Nonparallel theory

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Outline

- Parallel flow assumption
- Orr-Sommerfeld equations
- Discrepancy in boundary layer results
- Weakly non-parallel assumption
- Application to flat plate boundary layer stability problem
- Results of non-parallel theory vs. experiment

The parallel flow assumption

- Within stability theory, the assumption of parallel flow is vital to for deriving the celebrated Orr-Sommerfeld equation.
- The parallel flow assumption leads to separable PDE for the linear perturbations that can be reduced to a system of ODE's of second order through the traveling wave assumption.
- It assumes that normal derivatives are far more important than streamwise ones and through the continuity equation, it means that the normal velocity is identically zero.
- Plane Poiseuille flow fits these conditions exactly since the base flow is dependent on y only and is exactly parallel.

The parallel flow assumption

- In parallel flow the base flow is thought to be parallel and depend only on normal coordinate

$$\mathbf{V} = (U, V, W)^T = (U(y), V(y), W(y))^T$$

- From continuity (incompressible) we get that $\partial V/\partial y = 0 \Rightarrow V = 0$ to satisfy no-penetration boundary condition.
- We get then:

$$\mathbf{V} = (U(y), 0, W(y))^T$$

The Orr-Sommerfeld equation

- The linearized Navier-Stokes equations can be reduced to a system of 3 ODE equations after the traveling wave reduction i.e.

$$(u, v, w, p)^T = (\hat{u}(y), \hat{v}(y), \hat{w}(y), \hat{p}(y))^T \exp(i(\alpha x + \beta z - \omega t)).$$

- Therefore we get:

$$i\tilde{\alpha}\tilde{\hat{u}} + \frac{d\hat{v}}{dy} = 0, \;\; i\hat{\alpha}\tilde{\hat{u}}(\tilde{U} - \tilde{c}) + \tilde{U}'\frac{d\tilde{v}}{dy} = -\frac{i\tilde{\alpha}\hat{p}}{\rho} + \nu(\tilde{\hat{u}}'' - \alpha^2\tilde{\hat{u}})$$

$$i\hat{\alpha}\hat{v}'(\tilde{U} - \tilde{c}) = -\frac{\hat{p}'}{\rho} + \nu(\hat{v}'' - \alpha^2\hat{v})$$

The Orr-Sommerfeld equation

- For the special case of 2D flow, i.e. $W = 0$ then we can get the Orr-Sommerfeld equation for the streamfunction perturbation:

$$(U - c)(\hat{\psi}'' - \alpha^2\hat{\psi}) - U''\hat{\psi} = \frac{i}{\alpha Re}(\hat{\psi}^{(iv)} - 2\alpha^2\hat{\psi}'' + \alpha^4\hat{\psi})$$

  with homogeneous boundary conditions $\hat{\psi} = \hat{\psi}' = 0,\ y = 0$ and $\hat{\psi} \to 0$ as $y \to \infty$.
- This equation is far easier to solve than the system of PDE for the linearized Navier-Stokes equations.
- The linear stability theory for boundary layer based on this equation gives the critical Reynolds number of $Re_{\delta_1} = 520$ based on displacement thickness $\delta_1 \propto \sqrt{x}$.

Discrepancy in the stability theory for Blasius boundary layer

- For locally parallel assumption, critical Reynolds number Re_{δ_1} is over-predicted by 30% [1] with respect to the data of Schubauer and Skramstad[2] and that of Barry and Ross [3].
- The suspicion was that non-parallel effects could account for this discrepancy.
- Also, the assumption of parallel flow was not fully justified to begin with except as a requirement to allow separability [4].
- Boundary layer is growing $\propto \sqrt{x}$, and is not really parallel since $dU/dx \neq 0$ and therefore $dV/dy \neq 0 \Rightarrow V \neq 0$.

The non-parallel stability for 2D boundary layers

- Non-parallel effects were incorporated into stability analysis of the 2D flat plate boundary layer by Bouthier [5, 6] (1972,1973), Gaster (1974) [4], and Saric and Nayfeh (1975) [1].
- The method of multiple scales was utilized to obtain first order corrections to the growth rate of disturbances in the boundary layer.
- Bouthier and Gaster used similarity variables while Saric and Nayfeh developed their theory in normal Cartesian coordinates.
- This was a good setting for testing the non-parallel theory since data existed for comparison of results with experiment.

Weakly non-parallel assumption

- For fully non-parallel flow of course, then $\mathbf{V} = (U(x,y,z), V(x,y,z), W(x,y,z))^T$.
- For boundary layer case we relax the above by defining two slow coordinates $X = \epsilon x$ and $Z = \epsilon z$ and that

$$U(x,y,z) \rightarrow U_0(X,y,Z) + \epsilon U_1(X,y,Z) + \dots$$
$$W(x,y,z) \rightarrow W_0(X,y,Z) + \epsilon W_1(X,y,Z) + \dots$$

  for a small parameter controlling the non-parallel effects by ϵ.
- Assuming that the two velocities depend on the slow variables X and Z, then

$$\frac{\partial U}{\partial x} = \epsilon \frac{\partial U_0}{\partial X} + O(\epsilon^2), \frac{\partial W}{\partial z} = \epsilon \frac{\partial W_0}{\partial X} + O(\epsilon^2)$$

- Therefore from continuity of the base flow it must be that

$$\frac{\partial V}{\partial y} = \epsilon\left(\frac{\partial U_0}{\partial X} + \frac{\partial W_0}{\partial Z}\right) + O(\epsilon^2)$$

- It means that $V = \epsilon V_1(X, y, Z) + \epsilon^2 V_2(X, y, Z) + \ldots$.
- Therefore, we can obtain each of these corrections from the base flow equations.
- For the boundary layer, we can obtain the base flow in an expansion with $Re^{-1/2}$.
- For 2D boundary layer, in particular we need only $U = U_0(X, y) + \epsilon U_1(X, y) + \ldots$, and $V = \epsilon V_1(X, y) + \ldots$.

Reduction to a single equation

- The continuity, x-momentum, and y-momentum equations can be reduced to a single equation for the streamfunction. First let $u' = \partial\psi'/\partial y$ and $v' = -\partial\psi'/\partial x$, the continuity is automatically satisfied,

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = \frac{\partial^2\psi'}{\partial x\partial y} - \frac{\partial^2\psi'}{\partial x\partial y} = 0$$

- Now take the x-momentum equation and differentiate by y and subtract the y-momentum equation differentiated by x to eliminate the pressure perturbation:

$$\frac{\partial\nabla^2\psi'}{\partial t} - \frac{\partial\psi'}{\partial y}\frac{\partial\nabla^2\psi'}{\partial x} + \frac{\partial\psi'}{\partial x}\frac{\partial\nabla^2\psi'}{\partial y} = \nu\nabla^4\psi'$$

Superposition of base and disturbance flow

- Superimpose our base flow streamfunction Ψ with a small perturbation ψ, such that the full flow is given by

$$\psi' = \Psi + \psi$$

  and that non-linear terms in ψ can be ignored.
- It produces the following after subtracting out the base flow terms:

$$\boxed{\frac{\partial \nabla^2 \psi}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \Psi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \Psi}{\partial y} - \frac{\partial \Psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} + \frac{\partial \Psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} = \nu \nabla^4 \psi}$$

BC's: $\psi = 0$ and $\partial \psi / \partial y = 0$ at $y = 0$, and $\psi \to 0$ as $y \to \infty$.

Choice of coordinates

- At this point Gaster and Bouthier continue their analysis using the similarity variable $\eta = \left(U/\nu x\right)^{1/2} y$.
- On the other hand, Saric and Nayfeh utilize Cartesian coordinates. We proceed with the latter case to compare with the derivation used in class.

Weakly non-parallel flow assumption

- For locally parallel flow we would choose $\partial\Psi/\partial x = 0$ and so $\Psi = f(y)$, this reduces the above equation to the usual Orr-Sommerfeld equation.
- Non-parallel assumption dictates that the base flow is a weak function of x, i.e. define the slow coordinate $X = \epsilon x$ for ϵ small and so we write:

$$U = \frac{\partial\Psi}{\partial y} = U_0(X, y) + \epsilon U_1(X, y) + \dots$$

and

$$V = -\frac{\partial\Psi}{\partial x} = -\epsilon\frac{\partial\Psi}{\partial X} = \epsilon V_0(X, y) + \epsilon^2 V_1(X, y) + \dots$$

- To solve the equation above in terms of an asymptotic series with small parameter ϵ we utilize the method of multiple scales. We use the ansatz:

$$\psi = \phi(X, y; \epsilon)e^{Q}$$

and define

$$\frac{\partial Q}{\partial t} = -i\omega, \text{ and } \frac{\partial Q}{\partial x} = i\alpha_0(X)$$

- For locally parallel flow ϕ and α_0 are independent of the streamwise coordinate X, i.e. $\epsilon = 0$.
- From the iteration process it emerges that $\epsilon = Re^{-1/2} = (Ux_0/\nu)^{-1/2}$.
- In this case $\omega \in \mathbb{R}$ is the frequency, $\mathrm{Real}(\alpha_0)$ is the wavenumber and $\mathrm{Im}(\alpha_0) = \alpha_{0r}$ is the growth rate of the disturbance (where $\alpha_{0r} < 0$ causes growth of perturbations).

- We construct an approximate solution as an asymptotic sequence by defining $\phi = \phi_0(X, y) + \epsilon\phi_1(X, y) + \ldots$, and substituting into our governing equation and grouping terms according to ϵ:

$$\mathcal{L}_{OS}(\phi_0) + \epsilon(\mathcal{L}_{OS}(\phi_1) - \mathcal{L}_{NP}(\phi_0)) + \cdots = 0$$

- At first order we obtain the homogenous eigenvalue problem:

$$\mathcal{L}_{OS}(\phi_0) = (\phi_0^{(iv)} - 2\alpha_0^2\phi_0'' + \alpha_0^4\phi_0) \\ - i\alpha_0 Re\left[\left(U_0 - \frac{\omega}{\alpha_0}\right)(\phi_0'' - \alpha_0^2\phi_0) - U_0''\phi_0\right] = 0$$

$$\mathrm{BC}: \phi_0 = \phi_0' = 0, y = 0, \&\ \phi_0 \to 0, y \to \infty$$

Definition of $\mathcal{L}_{NP}$

It is given by:

$$\mathcal{L}_{NP}(\phi_0) = \underbrace{\left[Re(2\alpha_0\omega - 3U_0\alpha_0^2 - U_0'') + 4i\alpha_0^3\right]\frac{\partial\phi_0}{\partial X}}_{\text{streamwise variation of } \phi_0}$$

$$+ \underbrace{(ReU_0 - 4i\alpha_0)\frac{\partial^3\phi_0}{\partial y^2\partial X}}_{\text{streamwise variation of } \phi_0} + \underbrace{ReV\left(\phi_0''' - \alpha_0^2\phi_0'\right) - Re\frac{\partial^2 V}{\partial y^2}\frac{\partial\phi_0}{\partial y}}_{\text{due to normal velocity}}$$

$$+ \underbrace{\left[-2i\phi_0'' + (Re\omega - 3Re\alpha_0 U_0 + 6i\alpha_0^2)\phi_0\right]\frac{d\alpha_0}{dX}}_{\text{wavenumber and growth rate variation}}$$

$$+ \underbrace{i\alpha_0 Re\left[U_1(\phi_0'' - \alpha_0^2\phi_0) - U_1''\phi_0\right]}_{\text{higher order BL effects}}$$

- To account for the boundary layer growth we propose to solve $\mathcal{L}_{OS}(\phi_0) = 0$ assuming that $\phi_0 = A(X)\zeta(y;X)$ where A is a weak function of X and ζ depends parametrically on X.
- Therefore to a first approximation $\zeta(y;X)$ can be solved for in the usual way. It will depend implicitly on X, since we must have as input $U_0(X,y)$.
- The amplitude A will be determined in the next level of approximation in ϵ.

- At order ϵ we need to solve $\mathcal{L}_{OS}(\phi_1) = \mathcal{L}_{NP}(\phi_0 = A\zeta)$. The RHS can be evaluated:

$$\mathcal{L}_{NP}(A\zeta) = Re\left\{ \left[B_1\zeta + B_2\zeta''\right]\frac{dA}{dx_1} + \left[B_1\frac{\partial\zeta}{\partial X} + B_2\frac{\partial^3\zeta}{\partial y^2\partial X}\right.\right.$$
$$\left.\left. + \left(B_3\zeta + B_4\zeta''\right)\frac{d\alpha_0}{dX} + B_5\zeta + B_6\zeta' + B_7\zeta'' + B_8\zeta'''\right]\right\}A = Re\mathfrak{M}$$

- B_j are known functions of X and y.
- $d\alpha_0/dX$ and $\partial\zeta/\partial X$ are still to be determined.

Solvability

- The inhomogeneous problem defined by

$$\mathcal{L}_{OS}(\phi_1) = Re\mathfrak{M}$$
$$\text{BC: } \phi_1 = \phi_1' = 0, y = 0 \ \& \ \phi_1 \to 0, y \to \infty.$$

can only be solved if the inhomogeneity is orthogonal to the solution of the homogeneous adjoint problem

$$\mathcal{L}_{OS}^*(\zeta^*) = 0$$
$$\text{BC: } \zeta^* = (\zeta^*)' = 0, y = 0, \& \ \zeta^* \to 0, y \to \infty.$$

- This condition is expressed as

$$\int_0^\infty \mathfrak{M}\zeta^* dy = 0$$

Solution for $A(X)$ and ψ

- From the previous equation we obtain the evolution equation for A,
$$\frac{dA}{dX} = i\alpha_1(X)A$$
- Then we can get the correction due to the non-parallel effects to first order in ϕ_0,
$$A = A_0 \exp\left(i\epsilon \int \alpha_1(X)dx\right)$$
- The corrected leading order solution to our original problem is therefore
$$\psi = A(X)\zeta(y;X)e^Q = A_0\zeta(y;X)\exp\left[i\int(\alpha_0+\epsilon\alpha_1)dx - i\omega t\right]$$

Saric and Nayfeh's (1975) results

- Saric and Nayfeh's neutral stability boundaries were found by looking for the condition that

$$\mathrm{Im}(\alpha_0 + \epsilon\alpha_1) = 0$$

- This condition has no ambiguity, it depends only on α_0 and its correction α_1 and not on y nor the quantity whose growth/decay is being quantified, i.e. u or v, etc...
- However, no account is made of the distortion of the profile envelope ϕ_0 or any other quantity with downstream distance.

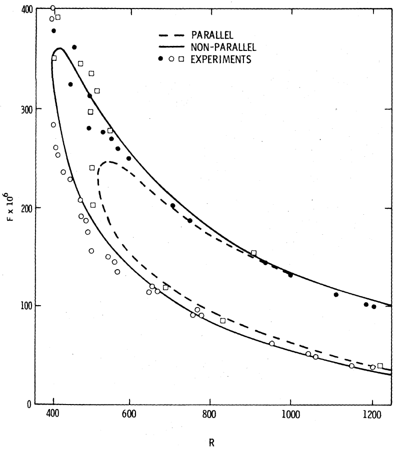

Figure: Results of Saric and Nayfeh (1975)

- As Drazin and Reid [7] point out, the exceptional results in Saric and Nayfeh (1975) are flawed due to the lack of consideration of the distortion the flow quantities with respect to streamwise coordinate x.
- The results are fortuitously close to the experimental data.
- However, if one does the calculation consistently then Saric and Nayfeh's analysis yields similar results for example to Gaster (1974).

Gaster's (1974) results

- Gaster found that stability curves for nearly parallel flows depend on the quantity being measured i.e. kinetic energy, $\int u^2 + v^2 dy$ or streamwise velocity.
- He developed a first order solution with the similarity variable $\eta \propto y/\sqrt{x}$,

$$\psi = A(\xi)\zeta(\xi,\eta)\exp i\left(Re^{1/2}\int_1^{\epsilon}\frac{\alpha(\xi)}{\xi^{1/2}}d\xi - \omega t\right)$$

where A, α are slow functions of ξ, ζ depends parametrically on ξ and $\xi = x/x_0$ and x_0 is the position of the introduction of the disturbance.

- Gaster incorporates the streamwise variation of the amplitude, eigenfunction into the condition for stability.

$$\alpha^{(c)} = \frac{\xi}{|u|}\frac{d|u|}{d\xi} = -\xi^{1/2}Re^{1/2}\alpha_i + \left[\left(\frac{\xi}{A}\frac{dA}{d\xi}\right)_r + \left(\frac{\xi}{\zeta'}\frac{\partial\zeta'}{\partial\xi}\right)_r - \frac{1}{2}\right]$$

- Total kinetic energy across the flow, $E = \int_0^\infty \overline{u^2} + \overline{v^2} d\eta$:

$$\alpha^{(a)} = \frac{\xi}{E}\frac{dE}{d\xi} = -\xi^{1/2}Re^{1/2}\alpha_i + \left[\left(\frac{\xi}{A}\frac{dA}{d\xi}\right)_r + \left(\frac{\xi}{e}\frac{\partial e}{\partial\xi}\right)_r - \frac{1}{4}\right]$$

where $e = \int_0^\infty [\zeta'\tilde{\zeta}' + \alpha\tilde{\alpha}\zeta\tilde{\zeta}]d\eta$.

- Integral of $\overline{u^2}$, i.e. $I = \int \overline{u^2} d\eta$,

$$\alpha^{(b)} = \frac{\xi}{I}\frac{dI}{d\xi} = -2\xi^{1/2}Re^{1/2}\alpha_i + 2\left[\left(\frac{\xi}{A}\frac{dA}{d\xi}\right)_r + \left(\frac{\xi}{h}\frac{\partial h}{\partial \xi}\right)_r - \frac{1}{2}\right]$$

where $h = \int_0^\infty [\zeta' \tilde{\zeta}'] d\eta$.

- Neutral stability was defined when $\alpha^{(c)} = \alpha^{(b)} = \alpha^{(a)} = 0$.
- The results using $|u|$ were directly compared to the data of Schubauer and Skramstad as well as for Barry and Ross.
- The comparison to experiment worsens at Reynolds numbers close to critical values and high frequencies of induced waves.

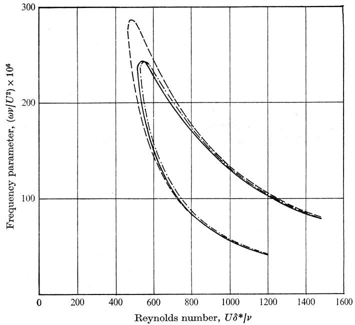

FIGURE 1. Neutral amplification curves based on integral parameters. ——, $\alpha_i = 0$, fo parallel flow; – – –, $\alpha^{(a)} = 0$, the kinetic energy; —·—, $\alpha^{(b)} = 0$, the integral of $\bar{u}^2$.

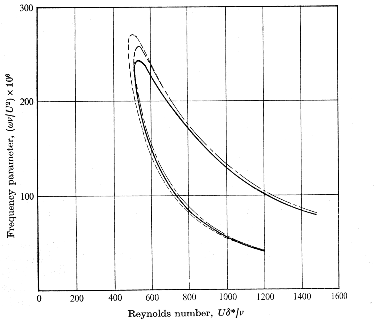

FIGURE 2. Neutral loops based on the points of maximum $|u|$. ——, $\alpha_i = 0$, for parallel flow; – – –, $\alpha^{(c)} = 0$, for inner maximum; —–—, $\alpha^{(c)} = 0$, for outer peak.

Figure: Results of Gaster (1974) showing dependence of stability curve on criteria used.

Figure: Results showing dependence on normal wall coordinate position of probe.

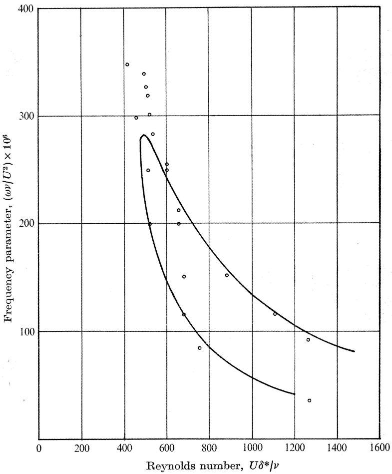

Figure: Results of Gaster (1974) showing comparison with data from Schubauer and Skramstad (1947)

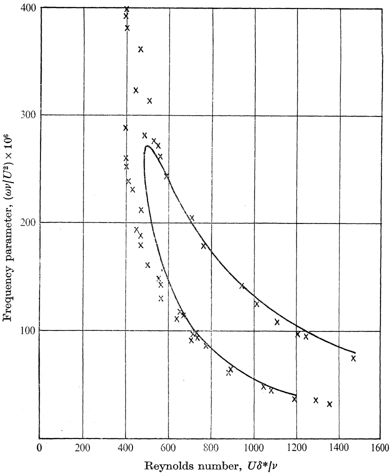

Figure: Results of Gaster compared with data from Barry and Ross (1970).

Results of Bouthier

- Based on $u^2 + v^2$ (local kinetic energy flux).
- Three criteria used:
  - **(I)** Region in (F, Re_{δ_1}) plane where kinetic energy is amplified for all η is called total instability.
  - **(II)** Region between region I and III or partially stable/unstable.
  - **(III)** Region where growth rate of local kinetic energy is damped for all η is called total stability.

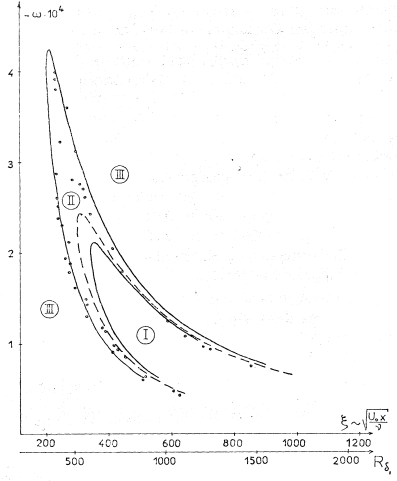

Figure: Results of Bouthier (1973) compared with data of Barry and Ross (1970)

- Particularly good agreement between experiment and energy flux neutral stability curves based on the curve bounding region III.
- However, Bouthier's results are comparing data taken by measuring the perturbation in u, and theory is based on $u^2 + v^2$.

Fasel and Konzelmann's results

- The authors solved the complete Navier-Stokes equations for a superimposed disturbance on the flat plate boundary layer [8].
- Objective was to investigate the non-parallel effects to compare with the various theoretical treatments and clarify the agreement and non-agreement with the experimental data.
- Results of CFD were compared to Gaster (1974), Saric and Nayfeh's (1975) and other non-parallel theory results.

Results

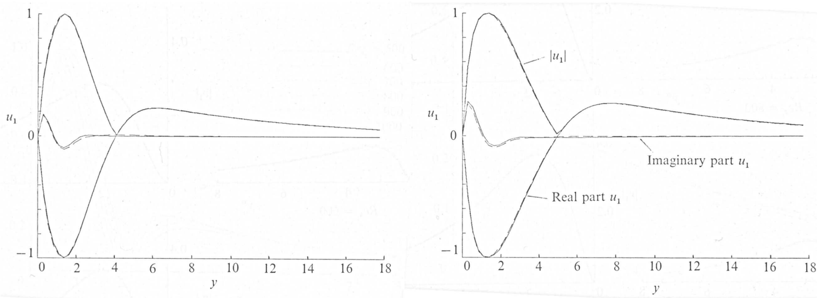

Figure: Left: Amplitude function for u for $Re_{\delta_1} = 600$. Right: The same for $Re_{\delta_1} = 800$. (−−)-Parallel stability theory. (−)-Numerical NS result.

Results

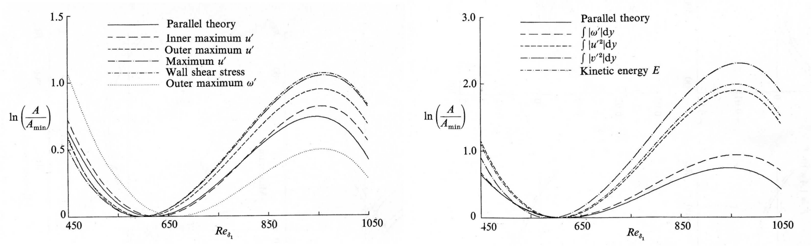

Figure: Amplification curves based on numerical NS calculation and various quantities and linear stability theory (−).

Results

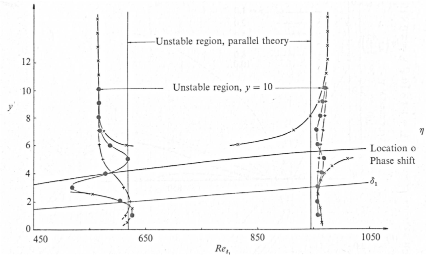

Figure: Results for neutral stability points as a function of y and flow quantities u, v and $E = u^2 + v^2$ for $F = 1.4 \times 10^{-4}$.

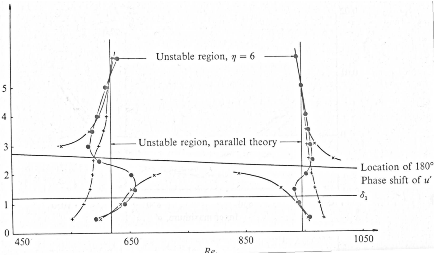

Figure: Results for neutral stability points as a function of $\eta = y/(2x)^{1/2}$. ($\times$) - u, ($+$) - v, and ($\bullet$) - E.

Results

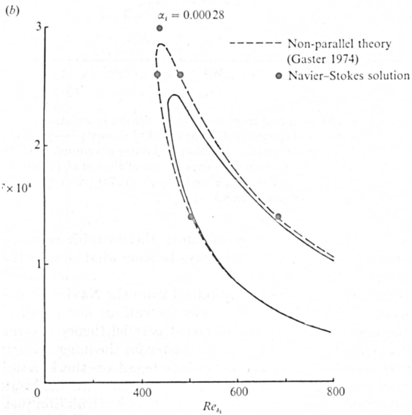

Figure: Neutral stability curve for the integral $\int u^2 + v^2 dy$

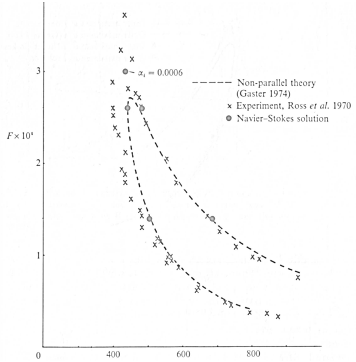

Figure: Neutral stability curve comparison of numerical, Gaster (1974) and data of Ross *et al* (1970).

Results

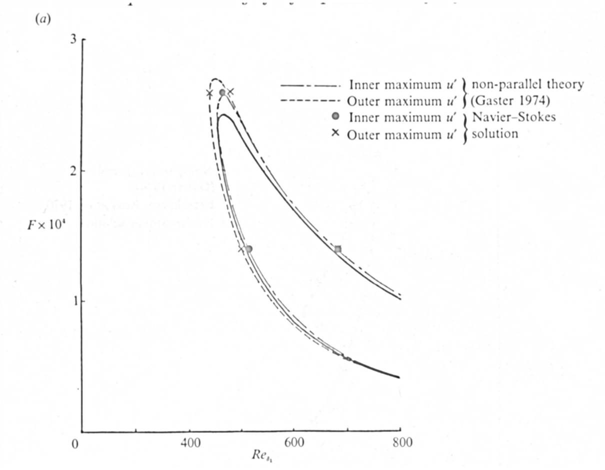

Figure: Neutral stability curves based on inner and outer maximum of u.

Conclusions

- The moral of the story is to have care when interpreting results derived from theory.
- Introduction of non-parallel effects produces ambiguity when determining neutral stability that is abset when we consider fully parallel theory.
- It can depend on the flow quantity through the streamwise variation of said quantity and the vertical position being used.
- Gaster's and Fasel and Konzelmann's results show that the nonparallel theory cannot account for all of the discrepancy especially at high frequencies and Reynolds numbers close to criticality.
- Also, locally parallel theory predicts amplitude functions (ζ) very well as per F & K (1990). It reaffirms that the parallel assumption is a good starting point for corrections due to non-parallel effects.

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