Example 1: Let S be the cylinder of radius 2 centered along the x-axis with $-1 \leq x \leq 3$. Determine the flux of $\vec{F} = \left(e^{x^2y^2} + \sec^2\theta\right)\vec{i} + (\cos\theta + \sin\theta)\,\vec{j} + (-\cos\theta)\,\vec{k}$ **out of** R.

Since the cylinder is centered along the x-axis instead of the z-axis (like we are accustomed to), we need to use a set of modified cylindrical coordinates. The orientation vector in this case is $\cos\theta\vec{j} + \sin\theta\vec{k}$ because this vector is normal to a cylinder centered along the x-axis. If there were a non-zero $\vec{i}$ component, this would not be true. We are also using dx (rather than dz) because the cylinder is centered along the x-axis.

Let T be the θx-region corresponding to S.

$$
\begin{aligned}
\int_S \vec{F}\cdot d\vec{A} &= \int_T \vec{F}(R,\theta,x)\cdot(\cos\theta\vec{j} + \sin\theta\vec{k})R\ d\theta\,dx \\
&= \int_T \left[\left(e^{x^2(2\cos\theta)^2} + \sec^2\theta\right)\vec{i} + (\cos\theta + \sin\theta)\,\vec{j} + (-\cos\theta)\,\vec{k}\right]\cdot(\cos\theta\vec{j} + \sin\theta\vec{k})(2)\ d\theta\,dx \\
&= \int_T (\cos\theta + \sin\theta)\,(2\cos\theta) + (-\cos\theta)\,(2\sin\theta)\ d\theta\,dx \\
&= \int_T 2\cos^2\theta + 2\sin\theta\cos\theta - 2\sin\theta\cos\theta\ d\theta\,dx \\
&= \int_T 2\cos^2\theta\ d\theta\,dx \\
&= \int_{-1}^{3}\int_0^{2\pi} 2\cos^2\theta\ d\theta\,dx \\
&= \int_{-1}^{3}\int_0^{2\pi} 2\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\ d\theta\,dx \\
&= \int_{-1}^{3}\int_0^{2\pi} 1 + \cos 2\theta\ d\theta\,dx \\
&= \int_{-1}^{3}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} dx \\
&= \int_{-1}^{3} 2\pi\ dx \\
&= \Big[2\pi x\Big]_{-1}^{3} \\
&= (2\pi\cdot 3) - (2\pi\cdot -1) \\
&= 6\pi + 2\pi \\
&\boxed{= 8\pi}
\end{aligned}
$$

Example 2: Consider the surface S which is the portion of the graph $f(x,y) = 2 + \sqrt{x^2+y^2}$, oriented **upward**, which lies above the disk of radius $\sqrt{2}$ in the xy-plane, D, centered at the origin. Determine the flux of $\vec{F}(x,y,z) = (y+5xy)\,\vec{i} + \left(\frac{\sqrt{x^2+y^2}}{y} - 5x^2 - x\right)\vec{j} + 9x\vec{k}$ through S.

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

$$\begin{aligned}
\int_S \vec{F}\cdot d\vec{A} &= \int_D \vec{F}(x,y,f(x,y))\cdot(-f_x\vec{i} - f_y\vec{j} + \vec{k})\ dA \\
&= \int_D \left[(y+5xy)\,\vec{i} + \left(\frac{\sqrt{x^2+y^2}}{y} - 5x^2 - x\right)\vec{j} + 9x\vec{k}\right]\cdot\left(-\frac{x}{\sqrt{x^2+y^2}}\right)\vec{i} + \left(-\frac{y}{\sqrt{x^2+y^2}}\right)\vec{j} + \vec{k}\ dA \\
&= \int_D \left(-\frac{xy}{\sqrt{x^2+y^2}} - \frac{5x^2y}{\sqrt{x^2+y^2}}\right) + \left(-1 + \frac{5x^2y}{\sqrt{x^2+y^2}} + \frac{xy}{\sqrt{x^2+y^2}}\right) + 9x\ dA \\
&= \int_D 9x - 1\ dA \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} (9r\cos\theta - 1)r\ dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} 9r^2\cos\theta - r\ dr\,d\theta \\
&= \int_0^{2\pi} \left[3r^3\cos\theta - \frac{r^2}{2}\right]_0^{\sqrt{2}} dr\,d\theta \\
&= \int_0^{2\pi} 6\sqrt{2}\cos\theta - 1\ d\theta \\
&= \left[6\sqrt{2}\sin\theta - \theta\right]_0^{2\pi} \\
&= (0-2\pi) - (0-0) \\
&\boxed{= -2\pi}
\end{aligned}$$

Example 3: Determine $\int_T \vec{F} \cdot d\vec{A}$ where $\vec{F} = (x+z)\vec{i} + (y-z)\vec{j} + (-z+2y)\vec{k}$ and T is a portion of $4x + 2y + z = 4$ in the first octant oriented **downward**.

The surface, T, is a portion of $f(x, y) = -4x - 2y + 4$. After calculating the partials, we achieve

$$f_x(x, y) = -4 \qquad\qquad f_y(x, y) = -2$$

Since T is oriented downward, we must dot our vector field with $f_x\vec{i} + f_y\vec{j} - \vec{k}$, which is the negation of when a surface is oriented upward $(-f_x\vec{i} - f_y\vec{j} + \vec{k})$.

Let R be the region in the xy-plane that lies under T.

$$\begin{aligned}
\int_T \vec{F} \cdot d\vec{A} &= \int_R \vec{F}(x, y, f(x, y)) \cdot (f_x\vec{i} + f_y\vec{j} - \vec{k})\ dA \\
&= \int_R \Big[(x + (4 - 4x - 2y))\vec{i} + (y - (-4x - 2y + 4))\vec{j} + (-(-4x - 2y + 4) + 2y)\vec{k}\Big] \cdot (-4\vec{i} - 2\vec{j} - \vec{k})\ dA \\
&= \int_R (-3x - 2y + 4)\vec{i} + (4x + 3y - 4)\vec{j} + (4x + 4y - 4)\vec{k} \cdot (-4\vec{i} - 2\vec{j} - \vec{k})\ dA \\
&= \int_R 12x + 8y - 16 - 8x - 6y + 8 - 4x - 4y + 4\ dA \\
&= \int_R -2y + 4\ dA \\
&= \int_0^1 \int_0^{-2x+2} -2y + 4\ dy\, dx \\
&= \int_0^1 \Big[-y^2 + 4y\Big]_0^{-2x+2}\ dx \\
&= \int_0^1 \big(-(-2x + 2)^2 + 4(-2x + 2)\big) - (0 + 0)\ dx \\
&= \int_0^1 -4x^2 + 8x - 4 - 8x + 8\ dx \\
&= \int_0^1 -4x^2 + 4\ dx \\
&= \left[-\frac{4x^3}{3} + 4x\right]_0^1 \\
&= -\frac{4}{3} + 4 \\
&= -\frac{4}{3} + \frac{12}{3} \\
&\boxed{= \frac{8}{3}}
\end{aligned}$$