ex: Jill wants to construct a rectangular box. She will use a material costing \$3 per square foot for the bottom and the sides of the box, and a material costing \$2/ft^2 for the top. Determine the dimensions of the box which will maximize the volume, given that she has \$200 to spend.

Creatively calling the dimensions of the box w, l and h, we have $V = wlh$. In order to get this to become an equivalent two-variable function, we need to determine a relationship among the variables, via the cost. Writing out an expression for the cost of the box by summing the costs of each side, we have

$$\begin{aligned} C &= 3wl + 3wh + 3lh + 3wh + 3lh + 2wl \\ &= 5wl + 6wh + 6lh \end{aligned}$$

Since we know she will spend \$200 on the box, we have

$$200 = 5wl + 6wh + 6lh$$

Solving this for h (we could equivalently have solved it for w or l):

$$\begin{aligned} 200 &= 5wl + 6wh + 6lh \\ 200 - 5wl &= (6w + 6l)h \\ h &= \frac{200 - 5wl}{6w + 6l} \qquad (1) \end{aligned}$$

Substituting this expression into $V = wlh$, we get

$$\begin{aligned} V(w, l) &= wl\left(\frac{200 - 5wl}{6w + 6l}\right) \\ &= \frac{200wl - 5w^2l^2}{6w + 6l} \end{aligned}$$

which is now a two-variable function, which we can optimize.

Taking partial derivatives:

$$\begin{aligned} \frac{\partial V}{\partial w} &= \frac{(200l - 10wl^2)(6w + 6l) - (200wl - 5w^2l^2)(6)}{(6w + 6l)^2} \\ &= \frac{1200wl - 60w^2l^2 + 1200l^2 - 60wl^3 - 1200wl + 30w^2l^2}{(6w + 6l)^2} \\ &= \frac{1200l^2 - 30w^2l^2 - 60wl^3}{(6w + 6l)^2} \end{aligned}$$

Similarly,

$$\frac{\partial V}{\partial l} = \frac{1200w^2 - 30l^2w^2 - 60w^3l}{(6w + 6l)^2}$$

Since both w and l must be positive (why?), the only critical points are when $\text{grad}V = \vec{0}$. So, we need to solve the simultaneous system of equations $\frac{\partial V}{\partial w} = 0$ *and* $\frac{\partial V}{\partial l} = 0$.

$$\begin{aligned} \frac{1200l^2 - 30w^2l^2 - 60wl^3}{(6w+6l)^2} &= 0 \\ 1200l^2 - 30w^2l^2 - 60wl^3 &= 0 \\ 40l^2 - w^2l^2 - 2wl^3 &= 0 \\ l^2(40 - w^2 - 2wl) &= 0 \\ 40 - w^2 - 2wl &= 0 \qquad (2) \\ &\text{(since } l \neq 0) \end{aligned}$$

$$\begin{aligned} \frac{1200w^2 - 30l^2w^2 - 60w^3l}{(6w+6l)^2} &= 0 \\ 1200w^2 - 30l^2w^2 - 60w^3l &= 0 \\ 40w^2 - l^2w^2 - 2w^3l &= 0 \\ 40 - l^2 - 2wl &= 0 \\ 40 &= l^2 + 2wl \qquad (3) \end{aligned}$$

Substituting (3) into (2):

$$\begin{aligned} (l^2 + 2wl) - w^2 - 2wl &= 0 \\ l^2 - w^2 &= 0 \\ (l-w)(l+w) &= 0 \end{aligned}$$

So, $l = w$. (Why don't we need $l = -w$?) Substituting $l = w$ back into (3):

$$\begin{aligned} 40 &= l^2 + 2(l)l \\ l &= \sqrt{\frac{40}{3}} \end{aligned}$$

Finally, substituting $l = \sqrt{\frac{40}{3}}$ and $w = \sqrt{\frac{40}{3}}$ into (1), we get $h = \frac{5\sqrt{10}}{3\sqrt{3}}$ (or some other equivalent form), and we can answer the question posed:

The dimensions of the box which maximize the volume are approximately
3.65 ft $\times$ 3.65 ft $\times$ 3.04 ft