List the intercepts, asymptotes, and domain of each of the following rational functions.

$$\#21 \text{ from section 3.3: } F(x) = \frac{2x}{x-4}.$$

First, notice that $\frac{2x}{x-4}$ is reduced, so we can proceed. The y-intercept occurs when $x = 0$, so we want to know what $F(0)$ is. $F(0) = \frac{2(0)}{0-4} = 0$, so the y-intercept is $(0, 0)$. To find the x-intercept, we need to set the function equal to zero and solve.

$$\begin{aligned} F(x) &= 0 \\ \frac{2x}{x-4} &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

Thus, the only x-intercept is $(0, 0)$.

Since the vertical asymptotes correspond to the zeros of the denominator, we are next interested in the zeros of $x - 4$. Clearly the only zero is $x = 4$, So, $F(x)$ has a vertical asymptote at $x = 4$. To find the other asymptote (remember that it can have **only one** non-vertical asymptote), we use the process laid out on page 197. That is, we want our function in the form $F(x) = q(x) + \frac{r(x)}{d(x)}$ with the degree of $r(x)$ less than the degree of $d(x)$. In order to achieve this we need to use polynomial division.

$$\begin{array}{r} 2 \\ x-4\overline{)2x+0} \\ \underline{-2x+8} \\ 8 \end{array}$$

Thus, $\frac{2x}{x-4} = 2 + \frac{8}{x-4}$, and the right hand side of this equality is in the desired form mentioned above [the $q(x)$ is 2, $r(x)$ is 8, and $d(x)$ is $x - 4$]. So, the line $y = 2$ is a horizontal asymptote.

Lastly, we find the domain. Recalling that the domain of a rational function is all real numbers except the zeros of the denominator, we easily see that the domain is $(-\infty, 4) \cup (4, \infty)$.

x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Vertical asymptote: $x = 4$
Horizontal asymptote: $y = 2$
Domain: $(-\infty, 4) \cup (4, \infty)$

$$\#25\text{: } f(x) = \frac{x-3}{(x+2)^2}.$$

Again, notice that $f(x) = \frac{x-3}{(x+2)^2}$ is reduced. $f(0) = \frac{0-3}{(0+2)^2} = \frac{-3}{4}$, so the y-intercept is $(0, -\frac{3}{4})$. Finding the x-intercept:

$$\begin{aligned} f(x) &= 0 \\ \frac{x-3}{(x+2)^2} &= 0 \\ x-3 &= 0 \\ x &= 3 \end{aligned}$$

Thus, the only x-intercept is $(3,0)$.

Next, we find the vertical asymptotes:

$$\begin{aligned} (x+2)^2 &= 0 \\ (x+2) &= 0 \\ x+2 &= 0 \\ x &= -2 \end{aligned}$$

So, $x = -2$ is a vertical asymptote. To find the other asymptote we need our function in the form $f(x) = q(x) + \frac{r(x)}{d(x)}$ with the degree of $r(x)$ less than the degree of $d(x)$. Luckily, since the degree of $x-3$ is 1, and the degree of $(x+2)^2$ is 2, it is already in that form ($q(x) = 0$). So, the line $y = 0$ is the horizontal asymptote.

x-intercept: $(3,0)$
y-intercept: $(0,-\frac{3}{4})$
Vertical asymptote: $x = -2$
Horizontal asymptote: $y = 0$
Domain: $(-\infty,-2) \cup (-2,\infty)$

$$f(x) = \frac{2x^2+10x+12}{3x-6}.$$

Once again, $f(x) = \frac{2x^2+10x+12}{3x-6}$ is reduced. $f(0) = \frac{2(0)^2+10(0)+12}{3(0)-6} = \frac{12}{-6} = -2$, so the y-intercept is $(0,-2)$. Now for the x-intercept:

$$\begin{aligned} 2x^2+10x+12 &= 0 \\ x^2+5x+6 &= 0 \\ (x+3)(x+2) &= 0 \\ x+3 = 0 \qquad \text{or} &\qquad x+2 = 0 \\ x = -3 \qquad \text{or} &\qquad x = -2 \end{aligned}$$

Thus, there are two x-intercepts, $(-3,0)$ and $(-2,0)$.

$$\begin{aligned} 3x-6 &= 0 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

So, $x = 2$ is the vertical asymptote. Again, to find the other asymptote we need our function in the form $f(x) = q(x) + \frac{r(x)}{d(x)}$ with the degree of $r(x)$ less than the degree of $d(x)$. So, we must once again do the polynomial division (yay!).

$$\begin{array}{r} \frac{2}{3}x + \frac{14}{3} \\ 3x-6 \overline{\smash{)}\,2x^2+10x+12} \\ \underline{-2x^2+4x} \\ 14x+12 \\ \underline{-14x+28} \\ 40 \end{array}$$

So, $\frac{2x^2+10x+12}{3x-6} = \frac{2}{3}x + \frac{14}{3} + \frac{40}{3x-6}$, and the line $y = \frac{2}{3}x + \frac{14}{3}$ is the slant asymptote.

x-intercepts: $(-3, 0)$ and $(-2, 0)$
y-intercept: $(0, -2)$
Vertical asymptote: $x = 2$
Slant asymptote: $y = \frac{2}{3}x + \frac{14}{3}$
Domain: $(-\infty, 2) \cup (2, \infty)$

$$f(x) = \frac{3x^3 + 2x^2 + 11x + 12}{3x^2 + 6x - 24}.$$

Yet again, $f(x)$ is reduced (this will **not** always happen, which is why it is important to check). $f(0) = \frac{12}{-24} = -\frac{1}{2}$, so the y-intercept is $(0, -\frac{1}{2})$. Since the numerator is a cubic (degree three) polynomial, we cannot simply set it equal to zero and solve (we don't have the necessary "tools"). We do however, have a way to find the zero(s) of a cubic: our calculators. Graphing $3x^3 + 2x^2 + 11x + 12$ and using the "zero" function, we see easily that -1 is a zero (in fact the only zero). Thus, the only x-intercept of $f(x)$ is $(-1, 0)$.

$$\begin{aligned} 3x^2 + 6x - 24 &= 0 \\ x^2 + 2x - 8 &= 0 \\ (x+4)(x-2) &= 0 \\ x + 4 = 0 \quad &\text{or} \quad x - 2 = 0 \\ x = -4 \quad &\text{or} \quad x = 2 \end{aligned}$$

So, there are two vertical asymptotes, $x = -4$ and $x = 2$. Once again, we need our function in the form $f(x) = q(x) + \frac{r(x)}{d(x)}$ with the degree of $r(x)$ less than the degree of $d(x)$. So, time to do the polynomial division.

$$\begin{array}{r} x - \frac{4}{3} \\ 3x^2 + 6x - 24 \overline{\big) 3x^3 + 2x^2 + 11x + 12} \\ \underline{-3x^3 - 6x^2 + 24x} \\ -4x^2 + 35x + 12 \\ \underline{4x^2 + 8x - 32} \\ 43x - 20 \end{array}$$

So, $\frac{3x^3+2x^2+11x+12}{3x^2+6x-24} = 43x - 20 + \frac{x - \frac{4}{3}}{3x^2+6x-24}$, and the line $y = x - \frac{4}{3}$ is the slant asymptote.

x-intercept: $(-1, 0)$
y-intercept: $(0, -\frac{1}{2})$
Vertical asymptotes: $x = -4$ and $x = 2$
Slant asymptote: $y = x - \frac{4}{3}$
Domain: $(-\infty, -4) \cup (-4, 2) \cup (2, \infty)$

$$f(x) = \frac{4x^2 - 36}{2x^2 - 12x + 21}.$$

$f(x)$ is reduced. $f(0) = \frac{-36}{21} = -\frac{12}{7}$, so the y-intercept is $(0, -\frac{12}{7})$. Now for the x-intercepts:

$$\begin{aligned} 4x^2 - 36 &= 0 \\ x^2 - 9 &= 0 \\ (x+3)(x-3) &= 0 \\ x + 3 = 0 \quad &\text{or} \quad x - 3 = 0 \\ x = -3 \quad &\text{or} \quad x = 3 \end{aligned}$$

Thus, there are two x-intercepts, $(-3, 0)$ and $(3, 0)$.

Next, we find the vertical asymptotes. Since $2x^2 - 12x + 21$ does not factor, we must use the quadratic formula.

$$\begin{aligned} x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(21)}}{2(2)} \\ x &= \frac{12 \pm \sqrt{144 - 168}}{4} \\ x &= \frac{12 \pm \sqrt{-24}}{4} \end{aligned}$$

Since, $\sqrt{-24}$ is not a real number, $2x^2 - 12x + 21$ has no real zeros. Since zeros of the denominator of a rational function correspond to the vertical asymptotes, $f(x) = \frac{4x^2-36}{2x^2-12x+21}$ has no vertical asymptotes.

Since the degree of the numerator is 2, and the degree of the denominator is also 2, we must once again do the polynomial division.

$$\begin{array}{r} 2 \\ 2x^2 - 12x + 21 \overline{\big) 4x^2 + \ 0x - 36} \\ \underline{-4x^2 + 24x - 42} \\ 24x - 78 \end{array}$$

So, $\frac{4x^2-36}{2x^2-12x+21} = 2 + \frac{24x-78}{2x^2-12x+21}$, and the line $y = 2$ is the horizontal asymptote.

x-intercepts: $(-3, 0)$ and $(3, 0)$
y-intercept: $(0, -\frac{12}{7})$
Vertical asymptote: none
Horizontal asymptote: $y = 2$
Domain: $(-\infty, \infty)$

If you find any errors, please let me know so that I can correct them.