ex: Determine the flux of $\vec{F} = 3x\vec{i} + (x^2(1 - xy^2) - z)\vec{j} + 2z\vec{k}$ through the surface S, which is the portion of $z = xy + x^2 - y^2x^3 + 8$ which lies above the rectangle R: $1 \le x \le 3$, $0 \le y \le 1$, oriented downward.

soln: Using our formula for computing flux through a portion of a graph, $\int_S \vec{F} \cdot d\vec{A} = \int_R \vec{F}(x, y, f(x, y)) \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\, dA$ (and paying attention to the orientation) we have:

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= -\int_R \left(3x\vec{i} + \left(x^2\left(1 - xy^2\right) - \left(xy + x^2 - y^2x^3 + 8\right)\right)\vec{j} + 2\left(xy + x^2 - y^2x^3 + 8\right)\vec{k}\right) \cdot \left(-\left(y + 2x - 3y^2x^2\right)\vec{i} - \left(x - 2yx^3\right)\vec{j} + \vec{k}\right)\, dA \\
&= -\int_R -3x\left(y + 2x - 3y^2x^2\right) - \left(x^2 - x^3y^2 - xy - x^2 + y^2x^3 - 8\right)\left(x - 2yx^3\right) + 2xy + 2x^2 - 2y^2x^3 + 16\, dA \\
&= -\int_R -3xy - 6x^2 + 9y^2x^3 - (-xy - 8)\left(x - 2yx^3\right) + 2xy + 2x^2 - 2y^2x^3 + 16\, dA \\
&= -\int_R -xy - 4x^2 + 7y^2x^3 + (xy + 8)\left(x - 2yx^3\right) + 16\, dA \\
&= \int_R xy + 4x^2 - 7y^2x^3 - \left(x^2y - 2y^2x^4 + 8x - 16yx^3\right) - 16\, dA \\
&= \int_1^3 \int_0^1 xy + 4x^2 - 7y^2x^3 - x^2y + 2y^2x^4 - 8x + 16yx^3 - 16\, dy\, dx \\
&= \int_1^3 \left[\frac{1}{2}xy^2 + 4x^2y - \frac{7}{3}y^3x^3 - \frac{1}{2}x^2y^2 + \frac{2}{3}y^3x^4 - 8xy + 8y^2x^3 - 16y\right]_0^1 dx \\
&= \int_1^3 \frac{1}{2}x + 4x^2 - \frac{7}{3}x^3 - \frac{1}{2}x^2 + \frac{2}{3}x^4 - 8x + 8x^3 - 16\, dx \\
&= \int_1^3 \frac{2}{3}x^4 + \frac{17}{3}x^3 + 4x^2 - 8x - 16\, dx \\
&= \left[\frac{2}{15}x^5 + \frac{17}{12}x^4 + \frac{4}{3}x^3 - 4x^2 - 16x\right]_1^3 \\
&= \left(\frac{2}{15}(3^5) + \frac{17}{12}(3^4) + \frac{4}{3}(3^3) - 4(3^2) - 16(3)\right) - \left(\frac{2}{15}(1^5) + \frac{17}{12}(1^4) + \frac{4}{3}(1^3) - 4(1^2) - 16(1)\right) \\
&= \frac{2}{15}(243) + \frac{17}{12}(81) + \frac{4}{3}(27) - 4(9) - 16(3) - \frac{2}{15} - \frac{17}{12} - \frac{4}{3} + 4 + 16 \\
&= \frac{486}{15} + \frac{1377}{12} + 36 - 36 - 48 - \frac{2}{15} - \frac{17}{12} - \frac{4}{3} + 20 \\
&= \frac{1744}{15}
\end{aligned}$$

ex: Determine the flux of $\vec{G} = (y+1)\vec{i} - x\vec{j} + z\vec{k}$ through the surface S, which is the portion of $z = 11 - (x^2+y^2)^2$ oriented in the positive z-direction, which lies above the $z = 2$ plane.

soln: The surface S intersects the $z = 2$ plane in the circle $x^2 + y^2 = 3$, so the region R in the xy-plane is the disk $x^2 + y^2 \leq 3$. Using the formula for flux trough a graph:

$$
\begin{aligned}
\int_S \vec{G} \cdot d\vec{A} &= \int_R \left((y+1)\vec{i} - x\vec{j} + \left(11 - \left(x^2+y^2\right)^2\right)\vec{k}\right) \cdot \left(4x(x^2+y^2)\vec{i} + 4y(x^2+y^2)\vec{j} + \vec{k}\right)\, dA \\
&= \int_R (y+1)\left(4x^3 + 4xy^2\right) - x\left(4yx^2 + 4y^3\right) + \left(11 - \left(x^2+y^2\right)^2\right)\, dA \\
&= \int_R 4x^3y + 4xy^3 + 4x^3 + 4xy^2 - 4yx^3 - 4xy^3 + 11 - \left(x^2+y^2\right)^2\, dA \\
&= \int_R 4x^3 + 4xy^2 + 11 - \left(x^2+y^2\right)^2\, dA \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(4(r\cos\theta)^3 + 4(r\cos\theta)(r\sin\theta)^2 + 11 - \left(r^2\right)^2\right) r\, dr d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} 4r^4\cos^3\theta + 4r^4\cos\theta\sin^2\theta + 11r - r^5\, dr d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} 4r^4\cos\theta(1-\sin^2\theta) + 4r^4\cos\theta\sin^2\theta + 11r - r^5\, dr d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} 4r^4\cos\theta + 11r - r^5\, dr d\theta \\
&= \int_0^{2\pi} \left[\frac{4}{5}r^5\cos\theta + \frac{11}{2}r^2 - \frac{1}{6}r^6\right]_0^{\sqrt{3}} d\theta \\
&= \int_0^{2\pi} \left(\frac{4}{5}(\sqrt{3})^5\cos\theta + \frac{11}{2}(\sqrt{3})^2 - \frac{1}{6}(\sqrt{3})^6\right) d\theta \\
&= \int_0^{2\pi} \frac{36\sqrt{3}}{5}\cos\theta + \frac{33}{2} - \frac{9}{2}\, d\theta \\
&= \left[\frac{36\sqrt{3}}{5}\sin\theta + 12\theta\right]_0^{2\pi} \\
&= 24\pi
\end{aligned}
$$

ex: Compute the flux of $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ out of a spherical shell of radius 4, centered at the origin.

soln: Using the formula for flux through a sphere:

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_T (4\sin\theta\sin\phi\vec{i} + 4\cos\phi\vec{j} + 4\cos\theta\sin\phi\vec{k}) \cdot (\cos\theta\sin\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\phi\vec{k}) 4^2 \sin\phi \, d\theta d\phi \\
&= \int_T 16(4\sin\theta\sin\phi\cos\theta\sin\phi + 4\cos\phi\sin\theta\sin\phi + 4\cos\theta\sin\phi\cos\phi)\sin\phi \, d\theta d\phi \\
&= \int_T 64(\sin\theta\cos\theta\sin^3\phi + \sin\theta\cos\phi\sin^2\phi + \cos\theta\sin^2\phi\cos\phi) \, d\theta d\phi \\
&= 64\int_0^\pi \int_0^{2\pi} \sin\theta\cos\theta\sin^3\phi + \sin\theta\cos\phi\sin^2\phi + \cos\theta\sin^2\phi\cos\phi \, d\theta d\phi \\
&= 64\int_0^\pi \left[\frac{1}{2}\sin^2\theta\sin^3\phi - \cos\theta\cos\phi\sin^2\phi + \sin\theta\sin^2\phi\cos\phi\right]_0^{2\pi} d\phi
\end{aligned}$$

$$= 64\int_0^\pi \left(\frac{1}{2}(\sin(2\pi))^2\sin^3\phi - \cos(2\pi)\cos\phi\sin^2\phi + \sin(2\pi)\sin^2\phi\cos\phi\right) - \left(\frac{1}{2}(\sin(0))^2\sin^3\phi - \cos(0)\cos\phi\sin^2\phi + \sin(0)\sin^2\phi\cos\phi\right) d\phi$$

$$\begin{aligned}
&= 64\int_0^\pi \frac{1}{2}(0)^2\sin^3\phi - (1)\cos\phi\sin^2\phi + (0)\sin^2\phi\cos\phi - \frac{1}{2}(0)^2\sin^3\phi + (1)\cos\phi\sin^2\phi - (0)\sin^2\phi\cos\phi \, d\phi \\
&= 64\int_0^\pi 0 \, d\phi \\
&= 0
\end{aligned}$$

ex: Compute the flux of $\vec{F} = z\vec{i} + y\vec{j} - x\vec{k}$ out of a spherical shell of radius 6, centered at the origin.

soln: Using the formula for flux through a sphere:

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_T (6\cos\phi\vec{i} + 6\sin\theta\sin\phi\vec{j} - 6\cos\theta\sin\phi\vec{k}) \cdot (\cos\theta\sin\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\phi\vec{k})6^2 \sin\phi \, d\theta d\phi \\
&= \int_T 36(6\cos\phi\cos\theta\sin\phi + 6\sin\theta\sin\phi\sin\theta\sin\phi - 6\cos\theta\sin\phi\cos\phi)\sin\phi \, d\theta d\phi \\
&= \int_T 216\sin^2\theta\sin^3\phi \, d\theta d\phi \\
&= \int_0^\pi \int_0^{2\pi} 216\sin^2\theta\sin^3\phi \, d\theta d\phi \\
&= \int_0^\pi 216\Big[\left(\frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta\right)\sin^3\phi\Big]_0^{2\pi} d\phi \\
&= \int_0^\pi 216\left(\left(\frac{2\pi}{2} - \frac{1}{2}\sin(2\pi)\cos(2\pi)\right)\sin^3\phi - \left(\left(\frac{0}{2} - \frac{1}{2}\sin(0)\cos(0)\right)\sin^3\phi\right)\right) d\phi \\
&= \int_0^\pi 216\pi\sin^3\phi \, d\phi \\
&= \int_0^\pi 216\pi\sin\phi(1 - \cos^2\phi) \, d\phi \\
&= \int_0^\pi 216\pi\sin\phi - 216\pi\sin\phi\cos^2\phi) \, d\phi \\
&= \Big[-216\pi\cos\phi + 72\pi\cos^3\phi\Big]_0^\pi \\
&= \big(-216\pi\cos(\pi) + 72\pi(\cos(\pi))^3\big) - \big(-216\pi\cos(0) + 72\pi(\cos(0))^3\big) \\
&= -216\pi(-1) + 72\pi(-1) + 216\pi(1) - 72\pi(1) \\
&= 216\pi - 72\pi + 216\pi(1) - 72\pi \\
&= 288\pi
\end{aligned}$$