1. Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y^2\vec{\imath} + xy\vec{\jmath}$ and $C$ is the line segment from $(10, 2)$ to $(6, 10)$.

First we need a parameterizations for the line segment $C$. As mentioned before, there is no unique choice of parameterization. There are three solutions to this problem given below, each using a different parameterization.

(a) Consider the parameterization for $C$: $\vec{r}(t) = (10 - 4t)\vec{\imath} + (2 + 8t)\vec{\jmath}$, $0 \le t \le 1$. Then we have $\vec{r}\,'(t) = -4\vec{\imath} + 8\vec{\jmath}$.

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \\
&= \int_0^1 \left((2 + 8t)^2\vec{\imath} + (10 - 4t)(2 + 8t)\vec{\jmath}\right) \cdot \left(-4\vec{\imath} + 8\vec{\jmath}\right)\, dt \\
&= \int_0^1 -4(4 + 32t + 64t^2) + 8(20 + 72t - 32t^2)\, dt \\
&= \int_0^1 -16 - 128t - 256t^2 + 160 + 576t - 256t^2\, dt \\
&= \int_0^1 144 + 448t - 512t^2\, dt \\
&= \left[144t + 224t^2 - \frac{512}{3}t^3\right]_0^1 \\
&= 144 + 224 - \frac{512}{3} - 0 \\
&= \frac{592}{3}
\end{aligned}$$

(b) In the previous solution, we may have noticed that the coefficients in the integrand got a bit large. This is due to our choice of parameterization. Thinking ahead about the fact that we will have to square our choice of $y$ in our parameterization, we may want to try to keep the coefficients smaller. This time, let's use the parameterization $\vec{r}(t) = (10 - t)\vec{\imath} + (2 + 2t)\vec{\jmath}$, $0 \le t \le 4$, giving $\vec{r}\,'(t) = -\vec{\imath} + 2\vec{\jmath}$.

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^4 \left((2 + 2t)^2\vec{\imath} + (10 - t)(2 + 2t)\vec{\jmath}\right) \cdot \left(-\vec{\imath} + 2\vec{\jmath}\right)\, dt \\
&= \int_0^4 -(4 + 8t + 4t^2) + 2(20 + 18t - 2t^2)\, dt \\
&= \int_0^4 -4 - 8t - 4t^2 + 40 + 36t - 4t^2\, dt \\
&= \int_0^4 36 + 28t - 8t^2\, dt
\end{aligned}$$

$$= \left[36t + 14t^2 - \frac{8}{3}t^3\right]_0^4$$
$$= 144 + 224 - \frac{512}{3} - 0$$
$$= \frac{592}{3}$$

(c) An alternative would be to choose a parameterization so that $y$ is even more straight forward. Consider $\vec{r}(t) = (11 - t)\vec{i} + (2t)\vec{j}$, $1 \le t \le 5$. Then $\vec{r}'(t) = -\vec{i} + 2\vec{j}$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^5 \left((2t)^2\vec{i} + (11-t)(2t)\vec{j}\right) \cdot \left(-\vec{i} + 2\vec{j}\right)\, dt$$
$$= \int_1^5 -4t^2 + 2(22t - 2t^2)\, dt$$
$$= \int_1^5 -4t^2 + 44t - 4t^2\, dt$$
$$= \int_1^5 -8t^2 + 44t\, dt$$
$$= \left[-\frac{8}{3}t^3 + 22t^2\right]_1^5$$
$$= \frac{8}{3}(125) + 22(25) - \left(-\frac{8}{3} + 22\right)$$
$$= -\frac{1000}{3} + 550 + \frac{8}{3} - 22$$
$$= \frac{592}{3}$$

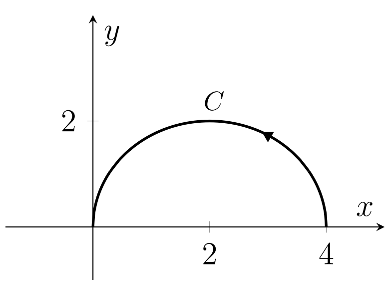
Notice that all 3 give the same result, with (a) having the simplest limits of integration, (c) having the most straight forward integrand, and (b) falling somewhere in between. The point being that computing a line integral in this fashion is usually a matter of "picking your poison" when it comes to choosing a parameterization.

2. Determine the value of $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (y+z)^2\vec{i} + \left(\frac{x-y}{z^2}\right)\vec{j} - xyz\vec{k}$ and $C$ is the line segment from $(0, 2, 1)$ to $(3, 0, 6)$.

Consider the parameterization:
$\vec{r}(t) = 3t\vec{i} - 2t\vec{j} + (1 + 5t)\vec{k}$, $0 \le t \le 1$
$\vec{r}'(t) = 3\vec{i} - 2\vec{j} + 5\vec{k}$

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 \left( (-2t+1+5t)^2\vec{i} + \frac{3t-(-2t)}{(1+5t)^2}\vec{j} - (3t)(2t)(1+5t)\vec{k} \right) \cdot \left( 3\vec{i} - 2\vec{j} + 5\vec{k} \right)\, dt \\
&= \int_0^1 3(3t+1)^2 + (-2)\frac{5t}{(1+5t)^2} + 5(6t^2)(1+5t)\, dt \\
&= \int_0^1 3(9t^2+6t+1) - \frac{10t}{(1+5t)^2} + 30t^2(1+5t)\, dt \\
&= \int_0^1 27t^2 + 18t - \frac{10t}{(1+5t)^2} + 30t^2 + 150t^3\, dt \\
&= \int_0^1 150t^3 + 57t^2 + 18t + 3 - \frac{10t}{(1+5t)^2}\, dt \\
&= \left[ \frac{75}{2}t^4 + 19t^3 + 9t^2 + 3t - \frac{2}{5}\left( \ln(1+5t) + \frac{1}{1+5t} \right)^{\dagger} \right]_0^1 \\
&= \frac{75}{2} + 19 + 9 + 3 - \frac{2}{5}\left( \ln(6) + \frac{1}{6} \right) - \left( 0+0+0+0 - \frac{2}{5}\left( \ln(1) + \frac{1}{1} \right) \right) \\
&= \frac{75}{2} + 31 - \frac{2}{5}\ln(6) - \frac{1}{15} + \frac{2}{5} \\
&= \frac{1125}{30} + \frac{930}{30} - \frac{2}{30} + \frac{12}{30} - \frac{2}{5}\ln(6) \\
&= \frac{413}{6} - \frac{2}{5}\ln(6)
\end{aligned}$$

3. Compute $\int_C \vec{F}\cdot d\vec{r}$ for the vector field $\vec{F} = y^2\vec{i} + xy\vec{j}$ and the curve $C$ shown below.




One possible parameterization is:

$r(t) = (2\cos t + 2)\vec{i} + (2\sin t)\vec{j}, 0 \le t \le \pi$

$r'(t) = (-2\sin t)\vec{i} + (2\cos t)\vec{j}$

† $\int \frac{10t}{(1+5t)^2}\, dt = \frac{2}{5}\left( \ln(1+5t) + \frac{1}{1+5t} \right)$

$u$-substitution with $u = 1+5t$:

$\int \frac{10t}{(1+5t)^2}\, dt = \frac{2}{5}\int \frac{u-1}{u^2}\, du$

Using the parameterization, we have

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_0^{\pi} \left((2\sin t)^2\vec{i} + (2\cos t+2)(2\sin t)\vec{j}\right)\cdot\left((-2\sin t)\vec{i} + (2\cos t)\vec{j}\right)\,dt \\
&= \int_0^{\pi} \left(4\sin^2 t\vec{i} + (4\cos t\sin t + 4\sin t)\vec{j}\right)\cdot\left((-2\sin t)\vec{i} + (2\cos t)\vec{j}\right)\,dt \\
&= \int_0^{\pi} -8\sin^3 t + 8\cos^2 t\sin t + 8\sin t\cos t\,dt \\
&= \int_0^{\pi} -8\sin t\left(1-\cos^2 t\right) + 8\cos^2 t\sin t + 8\sin t\cos t\,dt \\
&= \int_0^{\pi} -8\sin t + 16\cos^2 t\sin t + 8\sin t\cos t\,dt \\
&= \left[8\cos t - \frac{16}{3}\cos^3 t - 4\cos^2 t\right]_0^{\pi} \\
&= 8\cos(\pi) - \frac{16}{3}(\cos(\pi))^3 - 4(\cos(\pi))^2 - \left(8\cos(0) - \frac{16}{3}(\cos(0))^3 - 4(\cos(0))^2\right) \\
&= -8 + \frac{16}{3} - 4 - 8 + \frac{16}{3} + 4 \\
&= -\frac{16}{3}
\end{aligned}$$

4. Consider the vector field $\vec{F} = (x+12y)\vec{i} + x\vec{j} + \frac{1}{2}z^2x\vec{k}$ and the curve $C$ parameterized by $\vec{r}(t) = (t^2+1)\vec{i} - (t^3+2t)\vec{j} + 4t\vec{k}$, $1 \le t \le 3$. Compute $\int_C \vec{F}\cdot d\vec{r}$.

Using $\vec{r}(t)$ given and $\vec{r}\,'(t) = 2t\vec{i} - (3t^2+2)\vec{j} + 4\vec{k}$:

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_1^3 \left(\left(t^2+1-12t^3-24t\right)\vec{i} + (t^2+1)\vec{j} + \frac{1}{2}(4t)^2(t^2+1)\vec{k}\right)\cdot\left(2t\vec{i} - (3t^2+2)\vec{j} + 4\vec{k}\right)\,dt \\
&= \int_1^3 2t(-12t^3+t^2-24t+1) - (t^2+1)(3t^2+2) + 2(16t^2)(t^2+1)\,dt \\
&= \int_1^3 -24t^4 + 2t^3 - 48t^2 + 2t - (3t^4+5t^2+2) + 32t^4 + 32t^2\,dt \\
&= \int_1^3 8t^4 + 2t^3 - 16t^2 + 2t - 3t^4 - 5t^2 - 2\,dt \\
&= \int_1^3 5t^4 + 2t^3 - 21t^2 + 2t - 2\,dt \\
&= \left[t^5 + \frac{1}{2}t^4 - 7t^3 + t^2 - 2t\right]_1^3
\end{aligned}$$

$$
\begin{aligned}
&= 243 + \frac{81}{2} - 7(27) + 9 - 6 - (1 + \frac{1}{2} - 7 + 1 - 2) \\
&= 243 + \frac{81}{2} - 189 + 3 - \frac{1}{2} + 7 \\
&= 54 + \frac{80}{2} + 10 \\
&= 104
\end{aligned}
$$

5. Let $C$ be the curve parameterized by $\vec{r}(t) = t^3\vec{i} + (t^2 + 4t)\vec{j}$, $1 \le t \le 2$. Compute $\displaystyle\int_C \frac{y}{x}\,dx + (x - y)\,dy$.

From the parameterization, we have $x = t^3$ and $y = t^2 + 4t$. So, we also have $dx = 3t^2dt$ and $dy = (2t + 4)dt$.

Thus,

$$
\begin{aligned}
\int_C \frac{y}{x}\,dx + (x - y)\,dy &= \int_1^2 \left( \frac{t^2 + 4t}{t^3}(3t^2dt) + (t^3 - (t^2 + 4t))((2t + 4)dt) \right) \\
&= \int_1^2 \left( \frac{3(t^2 + 4t)}{t} + (t^3 - t^2 - 4t)(2t + 4) \right) dt \\
&= \int_1^2 3t + 12 + 2t^4 - 2t^3 - 8t^2 + 4t^3 - 4t^2 - 16t\,dt \\
&= \int_1^2 2t^4 + 2t^3 - 12t^2 - 13t + 12\,dt \\
&= \left[ \frac{2}{5}t^5 + \frac{1}{2}t^4 - 4t^3 - \frac{13}{2}t^2 + 12t \right]_1^2 \\
&= \frac{2}{5}(32) + \frac{1}{2}(16) - 4(8) - \frac{13}{2}(4) + 12(2) - \left( \frac{2}{5} + \frac{1}{2} - 4 - \frac{13}{2} + 12 \right) \\
&= \frac{64}{5} + 8 - 32 - 26 + 24 - (\frac{2}{5} - \frac{12}{2} + 8) \\
&= \frac{64}{5} - 26 - \frac{2}{5} + 6 - 8 \\
&= \frac{62}{5} - 28 \\
&= -\frac{78}{5}
\end{aligned}
$$

6. Determine $\int_C \vec{F} \cdot d\vec{r}$ where $C$ is the curve parameterized by $\vec{r}(t) = (e^t)\,\vec{i} + (e^{-t})\,\vec{j} + t^2\vec{k}$, $0 \le t \le 1$, and $\vec{F} = xy\vec{i} + x^2z\vec{j} + xyz\vec{k}$.

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \left( \left(e^t\right)\left(e^{-t}\right)\vec{i} + \left(e^t\right)^2 t^2\vec{j} + \left(e^t\right)\left(e^{-t}\right)t^2\vec{k} \right) \cdot \left( e^t\vec{i} - e^{-t}\vec{j} + 2t\vec{k} \right) dt \\
&= \int_0^1 \left(e^t\right)\left(e^t\right)\left(e^{-t}\right) - \left(e^{-t}\right)\left(e^{2t}\right)t^2 + 2t\left(e^t\right)\left(e^{-t}\right)t^2 \, dt \\
&= \int_0^1 e^t - e^t t^2 + 2t^3 \, dt \\
&= \int_0^1 e^t + 2t^3 \, dt - \int_0^1 e^t t^2 \, dt \\
&= \left[ e^t + \frac{1}{2}t^4 \right]_0^1 - \left( t^2 e^t \Big|_0^1 - \int_0^1 2te^t \, dt \right)^{\ddagger} \\
&= e^1 + \frac{1}{2} - \left(e^0 + 0\right) - \left(e^1 - 0\right) + \int_0^1 2te^t \, dt \\
&= e + \frac{1}{2} - 1 - e + 2te^t \Big|_0^1 - \int_0^1 2e^t \, dt \\
&= -\frac{1}{2} + 2e^1 - 0 - 2e^t \Big|_0^1 \\
&= -\frac{1}{2} + 2e - \left(2e^1 - 2e^0\right) \\
&= -\frac{1}{2} + 2e - 2e + 2 \\
&= \frac{3}{2}
\end{aligned}$$

‡ Integration by parts

7. Compute $\int_C \vec{G} \cdot d\vec{r}$, where $C$ is the line segment from $(0, 1, 2)$ to $(8, 6, 0)$ and $\vec{G} = (x + z)\vec{i} - (z^2 - 3)\vec{j} + 2xyz\vec{k}$.

One possible parameterization is:
$r(t) = 8t\vec{i} + (1 + 5t)\vec{j} + (2 - 2t)\vec{k}, 0 \leq t \leq 1$
$r'(t) = 8\vec{i} + 5\vec{j} - 2\vec{k}$

$$\begin{aligned}
\int_C \vec{G} \cdot d\vec{r} &= \int_0^1 \left((8t + 2 - 2t)\vec{i} - ((2 - 2t)^2 - 3)\vec{j} + 2(8t)(1 + 5t)(2 - 2t)\vec{k}\right) \cdot \left(8\vec{i} + 5\vec{j} - 2\vec{k}\right)\, dt \\
&= \int_0^1 \left((6t + 2)\vec{i} - (4 - 8t + 4t^2 - 3)\vec{j} + 16t(2 - 2t + 10t - 10t^2)\vec{k}\right) \cdot \left(8\vec{i} + 5\vec{j} - 2\vec{k}\right)\, dt \\
&= \int_0^1 8(6t + 2) - 5(1 - 8t + 4t^2) - 32t(2 + 8t - 10t^2)\, dt \\
&= \int_0^1 48t + 16 - 5 + 40t - 20t^2 - 64t - 256t^2 + 320t^3\, dt \\
&= \int_0^1 320t^3 - 276t^2 + 24t + 11\, dt \\
&= \left[80t^4 - 92t^3 + 12t^2 + 11t\right]_0^1 \\
&= 80 - 92 + 12 + 11 - 0 \\
&= 11
\end{aligned}$$