**Example 1:** Find and classify the critical points of $f(x, y) = \frac{1}{3}x^3 + y^3 - \frac{5}{2}x^2 + 3y^2 + 6x$.

We know that critical points are when the gradient vector, $\nabla f = f_x\vec{i} + f_y\vec{j}$, is the zero vector or undefined. Hence, we need to compute $f_x$ and $f_y$ and set each of them equal to 0 to find the critical points.

$$f_x(x, y) = x^2 - 5x + 6 \qquad\qquad f_y(x, y) = 3y^2 + 6y$$

$$x^2 - 5x + 6 = 0 \qquad \underline{\textit{and}} \qquad 3y^2 + 6y = 0$$

$$(x - 2)(x - 2) = 0 \qquad\qquad 3y(y + 2) = 0$$

$$x = 2,\ x = 3 \qquad\qquad y = 0,\ y = -2$$

Since the $x$-values found are independent of $y$ (there were no $y$'s in $x^2 - 5x + 6 = 0$), and the $y$-values found are independent of $x$ (there were no $x$'s in $3y^2 + 6y = 0$), we can mix and match the solutions. Hence, we obtain the following critical points:

$$\boxed{C.P.:\ (2, 0), (2, -2), (3, 0), (3, -2)}$$

Now, we must classify each critical point. To do this, we must calculate the second-order partial derivatives of $f$:

$$f_{xx} = 2x - 5 \qquad\qquad f_{yy} = 6y - 6 \qquad\qquad f_{xy} = 0$$

| Pt. | $f_{xx}$ | $f_{yy}$ | $f_{xy}$ | $D$ |
|---|---|---|---|---|
| (2,0) | $-1$ | 6 | 0 | $(-1)(6) - (0)^2 = -6$ |
| $(2, -2)$ | $-1$ | $-6$ | 0 | $(-1)(-6) - (0)^2 = 6$ |
| (3,0) | 1 | 6 | 0 | $(1)(6) - (0)^2 = 6$ |
| $(3, -2)$ | 1 | $-6$ | 0 | $(1)(-6) - (0)^2 = -6$ |

There is a local minimum at $(3, 0)$, a local maximum at $(2, -2)$, and saddle points at $(2, 0)$ and $(3, -2)$.

**Example 2:** Find and classify the critical points of $f(x,y) = x - 3y + xy - \frac{1}{2}y^2 + \frac{1}{3}x^3$.

$$f_x = 1 + y + x^2 \qquad\qquad f_y = -3 + x - y$$

$$(1)\quad 1 + y + x^2 = 0 \qquad \underline{\textit{and}} \qquad (2)\quad -3 + x - y = 0$$

By solving for $y$ in (2), we obtain $y = x - 3$. We can substitute this value of $y$ into (1):

$$1 + (x-3) + x^2 = 0$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x = -2,\ x = 1$$

By substituting $x = -2$ into (2), we obtain $y = -5$. By substituting $x = 1$ into (2), we obtain $y = -2$. Hence, we achieve the following critical points:

$$\boxed{C.P.:\ (-2,-5),(1,-2)}$$

Now, we must classify each critical point. To do this, we must calculate the second-order partial derivatives of $f$:

$$f_{xx} = 2x \qquad\qquad f_{yy} = -1 \qquad\qquad f_{xy} = 1$$

| Pt. | $f_{xx}$ | $f_{yy}$ | $f_{xy}$ | $D$ |
|---|---|---|---|---|
| $(-2,-5)$ | $-4$ | $-1$ | $1$ | $(-4)(-1) - (1)^2 = 3$ |
| $(1,-2)$ | $2$ | $-1$ | $1$ | $(2)(-1) - (1)^2 = -3$ |

There is a local minimum at $(-2,-5)$ and a saddle point at $(1,-2)$.