

Example 1: Consider the vector field $\vec{F}(x, y) = \left(\frac{-y}{x^2 + y^2} \right) \vec{i} + \left(\frac{x}{x^2 + y^2} \right) \vec{j}$.

(a) Calculate $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

$$\begin{aligned} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left(\frac{-y}{x^2 + y^2} \right) \\ &= \frac{(1)(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} - \frac{(-1)(x^2 + y^2) - (-y)(2y)}{(x^2 + y^2)^2} \\ &= \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} - \frac{-x^2 - y^2 + 2y^2}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{-x^2 + y^2}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2} \\ &= 0 \end{aligned}$$

(b) Does the vector field \vec{F} have the domain property required in order to apply the Curl Test?

The domain of \vec{F} is all of the xy -plane except for the origin. It is easy to find a closed curve, C , lying in the domain of \vec{F} which has points inside it which are not in the domain. For example, take the unit circle – the entirety of the unit circle is in the domain, but the origin (which is not in the domain of \vec{F}) is in the interior of C . Hence, the domain property does not hold for this vector field.

(c) What can we conclude about the path-independence or path-dependence of \vec{F} from parts (a) and (b)?

NOTHING! Solely from the results of the previous parts, we can not conclude whether the vector field is path-independent or path-dependent.

The fact that the curl $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$ is zero, does **not** imply that the vector field is path-independent.

The fact that the domain property does not hold for the vector field does **not** say anything about the path-independence or path-dependence of the vector field. It simply means that the Curl Test cannot be applied, since the hypotheses are not met, and we must show path-independence or path-dependence in another way.

(d) Let C be the unit circle oriented in the counterclockwise direction. Compute $\oint_C \vec{F} \cdot d\vec{r}$.

Everywhere on the curve C , \vec{F} is tangent to C , and $x^2 + y^2 = 1$. So, everywhere on C ,

the magnitude of \vec{F} is

$$\|\vec{F}\| = \sqrt{\left(\frac{-y}{x^2+y^2}\right)^2 + \left(\frac{x}{x^2+y^2}\right)^2} = \sqrt{\frac{y^2}{(x^2+y^2)^2} + \frac{x^2}{(x^2+y^2)^2}} = \sqrt{\frac{y^2+x^2}{(x^2+y^2)^2}} = 1$$

Hence, $\oint_C \vec{F} \cdot d\vec{r} = \|\vec{F}\|(\text{length of } C) = 1(2\pi) = 2\pi$.

Alternatively, this can be computed easily by taking the “usual” parameterization of C : $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$, $0 \leq t \leq 2\pi$.

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \left(\left(\frac{-\sin t}{\cos^2 t + \sin^2 t} \right) \vec{i} + \left(\frac{\cos t}{\cos^2 t + \sin^2 t} \right) \vec{j} \right) \cdot \left((-\sin t)\vec{i} + (\cos t)\vec{j} \right) dt \\ &= \int_0^{2\pi} \frac{\sin^2 t}{\cos^2 t + \sin^2 t} + \frac{\cos^2 t}{\cos^2 t + \sin^2 t} dt \\ &= \int_0^{2\pi} dt \\ &= 2\pi \end{aligned}$$

(e) What can be concluded about path-independence or path-dependence?

Since C is a closed curve and $\oint_C \vec{F} \cdot d\vec{r} \neq 0$, the vector field \vec{F} is path-dependent¹.

Example 2: Consider the vector field $\vec{F}(x, y) = \frac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}}$.

(a) Calculate $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

$$\begin{aligned} \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{y}{\sqrt{x^2 + y^2}} \right) & \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) \\ &= \frac{\partial}{\partial x} \left(y (x^2 + y^2)^{-1/2} \right) & &= \frac{\partial}{\partial y} \left(x (x^2 + y^2)^{-1/2} \right) \\ &= -\frac{1}{2} y (x^2 + y^2)^{-3/2} (2x) & &= -\frac{1}{2} x (x^2 + y^2)^{-3/2} (2y) \\ &= \frac{-xy}{(x^2 + y^2)^{3/2}} & &= \frac{-xy}{(x^2 + y^2)^{3/2}} \end{aligned}$$

$$\text{So, } \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

¹A vector field, \vec{F} , is path-independent if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C .

- (b) Does the vector field \vec{F} have the domain property required in order to apply the Curl Test?

No, the domain of \vec{F} is all of the 2-space except for the origin, just as in the previous example.

- (c) What can be concluded about the path-independence or path-dependence of \vec{F} from parts (a) and (b)?

Again, just as in the previous example, nothing can be concluded about the path-independence or path-dependence of \vec{F} yet.

- (d) Let C be the unit circle oriented in the counterclockwise direction. Compute $\oint_C \vec{F} \cdot d\vec{r}$.

The vector field \vec{F} is perpendicular to C at every point, so $\oint_C \vec{F} \cdot d\vec{r} = 0$.

Alternatively, for C : $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$, $0 \leq t \leq 2\pi$ and

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \left(\frac{(\cos t)\vec{i} + (\sin t)\vec{j}}{\sqrt{\cos^2 t + \sin^2 t}} \right) \cdot \left((-\sin t)\vec{i} + (\cos t)\vec{j} \right) dt \\ &= \int_0^{2\pi} \frac{-\cos t \sin t + \sin t \cos t}{\sqrt{\cos^2 t + \sin^2 t}} dt \\ &= \int_0^{2\pi} 0 dt \\ &= 0\end{aligned}$$

- (e) What can be concluded about path-independence or path-dependence?

Nothing. The fact that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for a particular closed curve says nothing about the path-independence or path-dependence of \vec{F} . In order to use this idea to conclude that \vec{F} is path-independent, we would have to show that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve in the domain. That is, we have one closed curve done, now only infinitely many more to go (this could take a while).

- (f) Consider the scalar function $f(x, y) = \sqrt{x^2 + y^2}$. Determine $\text{grad} f$.

$$\begin{aligned}\text{grad} f(x, y) &= f_x(x, y)\vec{i} + f_y(x, y)\vec{j} \\ &= \left(\frac{1}{2}(x^2 + y^2)^{-1/2}(2x) \right) \vec{i} + \left(\frac{1}{2}(x^2 + y^2)^{-1/2}(2y) \right) \vec{j} \\ &= \left(\frac{x}{\sqrt{x^2 + y^2}} \right) \vec{i} + \left(\frac{y}{\sqrt{x^2 + y^2}} \right) \vec{j}\end{aligned}$$

So, $\text{grad}f(x, y) = \vec{F}$

(g) What can be concluded about path-independence or path-dependence?

Since \vec{F} is a gradient vector field (we just showed that $f(x, y)$ is a potential function), we can conclude that \vec{F} is path-independent².

Example 3: Let \vec{F} be the vector field given by $\vec{F} = \left(\frac{x^2y^2}{x^2y^2+1} + \frac{1}{2} \ln(x^2y^2+1) \right) \vec{i} + \left(\frac{x^3y}{x^2y^2+1} \right) \vec{j}$.

(a) Calculate $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

$$\begin{aligned} \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{x^3y}{x^2y^2+1} \right) \\ &= \frac{3x^2y(x^2y^2+1) - x^3y(2xy^2)}{(x^2y^2+1)^2} \\ &= \frac{3x^4y^3 + 3x^2y - 2x^4y^3}{(x^2y^2+1)^2} \\ &= \frac{x^4y^3 + 3x^2y}{(x^2y^2+1)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{x^2y^2}{x^2y^2+1} + \frac{1}{2} \ln(x^2y^2+1) \right) \\ &= \frac{2x^2y(x^2y^2+1) - x^2y^2(2x^2y)}{(x^2y^2+1)^2} + \frac{1}{2} \left(\frac{1}{x^2y^2+1} \right) (2x^2y) \\ &= \frac{2x^4y^3 + 2x^2y - 2x^4y^3}{(x^2y^2+1)^2} + \frac{x^2y}{x^2y^2+1} \\ &= \frac{2x^2y}{(x^2y^2+1)^2} + \frac{x^2y(x^2y^2+1)}{(x^2y^2+1)^2} \\ &= \frac{2x^2y}{(x^2y^2+1)^2} + \frac{x^4y^3 + x^2y}{(x^2y^2+1)^2} \\ &= \frac{3x^2y + x^4y^3}{(x^2y^2+1)^2} \end{aligned}$$

²A continuous vector field, \vec{F} , defined on an open region is path-independent if and only if \vec{F} is a gradient vector field.

$$\text{So, } \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

- (b) Does the vector field \vec{F} have the domain property required in order to apply the Curl Test?

The domain of \vec{F} is the entire xy -plane (the part to be concerned with is $\ln(x^2y^2 + 1)$, however $x^2y^2 + 1 \geq 1$, so $\ln(x^2y^2 + 1)$, and hence \vec{F} , is defined for all points in 2-space). Since the domain is the entire xy -plane, it certainly has the required domain property.

- (c) What can we conclude about the path-independence or path-dependence of \vec{F} from parts (a) and (b)?

Since the curl of \vec{F} is zero, and the domain property holds for \vec{F} , we have *both* hypotheses of the Curl Test satisfied, and can use it to conclude that \vec{F} is path-independent³.

- (d) Let C be the unit circle oriented in the counterclockwise direction. Compute $\oint_C \vec{F} \cdot d\vec{r}$.

We just concluded that \vec{F} is path-independent, so with no computation required at all, we know that $\oint_C \vec{F} \cdot d\vec{r} = 0$ ¹.

Example 4: Determine whether or not the Curl Test can be applied to the vector field

$$\vec{F}(x, y) = (3x^2y^2 + 6(2x - 5y)^2 - 2x \sin(y^3)) \vec{i} + (2x^3y - 15(2x - 5y)^2 - 3x^2y^2 \cos(y^3)) \vec{j}.$$

There are two hypotheses to check (whether or not the curl is zero, and whether the domain property holds). Let's check the curl first.

$$\begin{aligned} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial x} (2x^3y - 15(2x - 5y)^2 - 3x^2y^2 \cos(y^3)) - \frac{\partial}{\partial y} (3x^2y^2 + 6(2x - 5y)^2 - 2x \sin(y^3)) \\ &= 6x^2y - 15(2)(2x - 5y)(2) - 6xy^2 \cos(y^3) - (6x^2y + 6(2)(2x - 5y)(-5) - 2x \cos(y^3)(3y^2)) \\ &= 6x^2y - 60(2x - 5y) - 6xy^2 \cos(y^3) - (6x^2y - 60(2x - 5y) - 6xy^2 \cos(y^3)) \\ &= 0 \end{aligned}$$

Since $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$, we now check the domain property. The domain of \vec{F} is all of the xy -plane, so the domain property holds.

We have that the domain property holds and $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$. Thus, since both hypotheses are met, the Curl Test can be applied, and we can conclude that \vec{F} is path-independent.

³It is not necessary to find it, but since \vec{F} is path-independent, there is a potential function. $f(x, y) = \frac{x}{2} \ln(x^2y^2 + 1)$ is one such potential function.

Example 5: Determine whether or not the Curl Test can be applied to the vector field

$$\vec{H}(x, y) = \left(-\frac{1}{y} + 3y - \tan^{-1}(y^2) + 18x\sqrt{x^2 - y^3} \right) \vec{i} + \left(\frac{x}{y^2} - \frac{2xy}{1 + y^4} - 27y^2\sqrt{x^2 - y^3} + xy \right) \vec{j}.$$

Computing $\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}$:

$$\begin{aligned} \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} &= \frac{\partial}{\partial x} \left(\frac{x}{y^2} - \frac{2xy}{1 + y^4} - 27y^2\sqrt{x^2 - y^3} + xy \right) - \frac{\partial}{\partial y} \left(-\frac{1}{y} + 3y - \tan^{-1}(y^2) + 18x\sqrt{x^2 - y^3} \right) \\ &= \frac{1}{y^2} - \frac{2y}{1 + y^4} - 27y^2 \left(\frac{1}{2} \right) (x^2 - y^3)^{-1/2} (2x) + y - \\ &= \left(-\left(-\frac{1}{y^2} \right) + 3 - \frac{1}{1 + (y^2)^2} (2y) + 18x \left(\frac{1}{2} \right) (x^2 - y^3)^{-1/2} (-3y^2) \right) \\ &= \frac{1}{y^2} - \frac{2y}{1 + y^4} - \frac{27xy^2}{\sqrt{x^2 - y^3}} + y - \left(\frac{1}{y^2} + 3 - \frac{2y}{1 + y^4} - \frac{27xy^2}{\sqrt{x^2 - y^3}} \right) \\ &= \frac{1}{y^2} - \frac{2y}{1 + y^4} - \frac{27xy^2}{\sqrt{x^2 - y^3}} + y - \frac{1}{y^2} - 3 + \frac{2y}{1 + y^4} + \frac{27xy^2}{\sqrt{x^2 - y^3}} \\ &= y - 3 \end{aligned}$$

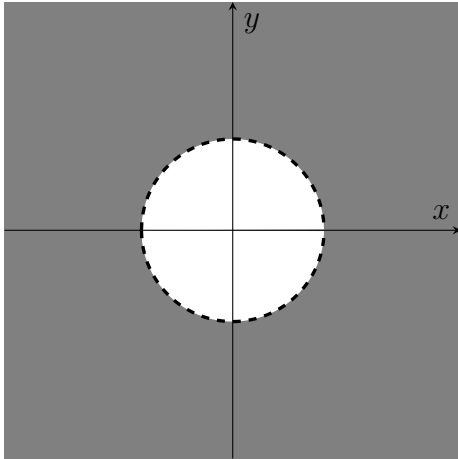
Since $\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \neq 0$, the curl test does not apply. There's no need to check whether or not the domain property holds for \vec{H} because no matter whether it does or not, the Curl Test cannot be used since we already know that (at least) one hypothesis is not met.

In fact, since $\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \neq 0$, we know that \vec{H} is path-dependent.

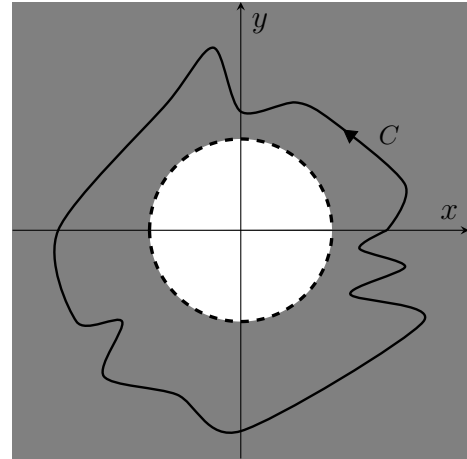
Example 6: Determine whether or not the Curl Test can be applied to the vector field

$$\vec{G} = \left(\frac{x^2y^3 - \sin(x + y)}{x^4 + y^2 + 2} \right) \vec{i} - \left(\frac{\cosh(xy) + e^{y \cos x}}{\sqrt{x^2 + y^2 - 1}} \right) \vec{j}.$$

For this one, let's take a look at the domain first. In order for a point (x, y) to be in the domain, it must be the case that $x^2 + y^2 - 1 > 0$. That is, the domain is all points in the xy -plane where $x^2 + y^2 > 1$. The domain of \vec{G} is the xy -plane minus the closed unit disk (or phrased differently, the domain is all points outside the unit disk).



The domain of \vec{G}



The domain of \vec{G} with a closed curve, C , drawn

After visualizing/sketching the domain of \vec{G} , as above on the left, we can check whether or not the domain property holds by seeing whether it's possible to draw a closed curve that lies in the domain of \vec{G} but contains points in its interior which are not in the domain (that is, can we draw a curve which shows that the domain property fails). On the right above is an illustration of one such closed curve (we don't have to do any integration along it, so who cares how "interesting" the curve is). Hence, the domain property does not hold for the vector field \vec{G} .

Since the domain property does not hold, the Curl Test cannot be applied. Thankfully, we do not need to compute $\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}$, since we already know that regardless of the result, we cannot use the Curl Test (which is all that was asked).

Note: Simply not being able to apply the Curl Test tells us nothing about whether the vector field is path-independent or path-dependent. In **Example 1** the Curl Test could not be applied and the vector field turned out to be path-dependent, yet in **Example 2** the Curl Test could not be applied and the vector field turned out to be path-independent.