

LIMITS - understanding them and some examples

Some extra resources:

<https://youtu.be/kfF40MiS7zA> (watch until about 9:55, this channel has a really good reputation for explaining the big concepts and the “why” we care about things in math)

<https://youtu.be/fOrOeZA-vdY> “Evaluating Limits by Factoring”

<https://youtu.be/M2BgBG2Su94> “Evaluating Limits with Fractions/Square Roots”

<https://youtu.be/NmLljBAg82o> “Evaluating Limits at Infinity”

<https://youtu.be/7Q2HwTHcxA0> “Evaluating Limits from a graph”

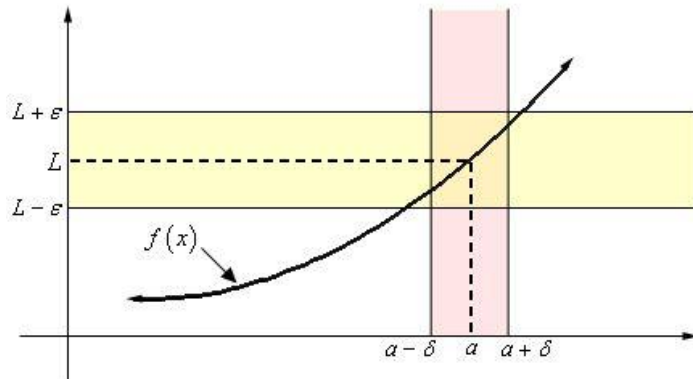
(these four I put above are less about the concepts, but go over specific example problems of different kinds of functions you will need to evaluate limits with, or finding limits from a graph)

For practice problems, and example write ups, scroll to the bottom

Where/When Does a Limit Exist?

Think of the pink and the yellow shading as bounds.

When we try to evaluate a limit, we're closing in those **bounds to be smaller and smaller** as we get close to the point that we're interested in.



(for example if we were evaluating the \lim as $x \rightarrow 2$, then we are trying to find what happens as we get closer to 2, NOT at 2!)

What makes a limit a limit is that **we don't ever get to 2**, but we get super super super (ie, infinitely) close to the x value 2.

This means getting to 2.000000000001 on the right side, and 1.99999999999 on the left

If our y-values for when x is 2.000000000001 and x is 1.99999999999 match up, (say both of them approach $y=6$) then we have found what the limit as $x \rightarrow 2$ is

If the values at $x = 2.000000000000001$ and $x = 1.99999999999$ DON'T match up, that's when **the limit does not exist**

Existence of a Limit

The limit of a function $f(x)$ exists if and only if the one sided limits of the function are equal.

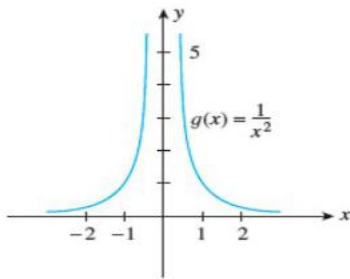
$$\lim_{x \rightarrow c} f(x) = L \quad \text{if and only if}$$

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$$

Cases where the limit does not exist:

1. If the graph stretches to infinity or negative infinity

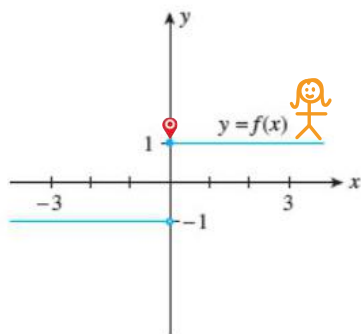
(In this example below, see that we get closer and closer to zero, and the graph points up towards infinitely large y-values and never touches the axis. So, as $x \rightarrow 0$ (as the limit approaches zero) the **limit does not exist**)



(b) $\lim_{x \rightarrow 0} g(x)$ does not exist.

2. If $f(x)$ come close to two different values, depending on what side you come from

(In this example below, we get close to zero from the right side, and the function approaches $y=1$. However when we try to do this from the left side, see that $y=-1$. There's no way that y can equal 1 and -1 at the same time, so this mean as $x \rightarrow 0$, the **limit does not exist** here)



Imagine you are walking along this graph, this graph is your path. Your destination is the end point (when $x=0$). At what y-value would you be at?

(a) $\lim_{x \rightarrow 0} f(x)$ does not exist.

*note though, that in number 2, both of the one sided limits **do exist** (more on one sided limits on the next page)

One sided limits:

Besides finding the limit at a single point, we can find limits on a specific side of that value, either on the right or the left

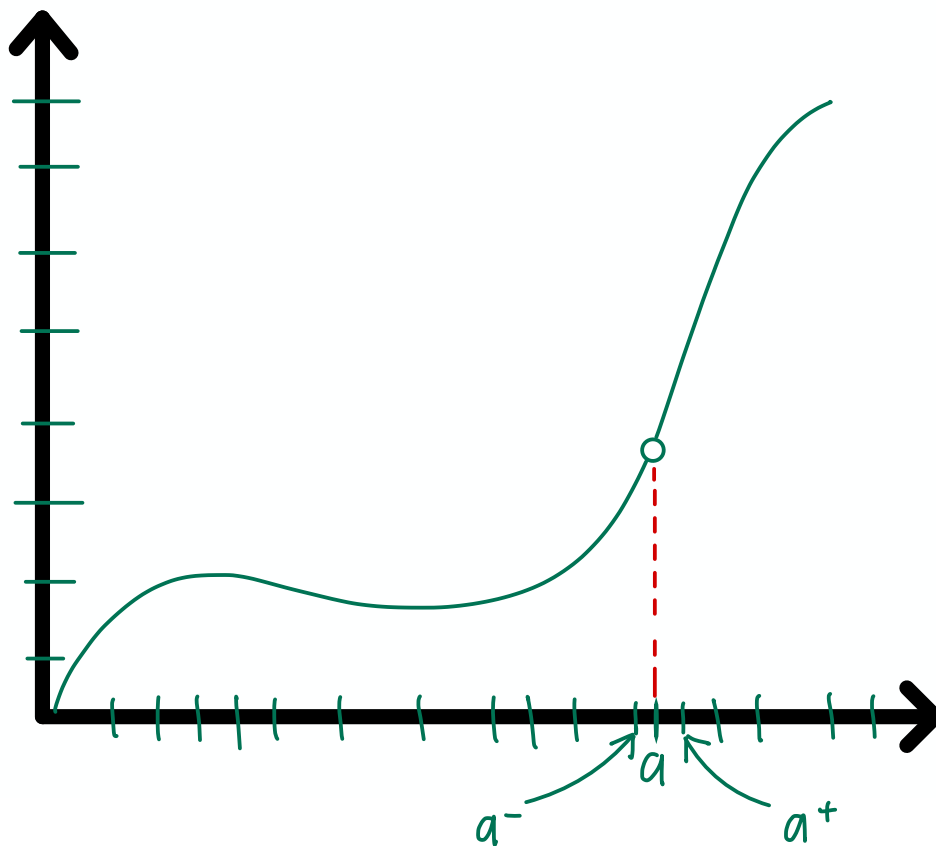
Here is how we notate which “side” we are coming from

$x \rightarrow a^+$ means x is approaching from the right

$x \rightarrow a^-$ means x is approaching from the left

See how in both, we are trying to get very close to 1, but just on different “sides” of it

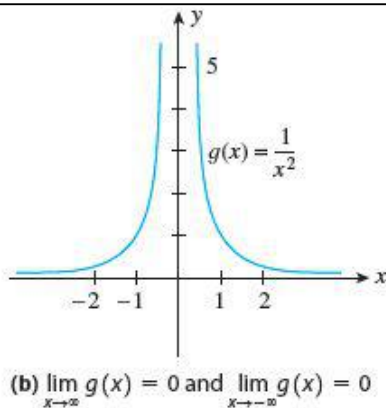
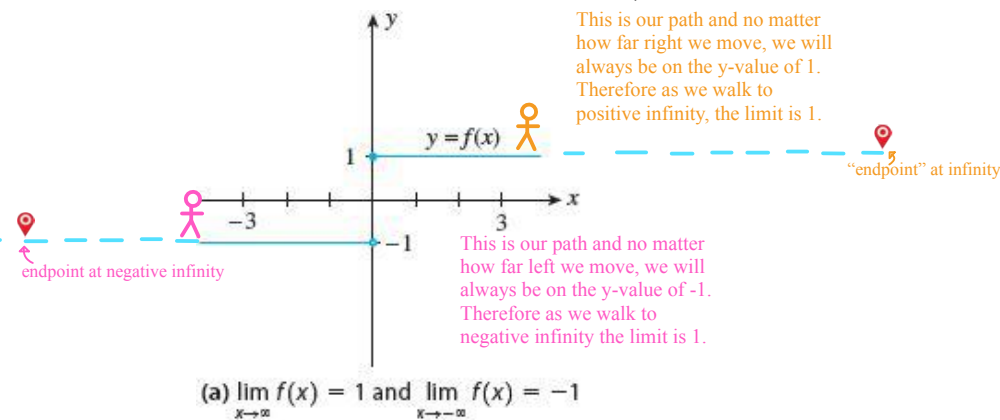
This is a graphical representation:



Limits to Infinity

In calculating limits that go to infinity, we don't actually plug in the value of "infinity" but think of the input as a number that continuously grows infinitely large

Since infinity is a concept and not a number, what we do is allow the input to keep growing without stopping (a really really large POSITIVE number when $x \rightarrow \infty$ and a very very large NEGATIVE number when $x \rightarrow -\infty$)



For limits where $x \rightarrow \infty$ or $x \rightarrow -\infty$, can see what would happen if we put a very large number in place of infinity

For example, in the function $g(x) = 1/x^2$, if we considered $x = 200$, then we get $\frac{1}{(200)^2}$, which equals 0.000025 (already pretty small)

But since we're evaluating a limit that goes to infinity, we keep pushing that number greater and greater, so we try $x = 2000$, where $g(x) = 0.00000025$ and $x = 20000$, where $g(x) = 0.0000000025$. And you will see that the result is numbers

getting closer and closer to zero.

The function will never reach 0, but as we go on for infinity the function gets closer and closer to 0.

We can make the same argument with $\lim_{x \rightarrow -\infty}$, as we get more and more negative, the resulting values will get extremely close to zero.

What to do with indeterminants (zero divided by zero):

An **indeterminate** happens when the numerator and the denominator of the function give us zeros when evaluating the function. Since $0/0$ is undefined, we have to use some of the following methods to evaluate the limit:

Situation 1: Expand the expression and factor

If we were asked to find:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}.$$

We can first try direct substitution, but quickly we see that we get an indeterminate form:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \frac{3^2 - 9}{3 - 3} = \frac{0}{0},$$

So we have to factor and expand the equation:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3} (x + 3) = 3 + 3 = 6.$$

The $(x-3)$ in the numerator and denominator cancelled out, which let us evaluate the limit and find the answer.

Situation 2: Rationalize the equation

If we were asked to find:

$$\lim_{x \rightarrow 0} \frac{\sqrt{4+3x}-2}{x} ?$$

We would quickly see we get an indeterminate (we would get 0 divided by 0) when we substitute x for 0

So instead, we need to rationalize the equation, by multiplying the numerator and denominator by a conjugate $(\sqrt{4+3x}+2)$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{4+3x}-2}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{4+3x}-2)(\sqrt{4+3x}+2)}{x \cdot (\sqrt{4+3x}+2)} \\ &= \lim_{x \rightarrow 0} \frac{(4+3x)-4}{x \cdot (\sqrt{4+3x}+2)} \\ &= \lim_{x \rightarrow 0} \frac{3x}{x \cdot (\sqrt{4+3x}+2)} \end{aligned}$$

By using the conjugate, we rewrote the numerator without a square root, and now can evaluate the limit to find the answer

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{3}{\sqrt{4+3x}+2} \\ &= \frac{3}{4} \cdot \square \end{aligned}$$

We multiply by the conjugate, because if we use the FOIL method, the terms cancel out in such a way that the radical (square root) is no longer a part of the expression:

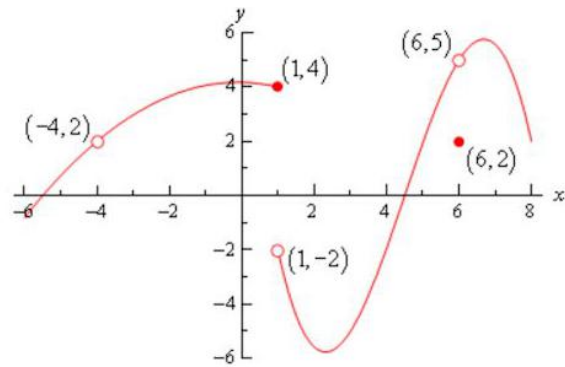
$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})=a-b$$

Properties of Limits

If a and c are real numbers, and $f(x)$ and $g(x)$ are functions, then

1. $\lim_{x \rightarrow a} c = c$
2. $\lim_{x \rightarrow a} x = a$
3. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
4. $\lim_{x \rightarrow a} [f(x)g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \cdot \left[\lim_{x \rightarrow a} g(x) \right]$
5. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$, provided $\lim_{x \rightarrow a} g(x) \neq 0$
6. $\lim_{x \rightarrow a} [c \cdot f(x)] = c \left[\lim_{x \rightarrow a} f(x) \right]$

PRACTICE PROBLEMS



compute each of the following.

(a) $f(-4)$ (b) $\lim_{x \rightarrow -4^-} f(x)$ (c) $\lim_{x \rightarrow -4^+} f(x)$ (d) $\lim_{x \rightarrow -4} f(x)$

(e) $f(1)$ (f) $\lim_{x \rightarrow 1^-} f(x)$ (g) $\lim_{x \rightarrow 1^+} f(x)$ (h) $\lim_{x \rightarrow 1} f(x)$

(i) $f(6)$ (j) $\lim_{x \rightarrow 6^-} f(x)$ (k) $\lim_{x \rightarrow 6^+} f(x)$ (l) $\lim_{x \rightarrow 6} f(x)$

Answers:

- Undefined (there is no closed dot, so the function doesn't exist at this point)
- 2
- 2
- 2
- 4
- 4
- 2
- Does not exist (The two one-sided limits both exist, however they are different and so the normal limit doesn't exist)
- 2
- 5
- 5
- 5 (remember that the limit doesn't care what is happening at the point and so it's possible for the limit to have a different value than the function at a point, Limits are only concerned with what the function is doing around the point)

MORE PRACTICE

What is $\lim_{x \rightarrow 5} \frac{x^2 - x - 20}{x - 5}$?

Answer- 9

Find $\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 4x} - x \right)$.

Answer- 2

What is $\lim_{h \rightarrow 0} \frac{(h+4)^2 - 16}{h}$?

Answer- 8