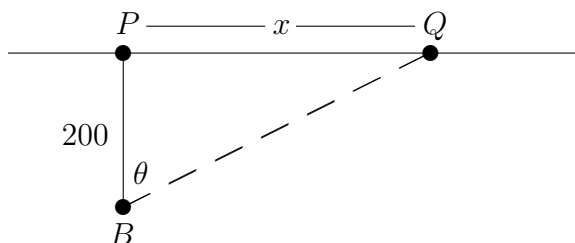


Example 1: A revolving beacon in a lighthouse makes one revolution every 15 seconds. The beacon is 200 ft from the nearest point  $P$  on a straight shoreline. Find the rate at which a ray from the light moves along the shore at a point 400 ft from  $P$ .

Solution: Diagram the problem, labeling the point  $P$ , the beacon as  $B$ , and  $Q$  as another point along the shore which is  $x$  ft from  $P$ . Also, let  $\theta$  be the angle in question (i.e.  $\angle PBQ$ ).



Since the light revolves four times per minute, the angle  $\theta$  changes at a rate of  $4 \cdot 2\pi$  radians per minute; that is,  $\frac{d\theta}{dt} = 8\pi$ . Using the triangle  $PBQ$  (and the fact that it is a right triangle), we see that

$$\tan \theta = \frac{x}{200} \quad \text{or equivalently} \quad x = 200 \tan \theta.$$

The rate at which the ray of light moves along the shore is

$$\frac{dx}{dt} = 200 \sec^2 \theta \frac{d\theta}{dt} = (200 \sec^2 \theta) (8\pi) = 1600\pi \sec^2 \theta.$$

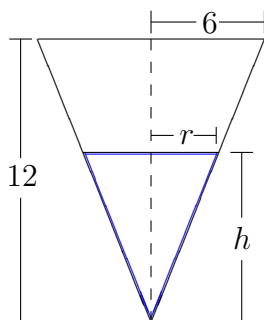
If  $x = 400$ , then  $\overline{BQ} = \sqrt{200^2 + 400^2} = 200\sqrt{5}$ , and

$$\sec \theta = \frac{200\sqrt{5}}{200} = \sqrt{5}.$$

$$\text{Hence, } \frac{dx}{dt} = 1600\pi (\sqrt{5})^2 = 8000\pi \approx 25,133 \text{ ft/min.}$$

Example 2: A water tank has the shape of an inverted right circular cone of height 12 ft and base radius 6ft. If water is being pumped into the tank at a rate of 10 gal/min, approximate the rate at which the water level is rising when the water is 3 ft deep. [1 gal  $\approx$  0.1337 ft<sup>3</sup>]

Solution: Begin by sketching and labeling the dimensions of the tank. Also sketch the water in the tank on the same diagram, letting  $r$  denote the radius of the surface of the water when the depth is  $h$ . Note that  $r$  and  $h$  refer to the **water level in the tank, not the tank itself**.



The problem can now be stated as follows:

$$\text{Given: } \frac{dV}{dt} = 10 \text{ gal/min}$$

$$\text{Find: } \frac{dh}{dt} \text{ when } h = 3 \text{ ft}$$

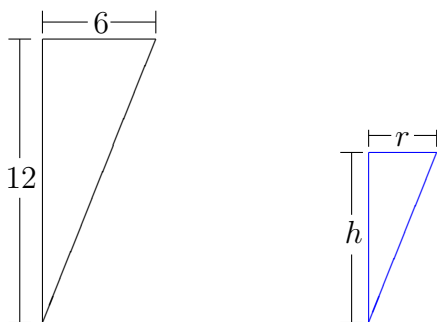
The volume  $V$  of water in the tank corresponding to depth  $h$  is

$$V = \frac{1}{3}\pi r^2 h.$$

This formula for  $V$  depends on both  $h$  and  $r$ . Differentiating this expression for the volume (of the water) with respect to  $t$ , we get

$$\frac{dV}{dt} = \frac{1}{3}\pi \left( 2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right)$$

We now have an equation with 5 variables ( $\frac{dV}{dt}$ ,  $r$ ,  $\frac{dr}{dt}$ ,  $h$  and  $\frac{dh}{dt}$ ). Thankfully, we can use the fact that we have similar triangles (the tank and the water in the tank) to reduce the number of variables.



So, we have  $\frac{r}{6} = \frac{h}{12}$ , or more helpfully,  $r = \frac{h}{2}$ . Differentiating this relationship with respect to  $t$  we have  $\frac{dr}{dt} = \frac{1}{2} \frac{dh}{dt}$ . Also, since we want to know what is happening when  $h = 3$ , we can use the relationship to see that this corresponds to  $r = \frac{3}{2}$ . Thus, we now have everything we need to find  $\frac{dh}{dt}$  when  $h = 3$ .

$$\begin{array}{ll} \text{When } h = 3: & \bullet \frac{dV}{dt} = 10 \text{ gal/min} \approx 1.337 \text{ ft}^3/\text{min} & \bullet h = 3 \\ & \bullet \frac{dr}{dt} = \frac{1}{2} \frac{dh}{dt} & \bullet r = \frac{3}{2} \end{array}$$

Substituting all of these into the expression  $\frac{dV}{dt} = \frac{1}{3}\pi (2r\frac{dr}{dt}h + r^2\frac{dh}{dt})$ :

$$\begin{aligned} 10 &= \frac{1}{3}\pi \left( 2 \left(\frac{3}{2}\right) \left(\frac{1}{2} \frac{dh}{dt}\right) (3) + \left(\frac{3}{2}\right)^2 \frac{dh}{dt} \right) \\ \frac{30}{\pi} &= \frac{9}{2} \frac{dh}{dt} + \frac{9}{4} \frac{dh}{dt} \\ \frac{30}{\pi} &= \frac{27}{4} \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{40}{9\pi} \text{ gal/min/ft}^2 \\ &\approx \frac{40}{9\pi} (.01377) \approx \underline{\underline{.189 \text{ ft/min}}} \end{aligned}$$

**Alternatively:** Before differentiating, we could have expressed  $V$  in terms of only one variable, using similar triangles. Recall  $r = \frac{h}{2}$ . Consequently, at depth  $h$ ,

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 (h) = \frac{1}{12}\pi h^3.$$

Differentiating the last equation implicitly with respect to  $t$  gives us the following general relationship between the rates of change of  $V$  and  $h$  at any time  $t$ :

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}.$$

If  $h \neq 0$ , an equivalent formula is

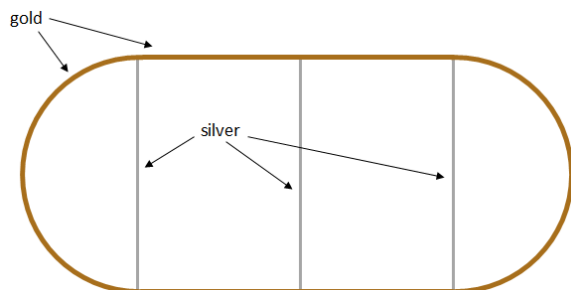
$$\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}.$$

Finally, we substitute  $h = 3$  ft and  $dV/dt = 10$  gal/min  $\approx 1.337$  ft<sup>3</sup>/min, and obtain

$$\frac{dh}{dt} \approx \frac{4}{\pi(9)} (1.337) \approx 0.189 \text{ ft/min.}$$

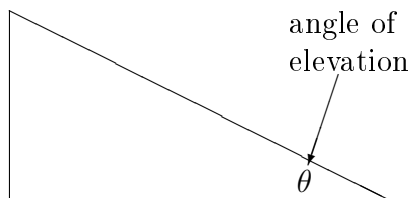
- There are a couple optimization problems intermixed within these, so buyer beware.

1. If  $A = x^2$  and  $dx/dt = 3$  when  $x = 10$ , find  $dA/dt$ .
2. If  $x^2 + 3y^2 + 2y = 10$  and  $dx/dt = 2$  when  $x = 3$  and  $y = -1$ , find  $dy/dt$ .
3. Gas is being pumped into a spherical balloon at a rate of  $5 \text{ ft}^3/\text{min}$ . Find the rate at which the radius is changing when the diameter is 18 inches.
4. A ladder 20 ft long leans against a vertical building. If the bottom of the ladder slides away from the building horizontally at a rate of 3 ft/sec, how fast is the ladder sliding down the building when the top of the ladder is 8 ft from the ground?
5. Boyle's law for confined gases states that if the temperature is constant,  $pv = c$ , where  $p$  is pressure,  $v$  is volume, and  $c$  is a constant. At a certain instant, the volume is  $75 \text{ in.}^3$ , the pressure is  $30 \text{ lb/in.}^2$ , and the pressure is decreasing at a rate of  $2 \text{ lb/in.}^2$  every minute. At what rate is the volume changing at this instant?
6. A company wishes to create a new large sign for their lobby. The design of the sign will be a rectangle with a semicircle at each end, as shown below. The exterior border of the sign will be made from gold piping which will cost \$12 per inch, and there will be three vertical strips of silver piping which costs \$5 per inch. The sign will have the greatest impact if the area that the sign occupies is maximized. If the company is willing to spend \$10,000 on the new sign, what is the radius of semicircular sections that will give the sign the greatest impact?

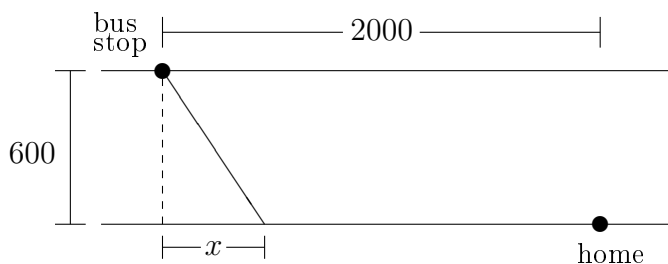


7. The ends of a water trough 8 ft long are equilateral triangles whose sides are 2 ft long. If water is being pumped into the trough at a rate of  $5 \text{ ft}^3/\text{min}$ , find the rate at which the water level is rising when the depth of the water is 8 in.
8. The area of an equilateral triangle is decreasing at a rate of  $4 \text{ cm}^2/\text{min}$ . Find the rate at which the length of a side is changing when the area of the triangle is  $200 \text{ cm}^2$ .
9. A large spherical balloon is being filled at a rate of  $\frac{\pi}{3} \text{ in}^3/\text{sec}$ . How fast is the surface area of the balloon increasing 30 seconds after the balloon began being filled?

10. An airplane which is flying at an altitude of 3 miles passes directly over an observer on the ground who tracks the plane's flight. As the plane moves away, the observer must keep decreasing the angle of elevation of her line of sight in order to view the plane. When the angle of elevation is exactly  $\frac{\pi}{6}$ , she notes that she is having to decrease the angle at a rate of 0.125 radians per second. How fast, in miles per hour, is the plane traveling?



11. When two resistors  $R_1$  and  $R_2$  are connected in parallel, the total resistance  $R$  is given by the equation  $1/R = (1/R_1) + (1/R_2)$ . If  $R_1$  and  $R_2$  are increasing at rates of 0.01 ohm/sec and 0.02 ohm/sec, respectively, at what rate is  $R$  changing at the instant that  $R_1 = 30$  ohms and  $R_2 = 90$  ohms?
12. Alaina (remember this example?) is now on her way home from work and needs to get back across the park in the shortest amount of time. Recall that she needs to travel 600 feet south and 2000 feet east from the bus stop in order to get home. She can travel through the park at a rate of 3 feet per second (it rained during the day, so the grass is wet which slows her down), and along the sidewalk at a rate of 6 feet per second. How far east of the bus stop should she travel through the park in order to get home the fastest?



13. A large boat is being pulled toward a dock via a rope attached to a winch. The rope is attached to the boat at a point which is 10 feet higher than the winch. If the winch is reeling in the rope at a constant rate of 5 feet per second, how fast is the boat moving toward the dock when 40 feet of rope are out? Give your answer in miles per hour.
14. An isosceles triangle has equal sides 6 in. long. If the angle  $\theta$  between the equal sides is changing at a rate of  $2^\circ$  per minute, how fast is the area of the triangle changing when  $\theta = 30^\circ$ ?

## Answers

1. 60

2. 3

3.  $\frac{20}{9\pi} \approx 0.707$  ft/min

4.  $-\frac{3}{8}\sqrt{336} \approx -6.9$  ft/sec

5. 5 in<sup>3</sup>/min (increasing)

6.  $r \approx 73.4$  in

7.  $\frac{15\sqrt{3}}{32} \approx 0.81$  ft/min

8.  $-\frac{\sqrt{2}}{5\sqrt[4]{3}} \approx -0.2149$  cm/min

9.  $\frac{dA}{dt}\bigg|_{t=30} = \frac{2\pi}{3\sqrt[3]{\frac{15}{2}}}$

10. 5400 mph

Editorial note: Oops! There is a typo in the problem, which is why the answer does not make practical sense.

11.  $\frac{11}{1600} = 0.006875$  ohm/sec

12.  $\approx 346.4$  ft

13.  $\approx 5.16$  ft/sec  $\approx 3.52$  mph

14.  $\frac{\pi}{5}\sqrt{3} \approx 0.54$  in<sup>2</sup>/min