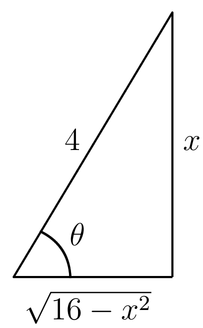
1. $\displaystyle\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$

Since the integrand contains $\sqrt{16-x^2}$, we use the substitution

$$x = 4\sin\theta, \qquad dx = 4\cos\theta\,d\theta.$$

Since we have $\sin\theta = \frac{x}{4}$, we can drawing and label the following triangle:



Then,

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{16-x^2}}\,dx &= \int \frac{1}{(4\sin\theta)^2\sqrt{16-(4\sin\theta)^2}}(4\cos\theta)\,d\theta \\
&= \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16(1-\sin^2\theta)}}\,d\theta \\
&= \int \frac{4\cos\theta}{16\sin^2\theta(4\sqrt{1-\sin^2\theta})}\,d\theta \\
&= \int \frac{4\cos\theta}{16\sin^2\theta(4\sqrt{\cos^2\theta})}\,d\theta \\
&= \int \frac{4\cos\theta}{16\sin^2\theta(4\cos\theta)}\,d\theta \\
&= \int \frac{1}{16\sin^2\theta}\,d\theta \\
&= \frac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta \\
&= \frac{1}{16}\int \csc^2\theta\,d\theta \\
&= -\frac{1}{16}\cot\theta + C
\end{aligned}$$

Referring to the triangle, we get $\cot\theta = \frac{\sqrt{16-x^2}}{x}$. Substituting this expression we get,

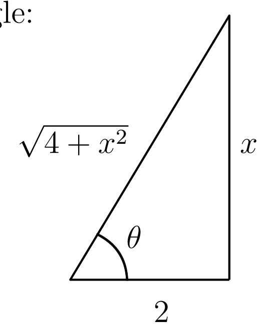
$$\begin{aligned}
\int \frac{1}{x^2\sqrt{16-x^2}}\,dx &= -\frac{1}{16}\cdot\frac{\sqrt{16-x^2}}{x} + C \\
&= -\frac{\sqrt{16-x^2}}{16x} + C
\end{aligned}$$

2. $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$

Here we use the substitution

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\,d\theta.$$

Drawing and labeling the triangle:



Consequently,

$$\begin{aligned}
\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{\sqrt{4+4\tan^2\theta}}(2\sec^2\theta)\,d\theta \\
&= \int \frac{2\sec^2\theta}{\sqrt{4\sec^2\theta}}\,d\theta \\
&= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta| + C
\end{aligned}$$

We know $\tan\theta = x/2$ (it's our substitution), and from the triangle and get

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Hence,

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

Using properties of logs, the right-hand side can be rewritten as $\ln\left|\sqrt{4+x^2}+x\right| - \ln 2 + C$. So,

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln\left(\sqrt{4+x^2}+x\right) + D,$$

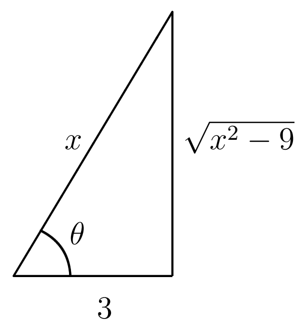
where $D = -\ln 2 + C$.

3. $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$

Here the substitution that we use is

$$x = 3\sec\theta, \qquad dx = 3\sec\theta\tan\theta\;d\theta.$$

The triangle is:



Consequently,

$$\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x}\,dx &= \int \frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta}\,(3\sec\theta\tan\theta)\;d\theta \\
&= \int \sqrt{9\tan^2\theta}\tan\theta\;d\theta \\
&= 3\int \tan^2\theta\;d\theta \\
&= 3\int \left(\sec^2\theta-1\right)\;d\theta \\
&= 3\int \sec^2\theta\;d\theta - 3\int d\theta \\
&= 3\tan\theta - 3\theta + C
\end{aligned}$$

The triangle then gives us $\tan\theta = \sqrt{x^2-9}/3$. From our original substitution $x = 3\sec\theta$, we get $\theta = \sec^{-1}(x/3)$. So,

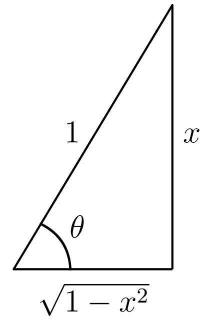
$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C$$

4. $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

The substitution we use here is

$$x = \sin\theta, \qquad dx = \cos\theta\;d\theta.$$

Drawing and labeling the triangle:



Then

$$\begin{aligned}\int \frac{(1-x^2)^{3/2}}{x^6}\,dx &= \int \frac{(1-\sin^2\theta)^{3/2}}{\sin^6\theta}\cos\theta\;d\theta \\ &= \int \frac{(\cos^2\theta)^{3/2}}{\sin^6\theta}\cos\theta\;d\theta \\ &= \int \frac{\cos^3\theta}{\sin^6\theta}\cos\theta\;d\theta \\ &= \int \frac{\cos^4\theta}{\sin^6\theta}\;d\theta\end{aligned}$$

As is often the case, we need to use a bit of creativity to rewrite this last integrand (using trig identities) in a form which is easier to integrate. One such way is:

$$\begin{aligned}\int \frac{\cos^4\theta}{\sin^6\theta}\;d\theta &= \int \frac{\cos^4\theta}{\sin^4\theta}\cdot\frac{1}{\sin^2\theta}\;d\theta \\ &= \int \cot^4\theta\csc^2\;d\theta\end{aligned}$$

This is now simply a u-substitution question. If we let $u=\cot\theta$, then $du=-\csc^2\theta d\theta$, and

$$\begin{aligned}\int \cot^4\theta\csc^2\;d\theta = -\int u^4\;du &= \frac{1}{5}u^5+C \\ &= -\frac{1}{5}\cot^5\theta+C\end{aligned}$$

Referring to the triangle, and returning to the variable x, we get

$$\begin{aligned}\int \frac{(1-x^2)^{3/2}}{x^6}\;dx &= -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5+C \\ &= -\frac{(1-x^2)^{5/2}}{5x^5}+C\end{aligned}$$

5. Find the area of the region bounded by the graph of $y = x^3(10 - x^2)^{-1/2}$, the x-axis, and the line $x = 1$.

Interpreting what is being asked, we see that we need to compute

$$\int_0^1 \frac{x^3}{\sqrt{10 - x^2}}\, dx$$

Here we need to use the substitution

$$x = \sqrt{10}\sin\theta, \qquad dx = \sqrt{10}\cos\theta\, d\theta.$$

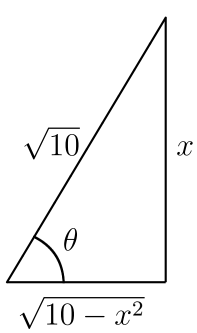
The corresponding triangle can be found on the next page.

Using the substitutions,

$$\begin{aligned}
\int \frac{x^3}{\sqrt{10 - x^2}}\, dx &= \int \frac{10\sqrt{10}\sin^3\theta}{\sqrt{10 - 10\sin^2\theta}}(\sqrt{10}\cos\theta\, d\theta) \\
&= 10\sqrt{10}\int \frac{\sqrt{10}\sin^3\theta\cos\theta}{\sqrt{10}\cos\theta}\, d\theta \\
&= 10\sqrt{10}\int \sin^3\theta\, d\theta \\
&= 10\sqrt{10}\int (1 - \cos^2\theta)\sin\theta\, d\theta \\
&= 10\sqrt{10}\left(\int \sin\theta\, d\theta - \int \cos^2\theta\sin\theta\, d\theta\right) \\
&= 10\sqrt{10}\left(-\cos\theta + \frac{1}{3}\cos^3\theta\right) + C \\
&= 10\sqrt{10}\left(-\frac{\sqrt{10 - x^2}}{\sqrt{10}} + \frac{1}{3}\left(\frac{\sqrt{10 - x^2}}{\sqrt{10}}\right)^3\right) + C \\
&= 10\sqrt{10}\left(-\frac{\sqrt{10 - x^2}}{\sqrt{10}} + \frac{1}{3}\left(\frac{\sqrt{10 - x^2}}{\sqrt{10}}\right)\left(\frac{10 - x^2}{10}\right)\right) + C \\
&= 10\sqrt{10}\frac{\sqrt{10 - x^2}}{\sqrt{10}}\left(-1 + \frac{1}{3}\cdot\frac{10 - x^2}{10}\right) + C \\
&= 10\sqrt{10 - x^2}\left(\frac{-30}{30} + \frac{10 - x^2}{30}\right) + C \\
&= -10\sqrt{10 - x^2}\left(\frac{20 + x^2}{30}\right) + C
\end{aligned}$$

Now that we have found the indefinite integral $\int \frac{x^3}{\sqrt{10-x^2}}\,dx$, we can easily evaluate the definite integral $\int_0^1 \frac{x^3}{\sqrt{10-x^2}}\,dx$:

$$\begin{aligned}
\int_0^1 \frac{x^3}{\sqrt{10-x^2}}\,dx &= \left[-10\sqrt{10-x^2}\left(\frac{20+x^2}{30}\right)\right]_0^1 \\
&= \left[-10\sqrt{10-1}\left(\frac{20+1}{30}\right)\right] - \left[-10\sqrt{10-0}\left(\frac{20+0}{30}\right)\right] \\
&= \left[-30\left(\frac{21}{30}\right)\right] + 10\sqrt{10}\left(\frac{2}{3}\right) \\
&= -21 + \frac{20\sqrt{10}}{3} \\
&\approx 0.0819
\end{aligned}$$



Here are some additional problems to practice:

1. $\displaystyle\int \frac{1}{x\sqrt{4-x^2}}\,dx$

2. $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

3. $\displaystyle\int \frac{1}{x\sqrt{9+x^2}}\,dx$

4. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+9}}\,dx$

5. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$

6. $\displaystyle\int \frac{1}{x^3\sqrt{x^2-25}}\,dx$

7. $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$

8. $\displaystyle\int \frac{x}{x^2+9}\,dx$

9. $\displaystyle\int \frac{1}{(x^2-1)^{3/2}}\,dx$

10. $\displaystyle\int \frac{1}{\sqrt{4x^2-25}}\,dx$

11. $\displaystyle\int \frac{1}{(36+x^2)^2}\,dx$

12. $\displaystyle\int \frac{1}{(16-x^2)^{5/2}}\,dx$

13. $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$

14. $\displaystyle\int \frac{1}{49+x^2}\,dx$

15. $\displaystyle\int \frac{x}{(16-x^2)^2}\,dx$

16. $\displaystyle\int x\sqrt{x^2-9}\,dx$

17. $\displaystyle\int \frac{x^3}{\sqrt{9x^2+49}}\,dx$

18. $\displaystyle\int \frac{1}{x\sqrt{25x^2+16}}\,dx$

19. $\displaystyle\int \frac{1}{x^4\sqrt{x^2-3}}\,dx$

20. $\displaystyle\int \frac{x^2}{(1-9x^2)^{3/2}}\,dx$

21. $\displaystyle\int \frac{(4+x^2)^2}{x^3}\,dx$

22. $\displaystyle\int \frac{3x-5}{\sqrt{1-x^2}}\,dx$

Some answers to the above problems.

1. $\frac{1}{2}\ln\left|\frac{2}{x}-\frac{\sqrt{4-x^2}}{x}\right|+C$ or equivalently, $\frac{1}{2}\ln\left|2-\sqrt{4-x^2}\right|-\frac{1}{2}\ln|x|+C$

3. $\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}}{x}-\frac{3}{x}\right|+C$ or equivalently, $\frac{1}{3}\ln\left|\sqrt{x^2+9}-3\right|-\frac{1}{3}\ln|x|+C$

5. $\frac{\sqrt{x^2-25}}{25x}+C$

7. $-\sqrt{4-x^2}+C$

9. $-\frac{x}{\sqrt{x^2-1}}+C$

11. $\frac{1}{432}\left(\tan^{-1}\left(\frac{x}{6}\right)+\frac{6x}{x^2+36}\right)+C$

13. $\sin^{-1}\left(\frac{x}{3}\right)+C$

15. $\frac{1}{2(16-x^2)}+C$

17. $\frac{1}{243}(9x^2+49)^{3/2}-\frac{49}{81}\sqrt{9x^2+49}+C$

19. $\frac{(3+2x^2)\sqrt{x^2-3}}{27x^3}+C$

21. $-\frac{8}{x^2}+8\ln|x|+\frac{1}{2}x^2+C$