Qualifying Exam Review

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`Based on M. Tabor's Principles and Methods in Applied Mathematics`

1 Dynamical Systems

1.1 Fixed point analysis

Types of fixed points (obtained from linearized theory). First real and distinct eigenvalues:

- if $\lambda_1 > 0, \lambda_2 > 0$, then we have **unstable node**.
- if $\lambda_1 < 0, \lambda_2 < 0$, then we have **stable node**.
- if $\lambda_1 > 0, \lambda_2 < 0$, then we have **hyperbolic fixed point (saddle)**.
- if $\lambda_1 = \lambda_2 > 0$, then we have **improper stable node**, or if $\lambda_1 = \lambda_2 < 0$ the **improper stable node**.

For complex eigenvalues we define the following terminology for $a, b > 0$:

- if $\lambda_{1,2} = -a \pm ib$ then we have **stable spiral**.
- if $\lambda_{1,2} = a \pm ib$, then we have **unstable spiral**.
- if $\lambda_{1,2} = \pm ib$, then we have **elliptic fixed point (center)**. The reliability of this linear analysis in this case is not really set in stone.

1.2 Homoclinic-Heteroclinic orbits

For **heteroclinic** or **homoclinic** orbits we have a conservative system in the form

$$u_x = v, v_x = f(u)$$

where $f(u)$ can be easily integrated, then we have the reduced system

$$u_{xx} = -\frac{\partial}{\partial u}V(u) = f(u)$$

for these types of systems it is easy to show that there is a conserved quantity E: $u_x^2/x + V(u)$. The minima and maxima of $V(u)$ represent the fixed points of the system. When one attains a specific energy E that depends on the initial conditions, then one can obtain separatices, that joint hyperbolic fixed points to each other (in infinite extents of time), i.e.

- For **heteroclinic** we join two different hyperbolic fixed points.
- For **homoclinic** we join the same fixed point to itself.

To obtain these orbits start with an initial condition from on the hyperbolic fixed point and solve the conservation equation by separation of variables.

1.3 Waves

We can introduce the so-called traveling wave reduction to PDE's i.e. $z = x - ct$, and obtain a ODE system which we may analyze if possible within the framework of the potential framework. It might not work.

Also, we should keep in mind the usefulness of linearizing equations and introducing the form $u \propto \exp(i(kx + \sigma(k)t))$ to obtain some information about solution behavior for certain values of k. The function $\sigma(k)$ is the dispersion relation. The phase speed is simply $c_p = \sigma(k)/k$ and the group velocity is

$$c_g = \frac{d\sigma}{dk}$$

The larger values of k represent waves that oscillate quickly, i.e. responsible for short term behavior. Conversely, for small values of k we obtain long term behavior of the solution.

2 Fourier Series

2.1 The coefficients

An important tool for decomposing periodic functions over finite domains is Fourier series. This is natural for functions in $L^2[a, b]$ for instance because it is a Hilbert[1] space and thus a guaranteed orthogonal basis. The recipe for the coefficients goes like

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

We can change variables pretty easy to these coefficients over say $[-1, 1]$ or whatever. Basically we replace the 'frequency' to $n\pi/(b-a)$. There is also a complex version

$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

The representation in the L^2 sense of the function is therefore either

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

or

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Therefore we can expect that the expansion preserves its energy or L^2 norm in this case. It follows that

$$||f||_2^2 = \sum |c_n = (f, \varphi_j)|^2$$

where $(,)$ represents the inner prod, and φ_j represent a member of a countable basis.

[1] a complete normed linear space whose norm is derived from the inner product

2.2 Theory: Riemann-Lebesgue Lemma etc.

An important result that tells us that for $f \in L^1$ that [2]

$$\lim_{n\to\infty} a_n = 0, \ \lim_{n\to\infty} b_n = 0$$

There is a result relating the smoothness of the space with the the decay of the Fourier coefficients:

$$|c_k| \leq C/|k|^n, |k| \geq 1$$

for $f \in C^n$.

2.3 More theory

If $f \in L^2$, then we have Parseval's theorem:

$$||f||_2^2 = 2\pi \sum_{n\in\mathbb{Z}} |c_k|^2$$

For $f \in L^2$ the partial sums converge to $f(x)$ almost everywhere.

If f is periodic, integrable and piecewise smooth on the interval of choice, then for all x,

1. $\lim_{n\to\infty} S_n(x) = f(x)$ for points of continuity of f.
2. $\lim_{n\to\infty} S_n(x) = \frac{1}{2}(f(x_+) + f(x_-))$ for points of discontinuity of f.

A related result is that if f is periodic, continuous and piecewise smooth the partial sums converge **uniformly**.

Smoothness gives us results related to the uniform convergence of the series almost everywhere in the interval:

$$f \in C^p \Rightarrow |c_k| = O(1/k^p) \Rightarrow ||f - S_n(f)||_\infty = O(1/k^{p-1})$$

This is for $p > 1$. For $p = 1$ we can get $||f - S_n(f)||_\infty \leq C/n^{1/2}$.

2.4 Application to solving equations

For linear PDE's we can define the solution as a sum of Fourier coefficients and obtain the coefficients subsequently.

3 Fourier Transforms

3.1 Definition

A tool to decompose functions into an *infinite* basis over an infinite domain. How to define it? There are various ways. I like the following:

$$\hat{f} = F(f) = \int_{\mathbb{R}} f(x)e^{ikx}dx, \ f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(k)e^{-ikx}dk$$

[2] On closed intervals $L^2 \subseteq L^1$

3.2 Operational stuff

- $\hat{\hat{f}} = 2\pi f(-y)$
- $\overline{f(-x)} = \overline{F^{-1}[\hat{f}(u)]}$
- $\hat{\bar{f}}(-k) = \overline{F[f]}$
- $F^4[f(y)] = (2\pi)^2 f(y)$
- $F(f(ax-b)) = \frac{1}{|a|}e^{ik(b/a)}\hat{f}(k/a)$
- $\frac{\partial^n}{\partial x^n} f = F^{-1}[(-ik)^n \hat{f}(k)]$
- $\frac{\partial^n}{\partial k^n} \hat{f} = F[(ik)^n \hat{f}(k)]$

3.3 Examples

Simple transforms:

$$\boxed{F(1/(a^2+x^2)) = \frac{\pi}{a}e^{-|k|a}, \ F(e^{-bx^2}) = \sqrt{\frac{\pi}{b}}e^{-k^2/4b}}$$

$$\boxed{F(e^{a|x|}) = \frac{2a}{x^2+a^2}, \ F(\mathrm{sech}(x)) = \pi\mathrm{sech}(k\pi/2)}$$

3.4 Riemann-Lebesgue lemma, etc...

- For $f \in L^1$, we know that
$$\lim_{|k|\to\infty} \hat{f}(k) = 0$$
- We also know that if $f \in L^1$ then $\hat{f} \in C(\mathbb{R})$.
- If $f \in L^2$, $\hat{f} \in L^2$. More generally we have **Parseval's theorem**:
$$||f||_2^2 = \frac{1}{2\pi}||\hat{f}||_2^2$$
- We have the nice result that
$$(f, \hat{g}) = (\hat{f}, g)$$
Also, we have the pointwise result that if $f \in L^1$ and piecewise continuous then the inverse transform converges pointwise and to the average of the left and right hand at points of discontinuity, i.e.
$$\lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \hat{f}e^{-ikx}dk = \frac{1}{2}(f(x_+) + f(x_-))$$
- Convolution has the nice property that in Fourier space:
$$h(x) = f * g(x) = \int_{\mathbb{R}} g(x-y)f(y)dy, \ \hat{h} = \hat{f}\hat{g}$$
We note that $f, g \in L^1 \mapsto h \in L^1$, and $f \in L^1, g \in L^2 \mapsto h \in L^2$.
- **Fourier mapping** for $f \in L^p$, i.e. for $1 \leq p \leq 2$, then the transform maps into the dual, i.e.
$$f \in L^p, \hat{f} \in L^q, q^{-1} + p^{-1} = 1.$$

4 Contour Integration

4.1 Parametrization

We can parametrize a contour integral,

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

where $C_{start} = a$ and $C_{end} = b$.

4.2 Cauchy's Theorem

$$\oint_C f(z)dz = 0$$

for f analytic and so satisfying, $u_x = v_y$ and $u_y = -v_x$. The contour C must be a Jordan contour (simple closed contour).

As a consequence, for and integral of the form,

$$I = \frac{1}{2\pi i}\oint_C \frac{dz}{(z-a)^m} = \begin{cases} 1 \text{ if } m = 1 \\ 0 \text{ if } m \neq 1 \end{cases}$$

4.3 Cauchy's Integral Formula

We have for a Jordan contour C,

$$f(z) = \frac{1}{2\pi i}\oint_c \frac{f(\xi)}{\xi - z}d\xi.$$

f must be analytic for the domain D enclosed by C. More generally we can write,

$$f^{(k)}(z) = \frac{k!}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^{k+1}}d\xi$$

An important result is that f has infinitely many derivatives so the Taylor series converges. The following are equivalent to saying f is analytic in a domain, D:

- f has a uniformly convergent Taylor series for all neighborhoods of $a \in D$.
- f has a derivative
- $\oint f(z)dz = 0$

4.4 Branch Cuts

Branch cuts is the domain where an inverse problem is not analytic. E.g. for $w^2 = z$, there is a discontinuity. Branch cuts are not unique while the branch points are. Branch points are points where a 2π circuit produces multi-valuedness. In example previously cited, the branch points occur at $z = 0$ and $z = \infty$. Branch cuts connect the branch points.

4.5 Poles and Residues

- (i) If $f(z)$ is analytic for $0 < |z - z_0| < R$ and not at z_0, then the point $z = z_0$ is an isolated singular point of $f(z)$.
- (ii) If $z = z_0$ is an isolated singular point and if $f(z)$ is bounded, then $z = z_0$ is a removable singularity.

4.5.1 Laurent's Series

We can write any such function as

$$f(z) = \sum_{n=-\infty}^{\infty} C_n(z - z_0)^n$$

Using the geometric series we can show that we get the following convergence,

if $|z| > 1$ $\dfrac{1}{1+1/z} = \sum_{n=0}^{\infty}(-1)^n(1/z)^n$ and if $|z| < 1$, $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

Under a contour integration, we know that the only surviving term of a Laurent expansion is $n = 1$, so it's coefficient is therefore,

$$C_{-1} = \lim_{z\to z_0} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[f(z)(z-z_0)^k\right]$$

where k is the multiplicity of the pole.

4.6 Cauchy's Residue Theorem

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{N} r_j$$

for N poles and r_j is the residue of the pole.

Jordan's Lemma

An integral of the form

$$I = \int_{-\infty}^{\infty} f(x)e^{ikx}dx$$

can be done by closing the path in the upper half plane and equating the result to the residues of that particular contour. It is a general result that:

- Suppose $f(z) \to 0$ uniformly and as $R \to \infty$ on C_R (upper half circle)

$$|f(Re^{i\theta})| \leq G(R) \text{ and } \lim_{R\to\infty} G(R) = 0$$

- Then we have the consequence that,

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{ikz}dz = 0, \;\; k > 0.$$

4.7 Principal value integral

For the integral

$$PV\int_{-\infty}^{\infty} e^{ikx}/xdx = i\pi\text{sign}(k)$$

4.8 Contour integrations of matrix functions

We can write that

$$f(\mathbf{A}) = \frac{1}{2\pi i}\oint \frac{f(z)}{z\mathbf{I}-A}dz$$

Introducing a matrix of the type

$$A = \lambda\mathbf{I} + \mathbf{E} = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

or its higher degree analogue produces

$$f(\mathbf{A}) = \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(\lambda)\mathbf{E}^k$$

since $E^n = 0$ we can terminate the addition (and via the residue theorem.

5 Distributions

5.1 The delta function

This function has the well known property of constant mass but is discontinuous, i.e.

$$\int_{\mathbb{R}} \delta(x)f(x)dx = f(0)$$

There are various sequences that give us the delta function in the distributional sense:

$$S_n(x) = \begin{cases} n, & |x| < 1/2n \\ 0, & |x| > 1/2n \end{cases}, \quad S_n(x) = \sqrt{\frac{n}{\pi}}e^{-nx^2},$$

$$S_n(x) = \frac{1}{\pi}\frac{n^2}{1+n^2x^2}, S_n(x) = \sin(nx)/\pi x$$

Here are some operational stuff about delta functions:

$$\delta(ax-b) = \frac{|a|}{\delta}(x-b/a), \delta(f(x)) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|}\delta(x-x_i)$$

Finally for 3 dimensions, we have that

$$\delta(\mathbf{r}) = \frac{1}{4\pi r^2}\delta(r),\ \delta(\mathbf{r}) = \delta(z)\delta(r)/2\pi r, \delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

for spherical, cylindrical and Cartesian coordinatesc.

5.2 Fourier transform of distributions

It is given as a result of the Parseval's relation:

$$T_{\hat{f}}(\phi) = T_f(\hat{\phi})$$

Note that this is only really properly defined in the sense of $S(\mathbb{R})$.

5.3 Other examples of distributions

5.3.1 Principal values

The principal value exists for continuous functions φ such that $\sup_{\mathbb{R}} |x\varphi(x)| < K < \infty$ and $\sup_R |\varphi'(x)| < K_1 < \infty$. Then,

$$PV\frac{1}{x}(\varphi) = \int_{|x|>1} \varphi(x)/xdx + \int_{-1}^{1} \frac{\varphi(x)-\varphi(0)}{x}dx$$

and likewise for suitable hypothesis on φ,

$$PV\frac{1}{x^2}(\varphi) = \int_{|x|>1} \frac{\varphi(x)-\varphi(0)}{x^2}dx + \int_{-1}^{1} \frac{\varphi(x)-\varphi(0)-x\varphi'(0)}{x^2}dx$$

etc... for $PV\frac{1}{x^n}$.

Also we can get the principal values using the results that

$$\frac{1}{x+i0} = -i\pi\delta + PV\frac{1}{x}, \text{ and } \frac{1}{x-i0} = i\pi\delta + PV\frac{1}{x}$$

5.4 Test function spaces

The following classes of functions are in 'size' order, i.e. there are the smooth functions contain the next two and there are obviously smooth functions not in S or D.

We define $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}$ on functions in $\mathbb{R}^n$ with $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_n)$.

5.4.1 Smooth functions or $C^\infty(\mathbb{R}^n)$

This is the linear space of complex valued infinitely differentiable functions on $\mathbb{R}^n$ such that $D^\alpha\varphi$ all exists for all n-tuples α.

Examples are e^{x^2} or $\sin(x)$.

Convergence

We define the following semi-norms

$$p_{m,k}(\varphi) = \max_{|\alpha|\le k} \sup_{|x|\le m} |D^\alpha\varphi|$$

We say a function $\phi_j \to \phi$ converges in this space if for all $\epsilon > 0, K, k \in \mathbb{N}$, such that for $j > M_{\epsilon,k,K}$

$$p_{K,k}(\phi_j - \phi) < \epsilon$$

5.4.2 Schwartz functions or $S(\mathbb{R}^n)$

These are smooth functions φ that have the property that

$$\lim_{|x|\to\infty} |x|^k |D^\alpha\varphi(x)| = 0$$

so decay faster than any power of x. Examples include $\exp(-x^2)$ or $\exp(-|x|^2)$ where $|x| = ||x||_2$.

Convergence

Define the following seminorms on $\phi \in S(\mathbb{R})$:

$$q_{m,k}(\phi) = \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} |x|^m |D^\alpha \varphi|$$

We say that $\phi_j \to \phi$ in $S(\mathbb{R})$ if

$$\lim_{j\to\infty} q_{m,k}(\phi_j - \phi) = 0$$

for all m, k.

5.5 Compactly supported functions or $C_0^\infty(\mathbb{R}^n)$

These functions have a closed region of support i.e

$$\text{supp}(\varphi) = \overline{\{x \in \mathbb{R}^n | \varphi(x) \neq 0\}}$$

which is the smallest closed set outside which $\varphi = 0$.

Therefore the compactly supported functions denoted either $C_0^\infty(\mathbb{R}^n)$ or $D(\mathbb{R}^n)$ are smooth functions with compact support i.e. its compact support is contained in some ball $\text{supp}(\varphi) = x||x| \leq R$.

Note: However, this does not mean these functions are analytic although the partial's all exist. It is a peculiar property to using the real line as the domain (or an n-tuple).

Convergence

A $\phi_j \to 0$ in $D(\mathbb{R}^n)$ if the following two conditions hold:

1. $\exists R$ such that $\phi_j(x) = 0$ when $|x| > R$ for all j.
2. For each k, the seminorms $p_{R,k}(\phi_j) \to 0$ as $j \to \infty$.

So for a general $\phi_j \to \phi$ then their difference must satisfy the above.

Note: The first two function spaces have countable seminorms which implies that we can a metric so this implies both these spaces are metrizable. However, for compactly supported functions the accompanying seminorm is not countable and therefore we do not have metrizable space but a 1st-countable one though.

5.6 Generalized functions

5.6.1 Definition of distribution

A distribution on a test function space χ is a linear, continuous (with respect to the topology on χ) real-valued operation on the test function space. This space is denoted χ'.

- $T(af + bf) = aT(f) + bT(f)$
- T is continuous w.r.t. the topology on χ.

The last property can be tested through the use of sequences of functions in χ, i.e. the distribution is continuous iff

$$\phi_j \to \phi \Rightarrow T(\phi_j) \to T(\phi) \text{ for } \phi_j \in \chi.$$

Further this can be probed by simply considering $\phi_j \to 0$ since T is a linear operation.

5.6.2 Convergence in distribution space

A distribution T_j in χ' converges to T iff only $T_j(\phi) \to T(\phi)$.

Given a continuous or smooth function F we can define an associated distribution $T_F \in D(\mathbb{R})'$ i.e.

$$T_F(g) = \int_{\mathbb{R}} F(x)g(x)dx$$

can be shown to be smooth. Likewise we may do this for $f \in D$ and get that T_f is distribution in $(C^\infty(\mathbb{R}))'$. As for $g \in S(\mathbb{R})$, we get tempered distribution T_g.

5.6.3 Derivatives of distributions

Through integration by parts we arrive at the results that in the sense of distribution we have that

$$(T_F(f))' = -T_F(f')$$

and continuity of the derivatives

$$T_j \to T \Rightarrow T_j' \to T'$$

5.7 Application: Green's function

An application for theory of generalized functions is Green's functions, i.e. Suppose we want to solve $u'' = -f$ with boundary conditions $u(1) = u(0) = 0$ then defining $G''(x, y) = \delta(x - y)$ can lead to a Green's function solution i.e.

$$u(x) = -\int_0^1 G(x, y)f(y)dy$$

satisfied the BC's and that the equation. We can also obtain G by a sequence that gives $G_j''(x, y) \to \delta(x - y)$.

6 Spectral theory

6.1 The setting

We study linear operators on a Hilbert space setting (usually). An operator norm is like

$$||L|| = \sup_{w\in X} \frac{||Lu||}{||u||}$$

A **bounded operator** is such that $||L|| < \infty$. For a bounded linear operator, we can show that

$$Lf = (f, u),\ u \in H$$

via the Riesz Representation Theorem. If L is a bounded linear operator then

- L^* exists,
- and it is a linear operator.

such that we have

$$(Lu, v) = (u, L^*v)$$

6.2 Theoretical results and definitions

- So a **normal** operator is such that $L^*L = LL^*$.
- A self-adjoint operator says $L^* = L$.
- A skew-adjoint operator is such that it is normal and $L^* = -L$.
- A hermitian operator is such that $L^* = L^{-1}$.

The definition of the spectrum of the operator relies on the resolvent set which is defined with respect to the resolvent operator:

$$R_\lambda(L) = (L - \lambda I)^{-1}$$

The spectrum is the complement of the set $\rho(L)$ of λ such that

- $R_\lambda(L)$ exists,
- $R_\lambda(L)$ is bounded,
- $R_\lambda(L)$ is defined on a set that is dense in X.

These values are known as regular values.

For the spectrum then, we have that three disjoint sets:

- **Point spectrum** Set where $R_\lambda(L)$ does not exist.
- **Continuous spectrum** Set where $R_\lambda(L)$ exists, and is defined on a dense set, but is **not bounded**.
- **Residual spectrum** Set where the resolvent operator is not defined on a dense set but it exists and may or not be bounded.

6.3 Theorems

- If L is normal (in a Hilbert space) then if λ is in the point spectrum then $\overline{\lambda}$ is in the point spectrum of the adjoint.
- If λ is in the residual spectrum then $\overline{\lambda}$ is in the point spectrum of L^*.
- Self-adjoint operators do not admit residual spectrum.
- The spectrum of a bounded linear operator on a (complex) Banach space is compact and lies on a disc given by

$$||\lambda|| \leq ||L||$$

- A compact linear bounded operator has a countable set of eigenvalues.
  - Every $\lambda \in \sigma(L)$ is an eigenvalue of L and $\lambda = 0$ is the only possible accumulation point in the spectrum.

7 Green's functions

In order to solve inhomogeneous systems of differential equations, the Green's function is a powerful tool. Suppose the problem is given by

$$Lu = f, \text{ and BC's}$$

then we can define the solution from the distributional solution $LG(x, y) = \delta(x - y)$ and imposing continuity at the parameter y as well as a unit jump in the derivative of the G at y:

$$\left.\frac{dG}{dx}\right|_{y_-}^{y+} = \frac{1}{p(y)}, \text{ where } L = p(x)\frac{d^n}{dx^n} + \ldots ...$$

Therefore we we can write:

$$u(x) = \int G(x, y)f(y)dy$$

A more general treatment basically says that the $n-1$ derivative must be discontinuous across the jump and all lower order derivative must be continuous at the jump.

Theoretical results

In general we have the relation between direct and adjoint Green's functions

$$G(x, y) = G^*(y, x)$$

If $L^* = L$ then we have further that

$$G(x, y) = G(y, x)$$

7.1 Solving when $(v, f) \neq 0$ and $L^*v = 0$

In this case we must subtract out the part of f that is not in the *range* of L. In this case we proceed as follows:

- Satisfy $LG = \delta(x-y) - \sum_{i=1}^n u_i(x)u_i(y)$ where u_i are the non-trivial nullspace functions of $Lu_i = 0$.
- $\frac{d^{n-1}}{dx^{n-1}}G\big|_{y_-}^{y+} = \frac{1}{p(y)}$ for order n differential operator with coefficient function $p(x)$.
- Impose continuity of $G^(k)(x, y)$ for $k = 0, 1, \ldots n - 2$.
- Impose that $\int_a^b K(x, y)u_i(x)dx = 0$ for $i = 1 \ldots n$.
- Impose BC's.

7.2 Higher dimension Green's functions

7.3 Useful trick

We have the result that

$$\int_a^x dx_n \int_a^{x_n} dx_{n-1} \dots \int_a^{x_2} h(x_1)dx_1 = \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1}h(y)dy$$

8 Integral equations

There are several classifications. First variable limits are called **Volterra** equations, and **Fredholm** type deal with fixed limits. Within this classification there are two types:

- $u(x) = \int_a^x K(x,y)f(y)dy$ - Volterra eq. of the first kind.
- $u(x) = f(x) + \int_a^x K(x,y)u(y)dy$ - Volterra eq. of the second kind.
- $u(x) = \int_a^b K(x,y)f(y)dy$ - Fredholm eq. of the first kind.
- $u(x) = f(x) + \int_a^b K(x,y)u(y)dy$ - Fredholm eq. of the second kind.

We can also introduce a parameter λ to the second Fredholm type integral which leads to a discussion of spectra.

8.1 Separable kernels

Cases which $G(x,y) = \sum_{i=1}^n g_i(x)f_i(y)$ are more simple and reduce to linear algebra problems. Cases where we obtain an answer of the form $u(x) = f(x) + f(\lambda)g(x)$ have several scenarios:

- If $f(\lambda) \neq 0$ then we have a unique solution.
- If $f(\lambda) \to \infty$, we no solution.
- If $f(\lambda) \to 0/0$ then we have non-unique solution where we have arbitrary multiples of $g(x)$.

9 Linear algebra

9.1 Jordan canonical form

To obtain the generalized eigenvectors we use that

$$(A - \lambda I)^p v = 0$$

for $p = 1$, this implies the vector v is an eigenvector. For higher p, we obtain generalized eigenvectors. For the matrix

$$A_1 = \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{pmatrix}$$

The number of ones that occur on the Jordan block of the eigenvalue represents the number of generalized eigenvectors needed to complete the *contribution* from each eigenvector.

9.2 Trace

Trace is additive. The trace of $Tr(QAQ^{-1}) = Tr(A)$

9.3 Fredholm alternative for matrices

For the matrix equations $Ax = b$ and $A^*y = c$ (although it works for bounded linear operators),

- if $Null(A) = 0, Null(A^*) = 0$ then the homogeneous problems always have solutions.
- if there exists the same number (n) of non-trivial homogeneous solutions for $A^*(y_i)$ and A (x_i) then no solution will exist unless

$$(x_i, c) = 0\&\ (y_i, b) = 0, \forall i = 1 \dots n.$$

i.e. $Ax = b$ has a solution if $A^*b = 0$.