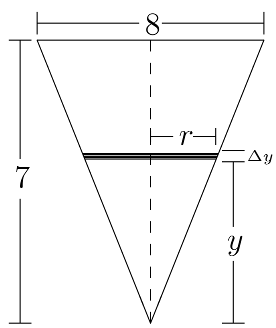
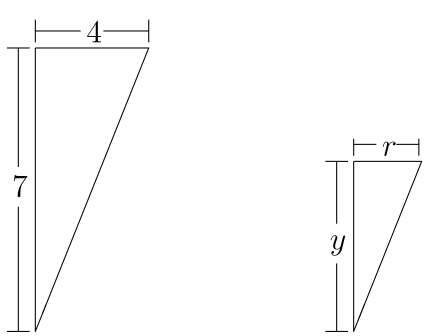
The goal of this handout is to emphasize that all of the things that we did in chapter 8 are really the same. Each type of problem is just about "slicing" and summing some quantity (volume, mass, work) attributed to each slice. There is a lot of redundancy throughout this handout. This is done for two reasons: 1) solutions were written to be self-contained and independent of each other, and 2) to illustrate that the problems are the same, by showing the same work in each solution.

1. A water tank is in the shape of an inverted (i.e. upside-down) right circular cone. The tank is 7 feet tall, and is 8 feet across the top. Use a definite integral to find the volume of the tank.

   We will find the volume of the tank by summing the volumes of arbitrarily small slices of the tank. So, we start by finding the volume of a slice. Because of the shape of the tank, it is far better to take horizontal slices. To help visualize the problem, we imagine cutting the tank in half (vertically, since we are summing horizontal slices) and looking at a 2-dimensional representation. Before we can actually determine the volume of a slice, we need to decide how to label the picture. In order to fully label the picture, we need to make a choice of variables, and a choice of where we will measure vertical distance from. For this problem, we choose to call the vertical distance from the bottom of the tank y (this choice is not unique, as will be illustrated later). We also need to choose a variable for the horizontal width of the slice. By thinking ahead, and realizing that each slice will be (approximately) a cylinder, we actually choose to label *half* the width as r. We can now draw and label the picture as follows:



   Since each slice is approximately a cylinder, we need to use the volume formula for a cylinder: $v = \pi r^2 h$. For our slice, pictured above, the height is Δy, and we must solve for r in terms of y. In order to do this, we cut the triangular region in half (along the dashed line) and use the fact that we now have similar right triangles:



$$\frac{r}{y} = \frac{4}{7}$$

So, $r = \frac{4}{7}\, y$

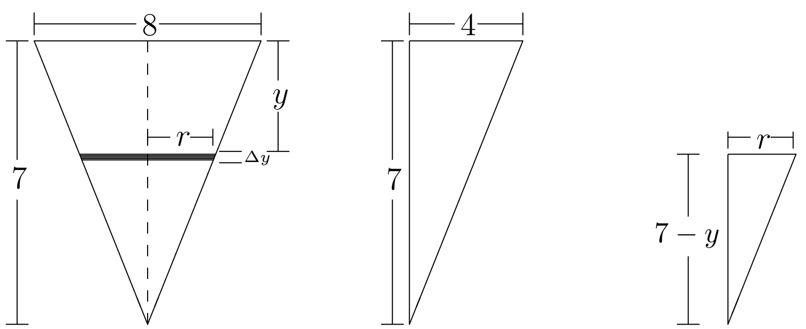
So, the volume of a slice is approximately $\pi\left(\frac{4}{7}y\right)^2 \Delta y$. Since we are taking horizontal slices, whose volume can be expressed in terms of their vertical distance from the bottom of the cone, we will integrate "in the vertical direction", and the limits of integration are 0 and 7 (the lowest a slice can be is 0 feet from the bottom of the cone, and the highest a slice can be is 7 feet above the bottom of the cone). So, the volume of the tank is given by $\int_0^7 \pi\left(\frac{4}{7}y\right)^2 dy$. Performing the integration:

$$\begin{aligned}
\int_0^7 \pi\left(\frac{4}{7}y\right)^2 dy &= \frac{16\pi}{49}\int_0^7 y^2\, dy \\
&= \frac{16\pi}{49}\cdot\frac{1}{3}y^3\bigg|_0^7 \\
&= \frac{16\pi}{49}\cdot\frac{1}{3}(7^3-0) \\
&= \frac{16\cdot 7\pi}{3} \\
&= \frac{112\pi}{3}
\end{aligned}$$

So, the volume of the tank is $\frac{112\pi}{3}$ ft^3, which we can check is correct via the usual volume formula for a right circular cone, $V = \frac{1}{3}\pi r^2 h$.

Alternate solution:

As was mentioned above, there is nothing special or unique about the choice to measure y as distance from the bottom of the cone. We could just as easily have called y distance from the *top* of the cone. The pictures for this alternate choice are as follows:



$$\frac{r}{7-y} = \frac{4}{7}$$

So, $r = \dfrac{4(7-y)}{7}$

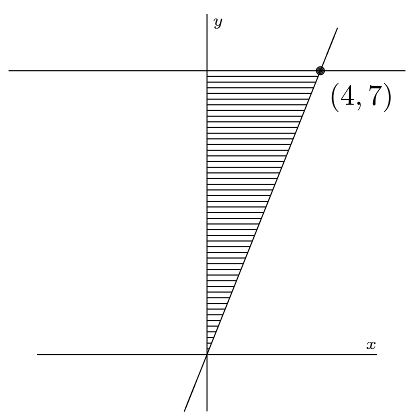
In this setup the volume of a slice is approximately $\pi\left(\frac{4(7-y)}{7}\right)^2 \Delta y$. Here y is the distance from the top of the cone, with down being the positive direction, so the limits of integration are still 0 and 7 (a slice must be between 0 feet and 7 feet from the top of the cone). So, the volume of the tank is given by $\int_0^7 \pi\left(\frac{4(7-y)}{7}\right)^2 dy$. Performing the integration:

$$\int_0^7 \pi \left(\frac{4(7-y)}{7}\right)^2 dy = \frac{16\pi}{49}\int_0^7 (7-y)^2\, dy$$
$$= \frac{16\pi}{49}\int_0^7 (49 - 14y + y^2)\, dy$$
$$= \frac{16\pi}{49} \cdot \left[49y - 7y^2 + \frac{1}{3}y^3\right]_0^7$$
$$= \frac{16\pi}{49} \cdot \left(49 \cdot 7 - 7 \cdot 7^2 + \frac{1}{3} \cdot 7^3 - 0\right)$$
$$= \frac{16\pi}{49} \cdot \left(\frac{1}{3} \cdot 7^3\right)$$
$$= \frac{16 \cdot 7\pi}{3}$$
$$= \frac{112\pi}{3}$$

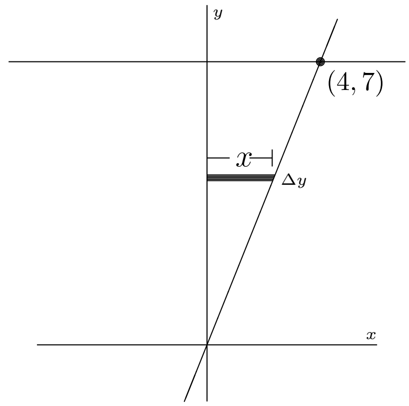
Which, not surprisingly, is exactly what we got before. So, it doesn't matter if choose to measure y as distance from the top of the cone, or from the bottom of the cone. In fact, it doesn't matter where we choose to measure y from, we could choose to measure y as distance from 12 feet above the top of the cone, if we were so inclined.

2. Find the volume of the solid created by revolving the area bounded by the y-axis, $y = \frac{7}{4}x$, and $y = 7$ around the y-axis.

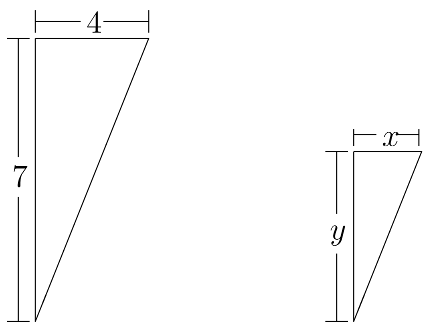
We want to start by drawing the region in question, so that we can better visualize what we must do.



Just as we did in the first problem, we will find the volume by summing the volumes of arbitrarily small slices of the region. Since we want to rotate the region about a vertical line, (the y-axis being the line $x = 0$), we will want to take horizontal slices. In order to visualize this better, the region is redrawn below (shading omitted for clarity) with an arbitrary slice drawn in.



The solid created by rotating this slice around the y-axis will be approximately a cylinder. Thus we will want to use the volume formula for a cylinder, $\pi r^2 h$, where the radius of our cylinder is x, and the height is Δy. So the volume of one slice approximately $\pi x^2 \Delta y$. Unfortunately this doesn't do us much good as we need the entire expression in terms of one variable (in this case, y). Thus, we need to solve for x in terms of y. We accomplish this by noting that we have similar triangles, and solving for x:



$$\frac{x}{y} = \frac{4}{7}$$

$$\text{So, } x = \frac{4}{7}\, y$$

So, the volume of a slice is approximately $\pi \left(\frac{4}{7}y\right)^2 \Delta y$. Representing the entire volume as an integral, we have: Volume $= \int_0^7 \pi \left(\frac{4}{7}y\right)^2 dy$; The limits of integration coming from the fact that, with respect to y, the region is bounded between $y = 0$ and $y = 7$. So,

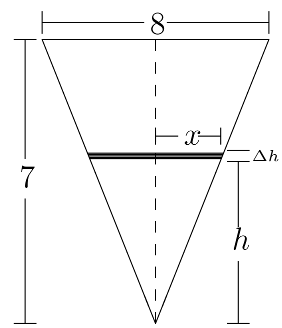
$$\begin{aligned}
\text{Volume of the solid} &= \int_0^7 \pi \left(\frac{4}{7}y\right)^2 \, dy \\
&= \frac{16\pi}{49} \int_0^7 y^2 \, dy \\
&= \frac{16\pi}{49} \cdot \frac{1}{3} y^3 \Big|_0^7 \\
&= \frac{16\pi}{49} \cdot \frac{1}{3}(7^3 - 0) \\
&= \frac{16 \cdot 7\pi}{3} \\
&= \frac{112\pi}{3}
\end{aligned}$$

Which, of course, makes sense, since the solid created is exactly the cone of problem 1.

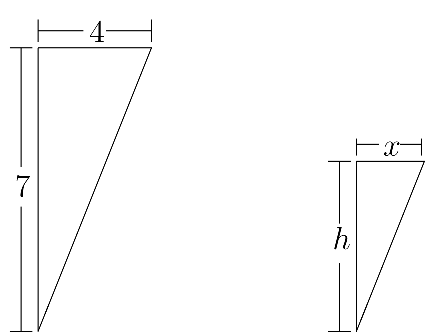
3. A water tank is in the shape of an inverted (i.e. upside-down) right circular cone. The tank is 7 feet tall, and is 8 feet across the top. Use a definite integral to find the weight of the water in a full tank. Recall that water weighs approximately 62.4 lb/ft^3.

In the first problem we were asked to find the total volume of the cone, so we started by finding the volume of an arbitrary slice. In this problem we are asked for the weight of the water in the tank, so we will start by finding the weight of an arbitrary slice.

Since we are told that water weighs approximately 62.4 lb/ft^3, in order to find the weight of a slice, we simply need to find the volume of a slice (in ft^3) and then multiply it by 62.4. We find the volume of the tank by summing the volumes of arbitrarily small slices of the tank. So, we start by finding the volume of a slice. Because of the shape of the tank, we will want to take horizontal slices. To help visualize the problem, we imagine cutting the tank in half (vertically, since we are summing horizontal slices) and looking at a 2-dimensional representation. Before we can actually determine the volume of a slice, we need to decide how to label the picture. In order to fully label the picture, we need to make a choice of variables, and a choice of where we will measure vertical distance from. For this problem, we choose to call the vertical distance from the bottom of the tank h. We also need to choose a variable for the horizontal width of the slice. Realizing that each slice will be (approximately) a cylinder, we actually choose to label *half* the width as x. We can now draw and label the picture as follows:



Since each slice is approximately a cylinder, we will want to use the volume formula for a cylinder: $v = \pi r^2 h$. For our slice, pictured above, the height is Δh, and the radius is x. In order to get something useful, we must solve for x in terms of h. In order to do this, we cut the triangular region in half and use the fact that we now have similar right triangles:



$$\frac{x}{h} = \frac{4}{7}$$

So, $x = \dfrac{4}{7}\, h$

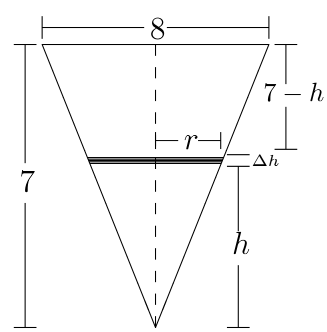
So, the volume of a slice is $\pi \left(\frac{4}{7}\, h\right)^2 \Delta h$. We want the approximate *weight* of a slice, which is simply $62.4 \cdot \pi(\frac{4}{7}h)^2 \Delta h$. The limits of integration are 0 and 7, so the weight of a full tank of water is:

$$\begin{aligned}\int_0^7 62.4\cdot\pi\left(\frac{4}{7}h\right)^2\,dh &= 62.4\cdot\frac{16\pi}{49}\int_0^7 h^2\,dh\\ &= 62.4\cdot\frac{16\pi}{49}\cdot\frac{1}{3}h^3\Big|_0^7\\ &= 62.4\cdot\frac{16\pi}{49}\cdot\frac{1}{3}(7^3-0)\\ &= 62.4\cdot\frac{16\cdot 7\pi}{3}\\ &= \frac{11648\pi}{5}\\ &\approx 7318.7\text{ lb}\end{aligned}$$

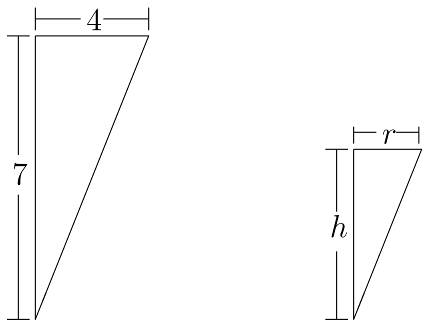
4. A water tank is in the shape of an inverted (i.e. upside-down) right circular cone. The tank is 7 feet tall, and is 8 feet across the top. How much work is required to pump all of the water out of a full tank?

   In this problem, we are asked to calculate work, so we start by looking at a "slice" and seeing how much work it requires to move it. Since work is force (weight) times distance, we find the weight of a slice, and how far we must move it. Water weighs approximately 62.4 lb/ft^3, so to find the weight, we really just need to find the volume of a slice.

   [This follows exactly as in problem 1]: Because of the shape of the tank, it is better to take horizontal slices. The help visualize the problem, we imagine cutting the tank in half (vertically, since we are summing horizontal slices) and looking at a 2-dimensional representation. Before we can actually determine the volume of a slice, we need to decide how to label the picture. In order to fully label the picture, we need to make a choice of variables, and a choice of where we will measure vertical distance from. For this problem, we choose to call the vertical distance from the bottom of the tank h. We also need to choose a variable for the horizontal width of the slice. By thinking ahead, and realizing that each slice will be (approximately) a cylinder, we actually choose to label *half* the width as r. We can now draw and label the picture as follows:



Since each slice is approximately a cylinder, we need to use the volume formula for a cylinder. For our slice, pictured above, the height is Δh, and we must solve for r in terms of h. In order to do this, we cut the triangular region in half and use the fact that we now have similar right triangles:



$$\frac{r}{h} = \frac{4}{7}$$

So, $r = \dfrac{4}{7}\, h$

So, the volume of a slice is approximately $\pi \left(\frac{4}{7}h\right)^2 \Delta h$. Thus the weight of a slice is $62.4\pi \left(\frac{4}{7}h\right)^2 \Delta h$. We want to know how much work it takes to move this slice out of the tank, so in addition to the force that it takes to move it, we also need to know how far we need to move it. If we glance back at the first picture (or just think about it), we realize that we need to move the slice $7 - h$ feet (the tank is 7 feet tall, and our arbitrary slice sits at a height of h feet). Again, work = force $\cdot$ distance, so the work required to move an arbitrary slice at height h out of the tank is $62.4\pi \left(\frac{4}{7}h\right)^2 \Delta h(7 - h)$. In order to find the total amount of work required to empty the tank, we will integrate "in the vertical direction", and since the tank is 7 feet tall, and we are measuring y as distance from the bottom of the tank, the limits of integration are 0 and 7. So, the work required to empty the tank is:

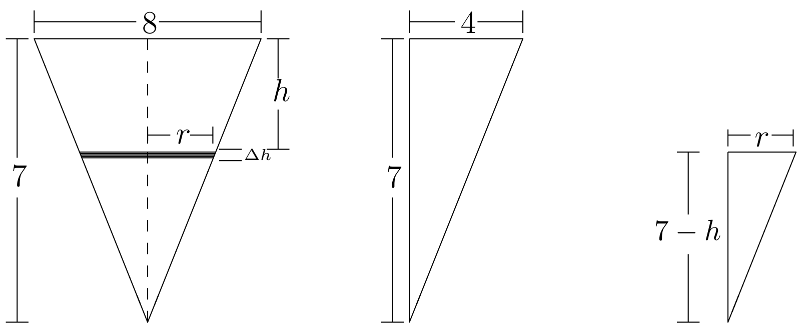
$$\begin{aligned}
W &= \int_0^7 62.4\pi \left(\frac{4}{7}h\right)^2 (7 - h)\, dh \\
&= 62.4\pi \frac{16}{49} \int_0^7 h^2(7 - h)\, dh \\
&= 62.4\pi \frac{16}{49} \int_0^7 (7h^2 - h^3)\, dh \\
&= 62.4\pi \frac{16}{49} \left[\frac{7}{3}h^3 - \frac{1}{4}h^4\right]_0^7 \\
&= 62.4\pi \frac{16}{49} \left(\frac{7}{3}7^3 - \frac{1}{4}7^4 - 0\right) \\
&= 62.4\pi \frac{16}{49} \cdot 7^4 \left(\frac{1}{3} - \frac{1}{4}\right) \\
&= 62.4\pi \cdot 16 \cdot 7^2 \cdot \frac{1}{12} \\
&= 4076.8\pi \\
&\approx 12808 \text{ ft-lb}
\end{aligned}$$

The following are examples which do not use a constant density. For the sake of brevity, their solutions may involve references to earlier solutions.

5. A water tank is in the shape of an inverted (i.e. upside-down) right circular cone. The tank is 7 feet tall, and is 8 feet across the top. The lid was accidently left off the top of tank and wind blew dirt into the tank causing the water to become muddied. As luck would have it, the dirt settled in such a way that the density of the water can be easily modeled. The density of the muddied water at a depth of h feet below the surface is given by $\delta(h) = 65 + h$ lb/ft^3. Use a definite integral to find the weight of the water in a full tank.

Again, since we are asked for the weight of the water in the tank, we will start by finding the weight of an arbitrary slice. Since we are given the density of the water (as a function of h) in lb/ft^3, we need to find the volume of a slice (as a function of h) in ft^3 and then multiply it by $\delta(h)$. Clearly this will give us the weight of the water (as a function of h) in lb.

Unlike earlier problems, we do not need to choose a variable, and zero point for measuring in the vertical direction, as the problem dictates that we will use h, and it will represent distance from the top of the tank. Thus, the pictures will be as they were in the second solution for problem 1:



$$\frac{r}{7-h} = \frac{4}{7}$$

$$\text{So, } r = \frac{4(7-h)}{7}$$

So, we have that the volume of the slice is $\pi\left(\frac{4(7-h)}{7}\right)^2 \Delta h$. Thus the weight of the slice is $(65+h)\pi\left(\frac{4(7-h)}{7}\right)^2 \Delta h$. Representing the weight of the entire tank as an integral, we have that the weight is $\int_0^7 (65+h)\pi\left(\frac{4(7-h)}{7}\right)^2\, dh$.

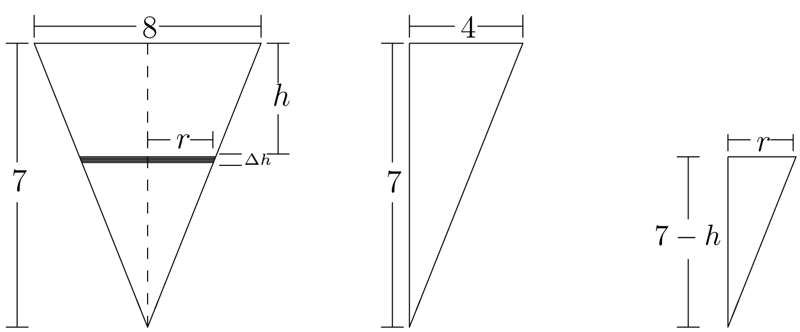
$$\begin{aligned}
\int_0^7 (65+h)\pi\left(\frac{4(7-h)}{7}\right)^2 dh &= \frac{16\pi}{49}\int_0^7 (65+h)(7-h)^2\, dh \\
&= \frac{16\pi}{49}\int_0^7 (65+h)(49-14h+h^2)\, dh \\
&= \frac{16\pi}{49}\int_0^7 (3185 - 861h + 51h^2 + h^3)\, dh \\
&= \frac{16\pi}{49}\left[3185h - \frac{861}{2}h^2 + 17h^3 + \frac{1}{4}h^4\right]_0^7
\end{aligned}$$

$$\begin{aligned}\int_0^7 (65+h)\pi\left(\frac{4(7-h)}{7}\right)^2\,dh &= \frac{16\pi}{49}\left(3185\cdot 7 - \frac{861}{2}7^2 + 17\cdot 7^3 + \frac{1}{4}7^4 - 0\right)\\ &= \frac{16\pi}{49}7^2\left(455 - \frac{861}{2} + 17\cdot 7 + \frac{1}{4}7^2\right)\\ &= 16\pi\left(\frac{623}{4}\right)\\ &= 2492\pi\\ &\approx 7829 \text{ lb}\end{aligned}$$

6. A water tank is in the shape of an inverted (i.e. upside-down) right circular cone. The tank is 7 feet tall, and is 8 feet across the top. The lid was accidently left off the top of tank and wind blew dirt into the tank causing the water to become muddied. As luck would have it, the dirt settled in such a way that the density of the water can be easily modeled. The density of the muddied water at a depth of h feet below the surface is given by $\delta(h) = 65+h$ lb/ft^3. The tank must be emptied and cleaned to get rid of the dirt. Find the amount of work required to pump all the water out a full tank.

   The first thing that we want to do here is find the amount of work needed to pump an arbitrary slice out of the tank. We next want to realize that this means that we want to find the weight of an arbitrary slice, and how far it must be moved to get out of the tank.

   As in the previous problem, we do not need to choose a variable, and zero point for measuring in the vertical direction, as the problem dictates that we will use h, and it will represent distance from the top of the tank. The pictures are exactly the same as those in the last problem:



$$\frac{r}{7-h} = \frac{4}{7}$$

So, $r = \dfrac{4(7-h)}{7}$

   Once again, the volume of the slice is $\pi\left(\frac{4(7-h)}{7}\right)^2\Delta h$, and the weight of the slice is $(65+h)\pi\left(\frac{4(7-h)}{7}\right)^2\Delta h$. Next we are interested in how far the slice must be moved. Luckily, it is clear that the slice must be moved h feet, since h is precisely the depth of the slice. So the work required to pump out an arbitrary slice is $h(65+h)\pi\left(\frac{4(7-h)}{7}\right)^2\Delta h$, and the work required to pump out all of the water is:

$$\begin{aligned}
W &= \int_0^7 h(65+h)\pi\left(\frac{4(7-h)}{7}\right)^2\,dh\\
&= \frac{16\pi}{49}\int_0^7 h(65+h)(7-h)^2\,dh\\
&= \frac{16\pi}{49}\int_0^7 (65h+h^2)(49-14h+h^2)\,dh\\
&= \frac{16\pi}{49}\int_0^7 (3185h-861h^2+51h^3+h^4)\,dh\\
&= \frac{16\pi}{49}\left[\frac{3185}{2}h^2-287h^3+\frac{51}{4}h^4+\frac{1}{5}h^5\right]_0^7\\
&= \frac{16\pi}{49}\left(\frac{3185}{2}7^2-287\cdot 7^3+\frac{51}{4}7^4+\frac{1}{5}7^5\right)\\
&= \frac{16\pi}{49}7^2\left(\frac{3185}{2}-287\cdot 7+\frac{51}{4}7^2+\frac{1}{5}7^3\right)\\
&= 16\pi\left(\frac{3185}{2}-2009+\frac{2499}{4}+\frac{343}{5}\right)\\
&= 16\pi\left(\frac{5537}{20}\right)\\
&= \frac{22148\pi}{5}\\
&\approx 13916 \text{ ft-lb}
\end{aligned}$$