1. $\displaystyle\int \frac{4x^3-3x^2+6x-27}{x^4+9x^2}\,dx$

Since the denominator factors as $x^2(x^2+9)$, we get that the partial fraction decomposition has the form

$$\frac{4x^3-3x^2+6x-27}{x^4+9x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+9}.$$

Multiplying by the lowest common denominator (x^4+9x^2), we get

$$4x^3-3x^2+6x-27=Ax(x^2+9)+B(x^2+9)+(Cx+D)x^2. \qquad (*)$$

Letting $x=0$, it is clear that $-27=9B$, so $B=-3$. Now (*) becomes

$$4x^3-3x^2+6x-27=Ax(x^2+9)-3(x^2+9)+(Cx+D)x^2,$$

which we can now multiply out, and equate coefficients.

$$\begin{aligned}4x^3-3x^2+6x-27&=Ax^3+9Ax-3x^2-27+Cx^3+Dx^2\\&=(A+C)x^3+(D-3)x^2+9Ax-27\end{aligned}$$

So, $A+C=4,\quad D-3=-3,\quad 9A=6$. We then get $D=0, A=2/3$, and $C=4-A=4-2/3=10/3$. The partial fraction decomposition is, therefore,

$$\frac{4x^3-3x^2+6x-27}{x^4+9x^2}=\frac{2}{3x}-\frac{3}{x^2}+\frac{10x}{3(x^2+9)}.$$

Integrating, we get

$$\begin{aligned}\int \frac{4x^3-3x^2+6x-27}{x^4+9x^2}\,dx&=\int\frac{2}{3x}\,dx-\int\frac{3}{x^2}\,dx+\int\frac{10x}{3(x^2+9)}\,dx\\&=\frac{2}{3}\int\frac{1}{x}\,dx-\frac{3}{2}\int x^{-2}\,dx+\frac{5}{3}\int\frac{2x}{x^2+9}\,dx\\&=\frac{2}{3}\ln|x|+\frac{3}{x}+\frac{5}{3}\ln(x^2+9)+C\\&=\frac{1}{3}\ln(x^2)+\frac{3}{x}+\frac{1}{3}\ln\left((x^2+9)^5\right)+C\\&=\frac{1}{3}\ln\left(x^2(x^2+9)^5\right)+\frac{3}{x}+C\end{aligned}$$

2. $\displaystyle\int \frac{x^3+6x^2+3x+16}{x^3+4x}\,dx$

The first thing that we must notice is that the degree of the numerator is not smaller than the degree of the denominator, so we must perform polynomial long division before we can do a partial fraction decomposition. Performing the long division, we get

$$\frac{x^3+6x^2+3x+16}{x^3+4x} = 1 + \frac{6x^2-x+16}{x^3+4x}.$$

Factoring the denominator, we see that the partial fraction decomposition has the form

$$\frac{6x^2-x+16}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

So,

$$6x^2-x+16 = A(x^2+4) + (Bx+C)x.$$

Letting $x=0$: $16 = 4A$, so $A=4$. Substituting this in, and multiplying out the right-hand side, we get $6x^2-x+16 = 4x^2+16+Bx^2+Cx$. Thus, $C=-1$ and $B=2$. The integral then becomes

$$\begin{aligned}
\int \frac{x^3+6x^2+3x+16}{x^3+4x}\,dx &= \int dx + \int \frac{4}{x}\,dx + \int \frac{2x-1}{x^2+4}\,dx \\
&= \int dx + \int \frac{4}{x}\,dx + \int \frac{2x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx \\
&= x + 4\ln|x| + \ln\left(x^2+4\right) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \\
&= x + \ln\left(x^4(x^2+4)\right) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C
\end{aligned}$$

3. $\displaystyle\int \frac{5x^2+11x+17}{x^3+5x^2+4x+20}\,dx$

In order to factor the denominator, we use 'factor by grouping'.

$$x^3+5x^2+4x+20 = x^2(x+5) + 4(x+5) = (x+5)(x^2+4)$$

So, the partial fraction decomposition has the form

$$\frac{5x^2+11x+17}{x^3+5x^2+4x+20} = \frac{A}{x+5} + \frac{Bx+C}{x^2+4}$$

So,

$$5x^2+11x+17 = A(x^2+4) + (Bx+C)(x+5)$$

Multiplying out grouping like terms:

$$5x^2 + 11x + 17 = (A + B)x^2 + (C + 5B)x + (5C + 4A)$$

This give us the following system of equations:

$$A + B = 5 \quad (1)$$
$$C + 5B = 11 \quad (2)$$
$$5C + 4A = 17 \quad (3)$$

Solving this system gives us $A = 3$, $B = 2$, and $C = 1$. So,

$$\begin{aligned}\int \frac{5x^2 + 11x + 17}{x^3 + 5x^2 + 4x + 20}\,dx &= \int \frac{3}{x+5}\,dx + \int \frac{2x+1}{x^2+4}\,dx \\ &= 3\ln|x+5| + \int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx \\ &= 3\ln|x+5| + \ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\end{aligned}$$

4. $\displaystyle\int \frac{x^5}{(x^2+4)^2}\,dx$

Again, the first thing that we need to notice is that the degree of the numerator is not less than the degree of the denominator. After long division we get

$$\frac{x^5}{(x^2+4)^2} = x - \frac{8x^3 + 16x}{(x^2+4)^2}.$$

We now find the partial fraction decomposition, which will have the form

$$\frac{8x^3 + 16x}{(x^2+4)^2} = \frac{Ax + B}{x^2+4} + \frac{Cx + D}{(x^2+4)^2}.$$

Multiplying by $(x^2+4)^2$, and grouping like terms:

$$8x^3 + 16x = Ax^3 + Bx^2 + (4A + C)x + 4B + D,$$

which gives us the system of equations:

$$A = 8, \qquad B = 0, \qquad 4A + C = 16, \qquad 4B + D = 0$$

Solving this system gives us $A = 8$, $B = 0$, $C = -16$, and $D = 0$. So,

$$\frac{8x^3 + 16x}{(x^2+4)^2} = \frac{8x}{x^2+4} + \frac{-16x}{(x^2+4)^2},$$

and

$$\frac{x^5}{(x^2+4)^2} = x - \frac{8x}{x^2+4} + \frac{16x}{(x^2+4)^2}$$

$$\begin{aligned}
\int \frac{x^5}{(x^2+4)^2}\,dx &= \int x\,dx - \int \frac{8x}{x^2+4}\,dx + \int \frac{16x}{(x^2+4)^2}\,dx \\
&= \frac{1}{2}x^2 - \int \frac{4}{u}\,du + \int \frac{8}{u^2}\,du \\
&= \frac{1}{2}x^2 - 4\ln|u| - \frac{8}{u} + C \\
&= \frac{1}{2}x^2 - 4\ln(x^2+4) - \frac{8}{x^2+4} + C
\end{aligned}$$

Where the substitution $u = x^2 + 4$, $du = 2x\,dx$ was used for both the integrals.

Alternate Solution:

Another (simpler) approach to this problem is to note that it can also be done simply via u-substitution. If we use the substitution $u = x^2 + 4$, $du = 2x\,dx$, and note that $x^2 = u - 4$, then we get:

$$\begin{aligned}
\int \frac{x^5}{(x^2+4)^2}\,dx &= \int \frac{(x^2)^2 x}{(x^2+4)^2}\,dx \\
&= \frac{1}{2}\int \frac{(u-4)^2}{u^2}\,du \\
&= \frac{1}{2}\int \frac{u^2 - 8u + 16}{u^2}\,du \\
&= \frac{1}{2}\int \left(1 - \frac{8}{u} + \frac{16}{u^2}\right) du \\
&= \frac{1}{2}\left(u - 8\ln|u| - 16\frac{1}{u}\right) + C \\
&= \frac{1}{2}\left(x^2 + 4 - 8\ln(x^2+4) - \frac{16}{x^2+4}\right) + C \\
&= \frac{1}{2}x^2 + 2 - 4\ln(x^2+4) - \frac{8}{x^2+4} + C \\
&= \frac{1}{2}x^2 - 4\ln(x^2+4) - \frac{8}{x^2+4} + D
\end{aligned}$$

The partial fraction decomposition that ate 129

$$\int \frac{2x^2+3x-7}{(x-3)^2(x^2+x+1)}\,dx$$

The partial fraction decomposition has the form

$$\frac{2x^2+3x-7}{(x-3)^2(x^2+x+1)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+x+1}.$$

Multiplying by the lowest common denominator $(x-3)^2(x^2+x+1)$,

$$2x^2+3x-7 = A(x-3)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x-3)^2.$$

Letting $x=3$: $18+9-7 = B(9+3+1)$, so $B = 20/13$. Substituting this in, $2x^2+3x-7 = A(x-3)(x^2+x+1) + (20/13)(x^2+x+1) + (Cx+D)(x-3)^2$. Multiplying out the right-hand side,

$$\begin{aligned}
2x^2+3x-7 =& A(x^3+x^2+x-3x^2-3x-3) + (20/13)x^2 \\
&+ (20/13)x + (20/13) + (Cx+D)(x^2-6x+9) \\
=& Ax^3 + Ax^2 + Ax - 3Ax^2 - 3Ax - 3A + (20/13)x^2 + (20/13)x \\
&+ (20/13) + Cx^3 - 6Cx^2 + 9Cx + Dx^2 - 6Dx + 9D \\
=& (A+C)x^3 + (20/13 - 2A - 6C + D)x^2 + (20/13 - 2A + 9C - 6D)x \\
&+ (20/13 + 9D - 3A)
\end{aligned}$$

So, by equating coefficients,

$$A + C = 0 \tag{1}$$
$$20/13 - 2A - 6C + D = 2 \tag{2}$$
$$20/13 - 2A + 9C - 6D = 3 \tag{3}$$
$$20/13 + 9D - 3A = -7 \tag{4}$$

Clearly, by equation (1), $-A = C$. We can use (4) to solve for D in terms of A.

$$\begin{aligned}
9D &= 3A - 7 - \frac{20}{13} \\
9D &= 3A - \frac{111}{13} \\
3D &= A - \frac{37}{13}
\end{aligned} \tag{5}$$

Solving for $3D$ will work just fine for us, as we can easily substitute this expression into (3). Doing this, and substituting $C = -A$, we get:

$$\begin{aligned}
\frac{20}{13} - 2A + 9(-A) - 2(A - \frac{37}{13}) &= 3 \\
-2A - 9A - 2A + \frac{74}{13} &= 3 - \frac{20}{13} \\
-13A &= \frac{19}{13} - \frac{74}{13} \\
A &= \frac{55}{169}
\end{aligned}$$

Thus, $C = -55/169$, and using (5):

$$\begin{aligned}
3D &= \frac{55}{169} - \frac{37}{13} \\
D &= -\frac{142}{169}
\end{aligned}$$

Now we can (finally) integrate:

$$\begin{aligned}
\int \frac{2x^2 + 3x - 7}{(x-3)^2(x^2+x+1)}\,dx &= \int \frac{55}{169(x-3)}\,dx + \int \frac{20}{13(x-3)^2}\,dx - \int \frac{55x + 142}{169(x^2+x+1)}\,dx \\
&= \frac{55}{169}\int \frac{dx}{x-3} + \frac{20}{13}\int \frac{dx}{(x-3)^2} - \frac{1}{169}\int \frac{55x}{x^2+x+1}\,dx \\
&\quad - \frac{1}{169}\int \frac{142}{x^2+x+1}\,dx \\
&= \frac{55}{169}\ln|x-3| - \left(\frac{20}{13}\right)\left(\frac{1}{x-3}\right) \\
&\quad - \frac{1}{169}\int \frac{(55/2)du}{u} \\
&\quad - \frac{142}{169}\int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}}\,dx \\
&= \frac{55}{169}\ln|x-3| - \left(\frac{20}{13(x-3)}\right) - \frac{1}{169}\ln|u| \\
&\quad - \left(\frac{142}{169}\right)\left(\frac{1}{\frac{\sqrt{3}}{2}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) \\
&= \frac{55}{169}\ln|x-3| - \frac{20}{13(x-3)} - \frac{1}{169}\ln(x^2+x+1) \\
&\quad - \left(\frac{284}{169\sqrt{3}}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \\
&= \frac{1}{169}\ln\left|\frac{(x-3)^{55}}{x^2+x+1}\right| - \frac{20}{13(x-3)} - \left(\frac{284}{169\sqrt{3}}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)
\end{aligned}$$

Here are some additional problems to practice:

1. $\int \frac{5x-12}{x(x-4)}\,dx$

2. $\int \frac{x+34}{(x-6)(x+2)}\,dx$

3. $\int \frac{37-11x}{(x+1)(x-2)(x-3)}\,dx$

4. $\int \frac{4x^2+54x+134}{(x-1)(x+5)(x+3)}\,dx$

5. $\int \frac{6x-11}{(x-1)^2}\,dx$

6. $\int \frac{-19x^2+50x-25}{x^2(3x-5)}\,dx$

7. $\int \frac{x+16}{x^2+2x-8}\,dx$

8. $\int \frac{11t+2}{2t^2-5t-3}\,dt$

9. $\int \frac{5x^2-10x-8}{x^3-4x}\,dx$

10. $\int \frac{4x^2-5x-15}{x^3-4x^2-5x}\,dx$

11. $\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx$

12. $\int \frac{2x^2-12x+4}{x^3-4x^2}\,dx$

13. $\int \frac{9x^4+17x^3+3x^2-8x+3}{x^5+3x^4}\,dx$

14. $\int \frac{5x^2+30x+43}{(x+3)^3}\,dx$

15. $\int \frac{2x^2+7x}{x^2+6x+9}\,dx$

16. $\int \frac{x^2+3x+1}{x^4+5x^2+4}\,dx$

17. $\int \frac{4x}{(x^2+1)^3}\,dx$

18. $\int \frac{2x^3+10x}{(x^2+1)^2}\,dx$

19. $\int \frac{x^4+2x^2+4x+1}{(x^2+1)^3}\,dx$

20. $\int \frac{x^3+3x-2}{x^2-x}\,dx$

21. $\int \frac{x^4+2x^2+3}{x^3-4x}\,dx$

22. $\int \frac{x^6-x^3+1}{x^4+9x^2}\,dx$

23. $\int \frac{2x^3-5x^2+46x+98}{(x^2+x-12)^2}\,dx$

24. $\int \frac{-2x^4-3x^3-3x^2+3x+1}{x^2(x+1)^3}\,dx$

Some answers to the above problems.

1. $3\ln|x| + 2\ln|x-4| + C$

3. $4\ln|x+1| - 5\ln|x-2| + \ln|x-3| + C$

5. $6\ln|x-1| + \dfrac{5}{x-1} + C$

7. $3\ln|x-2| - 2\ln|x+4| + C$

9. $2\ln|x| - \ln|x-2| + 4\ln|x+2| + C$

11. $5\ln|x+1|\dfrac{1}{x+1} - 3\ln|x-5| + C$

13. $5\ln|x| - \dfrac{2}{x} + \dfrac{3}{2x^2} - \dfrac{1}{3x^3} + 4\ln|x+3| + C$

16. $-\dfrac{1}{2}\ln(x^2+4) + \dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + \dfrac{1}{2}\ln(x^2+1) + C$

18. $\ln(x^2+1) - \dfrac{4}{x^2+1} + C$

20. $\dfrac{1}{2}x^2 + x + 2\ln|x| + 2\ln|x-1| + C$

22. $\dfrac{1}{3}x^3 - 9x - \dfrac{1}{9x} - \dfrac{1}{2}\ln(x^2+9) + \dfrac{728}{27}\tan^{-1}\left(\dfrac{x}{3}\right) + C$

23. $2\ln|x+4| + \dfrac{6}{x+4} - \dfrac{5}{x-3} + C$