100,000}t$$

$$20,000,000 - S = ke^{-\frac{3}{100,000}t}$$

$$S = 20,000,000 - ke^{-\frac{3}{100,000}t}$$

(c) As $t \to \infty$, $S(t) = 20,000,000 - ke^{-\frac{3}{100,000}t} \to 20,000,000$

- In the process of conducting a new series of experiments, Dennis the mad scientist is polluting one of the not-so-great lakes, Brown Lake. The volume of Brown Lake is 80 million gallons. Through an overly complicated system of dams, canals and other waterways that Dennis has constructed, .5 million gallons of water runs out of the lake each day. Coincidentally, each day Dennis pumps .5 million gallons of water out of a nearby spring, uses it for his nefarious means and dumps the water into Brown Lake. Unfortunately for the environment, 3% of the water that Dennis returns to the lake is toxic pollution. Via some questionable means, Dennis has ensured that there are no other ways that water enters of exits the lake.

- Write a differential equation which represents the amount of toxic pollution in the lake, assuming that there was initially no pollution in the lake. Solve this differential equation.

  Similar to above, we have,

$$\begin{pmatrix}\text{Rate of change of}\\ \text{quantity of pollution}\end{pmatrix} = \begin{pmatrix}\text{Rate of pollution}\\ \text{entering}\end{pmatrix} - \begin{pmatrix}\text{Rate of pollution}\\ \text{exiting}\end{pmatrix}$$

The rate of pollution entering is a constant: 3% of the .5 millions gallons of water that enter the lake each day is pollution. So each day $(.5)(.03) = .015$ million gallons of pollution enter the lake.

Like above, the amount of pollution that exits the lake each day depends on the current level of pollution in the lake. If we let P denote the amount of pollution in the lake (in millions of gallons), then the concentration of pollution in the lake is $\frac{P}{80}$. So, the amount of pollution that exits the lake is $\left(\frac{P}{80}\right)(.5) = \frac{P}{160}$ million gallons per day. Thus,

$$\frac{dP}{dt} = .015 - \frac{P}{160}$$

Now we get to solve it, so

$$\begin{aligned} \frac{dP}{dt} &= -\frac{1}{160}(P-2.4) \\ \int \frac{dP}{P-2.4} &= \int -\frac{1}{160} dt \\ \ln(P-2.4) &= -\frac{1}{160}t + C \\ P - 2.4 &= e^{-\frac{1}{160}t+C} \\ P &= ke^{-\frac{1}{160}t} + 2.4 \end{aligned}$$

Using the initial condition $P(0) = 0$:

$$\begin{aligned} 0 &= k + 2.4 \\ k &= -2.4 \end{aligned}$$

So,

$$P(t) = -2.4e^{-\frac{1}{160}t} + 2.4$$

It should be noted that 2.4 million gallons of toxic pollution is the equilibrium solution for this model. This can be seen by seeing that 2.4 is obviously the zero of the differential equation $\frac{dP}{dt} = -\frac{1}{160}(P-2.4)$. Another way is to realize that the pollution concentration in the lake should increase toward 3% (since that's the concentration of what's being pumped in), but if it ever got above 3%, then more pollution would be pumped out than was being pumped in, causing the concentration to decrease. So a 3% concentration, and thus 2.4 million gallons, is the equilibrium. It is a stable equilibrium, since as $t \to \infty$, $P(t) \to 2.4$.

- How long will it take for 2% of the lake to be toxic pollution?

  Since the volume of the lake is 80 million gallons, we want to find when there is $(80)(.02)=1.6$ million gallons of pollution in the lake. This means solving the equation

$P(t) = 1.6$.

$$\begin{aligned} 1.6 &= -2.4e^{-\frac{1}{160}t} + 2.4 \\ -.8 &= -2.4e^{-\frac{1}{160}t} \\ \frac{1}{3} &= e^{-\frac{1}{160}t} \\ \ln(1/3) &= -\frac{1}{160}t \\ t &= 160\ln(3) \approx 176 \text{ days} \end{aligned}$$

- What if Brown Lake had 100,000 gallons of pollution in it before Dennis started his polluting? Write and solve a differential equation to represent this scenario.

  The differential equation in this case is exactly the same as above, since the initial level of pollution does not affect the rate of change of the pollution. So,

  $$\frac{dP}{dt} = .015 - \frac{P}{160}$$

  and

  $$P = ke^{-\frac{1}{160}t} + 2.4$$

  The initial condition this time is $P(0) = .1$, so

  $$\begin{aligned} .1 &= k + 2.4 \\ k &= .1 - 2.4 = -2.3 \end{aligned}$$

  Thus,

  $$P = -2.3e^{-\frac{1}{160}t} + 2.4$$

- Exactly one year after Dennis begins polluting the lake, he is forced to cease operations. Immediately after he shuts down operations, the especially-fond-of-lakes environmentalist Diana snaps into action and begins a campaign to clean up the lake (Brown Lake is no Lake Michigan, but still worth saving). As luck would have it, there is an old water treatment facility near Brown Lake. Diana has reconfigured the plant to remove water from the lake, treat it and return the clean, pollution-free water to the lake. She has also closed off all other waterways to and from the lake to avoid further contamination of surrounding areas. So, no water flows in or out of the lake except that which goes through the treatment plant. At full capacity, the treatment plant can process 400,000 gallons a day.

  Write and solve a differential equation to represent the amount of pollution in the lake as a function of time since Diana started running the treatment plant. (Again assume that there was no pollution in the lake before Dennis began operations.)

There is now no pollution entering the lake, just pollution exiting. Just as before, the amount of pollution exiting the lake is the concentration of pollution in the water times the amount of water exiting. Thus,

$$\frac{dQ}{dt} = -\frac{Q}{80}(.4) = -\frac{Q}{200}$$

Solving this is an easy matter:

$$\begin{aligned} \frac{dQ}{dt} &= -\frac{Q}{200} \\ \int \frac{dQ}{Q} &= -\int \frac{1}{200} dt \\ \ln|Q| &= -\frac{t}{200} + C \\ Q &= ke^{-\frac{t}{200}} \end{aligned}$$

The initial condition is not given to us explicitly this time, but it is still given. We are told that Dennis polluted the lake for exactly one year, so we can find our current initial condition by finding out how much Dennis polluted the lake in one year. This simply means evaluating the function $P(t) = -2.4e^{-\frac{1}{160}t} + 2.4$ at $t = 365$ (we'll assume it wasn't a leap year).

$$P(365) = -2.4e^{-\frac{365}{160}} + 2.4 \approx 2.155$$

So, this is the initial condition for our current function, i.e. $Q(0) = 2.155$. So, $k = 2.155$, and our function is

$$Q = 2.155e^{-\frac{t}{200}}$$

- Researchers have decided that Brown Lake will be safe again once the concentration of pollution is less that 1/20 of 1%. How long will it take to reach this level?

  If the concentration is 1/20 of 1%, then this means there is (.0005)(80) = .04 million gallons of pollution in the lake.

$$\begin{aligned} .04 &= 2.155e^{\frac{t}{200}} \\ \frac{.04}{2.155} &= e^{-\frac{t}{200}} \\ \ln(\frac{.04}{2.155}) &= -\frac{t}{200} \\ t &= -200\ln(\frac{.04}{2.155}) \approx 797 \text{ days} \end{aligned}$$