Example 1: Let R be the cone $z = -2 + \sqrt{x^2+y^2}$, which opens toward the positive z-axis, combined with the disk of radius 2 in the xy-plane, centered at the origin. Determine the flux of $\vec{F} = \left(3yz^2 + \frac{x^3}{3}\right)\vec{i} + \left(-5x^4z + \frac{y^3}{3}\right)\vec{j} + \left(x^2 - 7y^2 + \frac{3}{2}\right)\vec{k}$ **into** R.

By computing the divergence of the vector field, $\vec{F}$, we obtain

$$\text{div}\vec{F} = x^2 + y^2$$

Let W be the solid whose boundary is R. By the Divergence Theorem,

$$\begin{aligned}
\oint_R \vec{F}\cdot d\vec{A} &= -\int_W \text{div}\vec{F}\, dV \\
&= -\int_W x^2 + y^2\, dV \\
&= -\int_0^{2\pi}\int_0^2\int_{r-2}^0 \left(r^2\right) r\, dz\, dr\, d\theta \\
&= -\int_0^{2\pi}\int_0^2\int_{r-2}^0 r^3\, dz\, dr\, d\theta \\
&= -\int_0^{2\pi}\int_0^2 \Big[r^3 z\Big]_{r-2}^0\, dr\, d\theta \\
&= -\int_0^{2\pi}\int_0^2 0 - r^3(r-2)\, dr\, d\theta \\
&= -\int_0^{2\pi}\int_0^2 -r^4 + 2r^3\, dr\, d\theta \\
&= -\int_0^{2\pi}\left[\frac{-r^5}{5} + \frac{2r^4}{4}\right]_0^2 d\theta \\
&= -\int_0^{2\pi}\left[\frac{-r^5}{5} + \frac{r^4}{2}\right]_0^2 d\theta \\
&= -\int_0^{2\pi}\left(\frac{-(2)^5}{5} + \frac{2^4}{2}\right) - 0\, d\theta \\
&= -\int_0^{2\pi}\left(\frac{-32}{5}\right) + 8\, d\theta \\
&= -\int_0^{2\pi}\left(\frac{-32}{5}\right) + \frac{40}{5}\, d\theta \\
&= -\int_0^{2\pi}\frac{8}{5}\, d\theta \\
&= \left[-\frac{8}{5}\theta\right]_0^{2\pi} \\
&\boxed{= -\frac{16\pi}{5}}
\end{aligned}$$

Example 2: Let S be the "Open bottom" cylindrical surface given by $x^2 + y^2 = 16$ for $0 \le z \le 3$. Determine the flux of $\vec{F} = (3z^2y)\vec{i} + (xy + y)\vec{j} + 2\vec{k}$ **out of** S.

Let D be the disk of radius 4 in the $z = 0$ plane, oriented downward, whose center is (0, 0, 0). Since $S + D$ is a closed surface, we can use the Divergence theorem to compute the flux out of $S + D$. Then, we will subtract this value by the flux of $\vec{F}$ out of D to obtain the value of the flux of $\vec{F}$ out of S.

By computing the divergence of the vector field, $\vec{F}$, we obtain

$$\text{div}\vec{F} = x + 1$$

Let W be the solid whose boundary is $S + D$. By the Divergence Theorem,

$$\begin{aligned}
\oint_{S+D} \vec{F} \cdot d\vec{A} &= \int_W x + 1 \; dV \\
&= \int_0^{2\pi} \int_0^4 \int_0^3 (r\cos\theta + 1) r \; dz \; dr \; d\theta \\
&= \int_0^{2\pi} \int_0^4 \int_0^3 r^2\cos\theta + r \; dz \; dr \; d\theta \\
&= \int_0^{2\pi} \int_0^4 \Big[zr^2\cos\theta + zr \Big]_0^3 dr \; d\theta \\
&= \int_0^{2\pi} \int_0^4 3r^2\cos\theta + 3r \; dr \; d\theta \\
&= \int_0^{2\pi} \Big[r^3\cos\theta + \frac{3}{2}r^2 \Big]_0^4 d\theta \\
&= \int_0^{2\pi} 64\cos\theta + 24 \; d\theta \\
&= \Big[64\sin\theta + 24\theta \Big]_0^{2\pi} \\
&= (0 + 48\pi) - (0 + 0) \\
&= 48\pi
\end{aligned}$$

$$\begin{aligned}
\int_D \vec{F} \cdot d\vec{A} &= \int_D \Big[(3z^2y)\vec{i} + (xy + 1)\vec{j} + 2\vec{k} \Big] \cdot -\vec{k} \; dA \\
&= \int_D -2 \; dA \\
&= -2\left(\pi(4)^2\right) \\
&= -32\pi
\end{aligned}$$

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \oint_{S+D} \vec{F} \cdot d\vec{A} \; - \int_D \vec{F} \cdot d\vec{A} \\
&= 48\pi - (-32\pi) \\
&\boxed{= 80\pi}
\end{aligned}$$

Example 3: Let P be the closed hemisphere of radius 3, centered at (3, 4, -5), for $y \geq 4$. Determine the flux of

$$\vec{G} = \left(e^{x^2y^2} + \frac{1}{\sqrt{x^2 - z^2}} + 8x\right)\vec{i} + \left(xyz^2 + 9x - 3yz + \frac{xy}{(x^2 - z^2)^{\frac{3}{2}}}\right)\vec{j} + \left(-\frac{xz^3}{3} - 2xy^2ze^{x^2y^2} + \frac{3z^2}{2}\right)\vec{k}$$

out of P.

By computing the divergence of the vector field, $\vec{G}$, we obtain

$$\begin{aligned}\text{div}\vec{G} &= \left(e^{x^2y^2} \cdot 2y^2x - \frac{x}{(x^2 - z^2)^{\frac{3}{2}}} + 8\right) + \left(xz^2 - 3z + \frac{x}{(x^2 - z^2)^{\frac{3}{2}}}\right) + \left(-xz^2 - 2e^{x^2y^2}xy^2 + 3z\right)\\ &= 8\end{aligned}$$

Let W be the solid whose boundary is P. By the Divergence Theorem,

$$\begin{aligned}\oint_P \vec{G} \cdot d\vec{A} &= \int_W \text{div}\vec{G}\; dV\\ &= \int_W 8\; dV\\ &= 8\left(\frac{2}{3}\pi(3)^3\right)\\ &\boxed{= 144\pi}\end{aligned}$$

Because the divergence is constant, we can multiply the divergence of the vector field by the volume of the solid we are integrating over. Note that the volume of a hemisphere is given by $V = \frac{2}{3}\pi r^3$.