

Biorthogonal eigenfunction expansion: a study case or case study, whichever.

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The Brown Bag

Motivation ×3

- ▶ For the linearized Navier-Stokes equations, we can think of the solution a superposition of *normal modes* of continuous and discrete spectra.
- ▶ The problem arises of decomposing numerical data into these modes to gain insight into the mechanism behind the flow.
- ▶ We want a method that can gives us the amplitudes of unstable and stable modes.

The biorthogonal eigenfunction system

- ▶ A possible method to resolve this issue is known as the **biorthogonal eigenfunction system** (BES) expansion.
- ▶ To show the *algorithm* of its application it is hopefully helpful to consider a simplified case.
- ▶ This study case will show the steps of the BES decomposition technique with all steps performed analytically.
- ▶ The structure of this presentation is as follows:
 1. Study problem solution by traditional means
 2. Develop the BES system and obtain solution
 3. Show that the approaches are equivalent
 4. Present numerical example of receptivity problem

The test problem

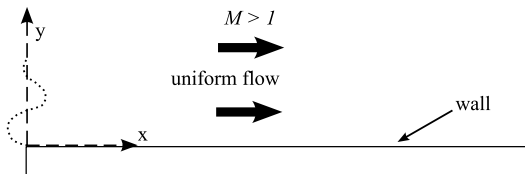


Figure: Uniform flow is in the x direction and there is periodic-in-time initial data $\mathbf{A}_0(y)$ along $x = 0$.

We consider supersonic, uniform, inviscid flow past an infinite flat plate for air $\gamma = 1.4$. The normal coordinate is y and the streamwise coordinate is x . There is also periodic in time initial data at $x = 0$.

The (governing) Euler equations of fluid mechanics

The Euler equations refer to a system of partial differential equations that are derived from conservation of mass, momentum and energy.

$$\begin{cases} D\rho'/Dt + \rho'\nabla \cdot \mathbf{u}' = 0, & \text{(the continuity equation),} \\ \rho' D\mathbf{u}'/Dt + \nabla p' - \rho'\mathbf{f} = 0, & \text{(momentum equation),} \\ \rho D e'/Dt + p\nabla \cdot \mathbf{u}' = 0, & \text{(energy equation).} \end{cases} \quad (1)$$

$D/Dt = \partial/\partial t + \mathbf{u}' \cdot \nabla$ (the material derivative).

The linearization

- ▶ Linearize using the form $q(x, y, t) = q_\infty + \exp(-i\omega t)q(x, y)$.
- ▶ Supposing a ideal gas and through the linearized equation of state can obtain:

$$-i\omega(\gamma M^2 p - T) + \frac{\partial}{\partial x}(\gamma M^2 p - T) + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$-i\omega u + \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}, \quad -i\omega v + \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}$$

$$-i\omega T + \frac{\partial T}{\partial x} = (\gamma - 1)M^2 \left(-i\omega p + \frac{\partial p}{\partial x} \right)$$

where we have scales time L/U_∞ , length scale L , velocity scale U_∞ , pressure scale $\rho_\infty U_\infty^2$, and temperature scale T_∞ .

Reformulation of the Euler equations

- ▶ The system has 4 equations and 4 unknown functions of x and y .
- ▶ The equations can be reformulated in terms of the vector function $\mathbf{A} = (u, v, p, T)^T$ and three 4×4 constant coefficient matrices H_1, H_2 and E .

$$E \frac{\partial \mathbf{A}}{\partial y} = H_1 \mathbf{A} + H_2 \frac{\partial \mathbf{A}}{\partial x}$$

- ▶ This form facilitates solving the system.

Boundary conditions and initial value problem

The boundary conditions using the new vector notation are as follows:

- ▶ $y = 0, A_2 = 0.$
- ▶ $y \rightarrow \infty, |A_j| \rightarrow 0$ for $j = 1, 2, 3$ and $4.$
- ▶ $x = 0, \mathbf{A}(0, y) = \mathbf{A}_0(y).$

and we require that the $\mathbf{A}_0(y)$ be localized disturbances in $y.$
This defines the initial value problem in x that we refer to as the study case.

The Laplace solution

- ▶ The system of partial differential equations are linear with constant coefficients.
- ▶ Suggests the Laplace transform of the streamwise coordinate:

$$\mathbf{A}_s(y) = \int_0^{\infty} \mathbf{A}(x, y) e^{-sx} dx$$

which is defined assume that the perturbations have a finite growth rate in x (a healthy assumption).

The ODE system

- ▶ The transformation of the x -coordinate leads to a system of inhomogeneous linear ODE's that we know how to solve explicitly:

$$\begin{aligned} E \frac{d\mathbf{A}_s}{dy} - H_1 \mathbf{A}_s - sH_2 \mathbf{A}_s &= \mathbf{F} \\ y = 0 : \quad A_{s,2} &= 0 \\ y \rightarrow \infty : \quad |A_{s,j}| &\rightarrow 0 \end{aligned} \tag{2}$$

where $\mathbf{F} = H_2 \mathbf{A}_0(y)$.

- ▶ We introduce solutions of the form $\mathbf{z} \exp(-\lambda y)$.

The fundamental solutions

- ▶ It leads to matrix equation

$$(E\lambda - H_1 - i\alpha H_2)\mathbf{z} = 0$$

- ▶ It leads to the condition:

$$\lambda = \pm \sqrt{M^2(s - i\omega)^2 - s^2} = \pm \mu(s)$$

- ▶ As a result we get fundamental vectors $\mathbf{z}_1, \mathbf{z}_2$,

$$\mathbf{z}_1 = \left(-\frac{s}{s - i\omega}, \frac{\mu}{s - i\omega}, M^2(\gamma - 1), 1 \right)^T,$$

$$\mathbf{z}_2 = \left(-\frac{s}{s - i\omega}, -\frac{\mu}{s - i\omega}, M^2(\gamma - 1), 1 \right)^T.$$

The full solution

- ▶ Using the fundamental solutions we form the matrix $M = [\mathbf{z}_1 \exp(-\mu y), \mathbf{z}_2 \exp(\mu y)]$ and write our full solution in the form,

$$\mathbf{A}_s(y) = M\mathbf{Q} + \mathbf{G}(y)$$

where \mathbf{Q} is determined from a differential relation and boundary conditions, and \mathbf{G} results from an algebraic one.

- ▶ The full solution is therefore:

$$\mathbf{A}_s(y) = \left(\int_0^y C_1(y'; s) dy' - c_1(s) \right) \mathbf{z}_1 e^{-\mu(s)y} + \left(\int_{\infty}^y C_2(y'; s) dy' \right) \mathbf{z}_2 e^{\mu(s)y} + \mathbf{G}(y)$$

Definitions

We make the following definitions:

$$c_1(s) = \int_0^{\infty} C_2(y'; s) dy',$$

$$C_1(y; s) = \frac{e^{\mu y}}{2\mu} [\mu v_0 - i\omega u_0 + ([M^2(s - i\omega) - s] p_0)],$$

$$C_2(y; s) = \frac{e^{-\mu y}}{2\mu} [\mu v_0 + i\omega u_0 - ([M^2(s - i\omega) - s] p_0)],$$

$$\mathbf{G}(y) = \left(\frac{p_0 + u_0}{s - i\omega}, 0, -\frac{M^2(\gamma - 1)p_0 - T_0}{s - i\omega}, 0 \right)^T.$$

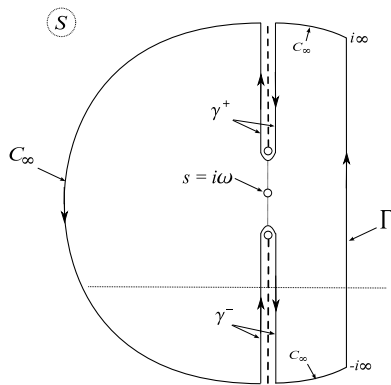
Complex plane of s 

Figure: The complex plane of s with branch cuts as shown and pole at $s = i\omega$.

- ▶ The formal or Laplace solution is therefore

$$\mathbf{A}(x, y) = \frac{1}{2\pi i} \int_{\Gamma} \mathbf{A}_s(y) e^{sx} ds$$

- ▶ Defining $\sqrt{re^{i\theta}} = |r|^{1/2} e^{i\theta/2}$, produces branch cuts γ^- and γ^+ .
- ▶ There is a pole at $s = i\omega$.

Cauchy's Residue Theorem

- ▶ Through Cauchy's residue theorem as a sum of residues and branch cut integrals,

$$\mathbf{A}(x, y) = \sum_n \text{Res}(\mathbf{A}_s e^{sx}) - \frac{1}{2\pi i} \left(\int_{\gamma^-} \mathbf{A}_s e^{sx} ds + \int_{\gamma^+} \mathbf{A}_s e^{sx} ds \right).$$

- ▶ We can further simplify the branch cut integrals to two integrals over the real line.

- ▶ The initial value problem solution is superposition of terms $g(y, s) \exp(sx + \mu(s)y)$.
- ▶ We rewrote as sum of residues and integrals over branch cuts.
- ▶ However it is difficult to obtain any physical insight from the solution in terms of normal modes.

The definition of the direct problem

The BES is derived from analysis of the so called direct problem:

$$E \frac{d\mathbf{A}_\alpha}{dy} - H_1 \mathbf{A}_\alpha - i\alpha H_2 \mathbf{A}_\alpha = 0$$

$$y = 0 : A_{\alpha,2} = 0$$

$$y \rightarrow \infty : |A_{\alpha,j}| < \infty$$

As $y \rightarrow \infty$ we need a bounded solution.

Eigenfunctions

- ▶ Supposing solutions of the form $\mathbf{z} \exp(\lambda y)$ leads to

$$(E\lambda - H_1 - i\alpha H_2)\mathbf{z} = 0 \Rightarrow \det(E\lambda - H_1 - i\alpha H_2) = 0$$

- ▶ From the determinant condition we obtain
$$-(\alpha - \omega)^2(\lambda^2 + (M^2 - 1)\alpha^2 - 2M^2\omega\alpha + M^2\omega^2) = 0.$$
- ▶ To satisfy boundary conditions as $y \rightarrow \infty$ it is necessary to assume that $\lambda = \pm ik$ for $k \in \mathbb{R}^+$. Therefore we obtain that

$$\alpha_{1,2}(k) = \frac{M^2\omega \pm \sqrt{(M^2 - 1)k^2 + M^2\omega^2}}{M^2 - 1}$$

$$\alpha_{3,4}(k) = \omega$$

- ▶ Each α_j has an associated $\mathbf{z}_j(\lambda)$ that satisfies the matrix equation. Therefore each mode has the form,

$$\mathbf{A}_{\alpha_j} = C_{1j} \mathbf{z}_j(ik) e^{iky} + C_{2j} \mathbf{z}_j(-ik) e^{-iky}$$

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 - ▶ Mode $m = 3$ is called the **vorticity** mode since it generates rotation in the fluid.
 - ▶ Mode $m = 4$ is called the **entropy** mode since its only non-zero component is in the temperature component.

- ▶ Now construct the solution to the problem as follows:

$$\mathbf{A}(x, y) \stackrel{??}{=} \sum_{m=1}^4 \int_0^{\infty} C_m(k) \mathbf{A}_{\alpha_m}(y; k) e^{i\alpha_m x} dk$$

- ▶ Is this equality true? Firstly, how do we define $C_m(k)$? Can we simply project the eigenfunctions on $\mathbf{A}(x, y)$ to find them?

- ▶ The inner product can be defined:

$$\langle \mathbf{A}(y), \mathbf{B}(y) \rangle = \lim_{\epsilon \rightarrow 0^+} \int_0^{\infty} e^{-\epsilon y^2} (\mathbf{B}^* \mathbf{A}) dy$$

This *regularization* ensures that inner product of \mathbf{A}, \mathbf{B} above makes sense and is bounded.

- ▶ $\mathbf{A}_{\alpha_m}(y; k)$ are not orthogonal.
- ▶ We can think of the direct problem as a linear differential operator on a 4×1 -column vector of complex valued functions.
- ▶ Finally, the fancy word *biorthogonal* finally will have a use.

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- ▶ It leads to orthogonality relation $v_i \cdot w_j = Q_i \delta_{ij}$. Therefore we can *decompose* any vector in \mathbb{R}^2 say x in terms of v_1, v_2 with help of the orthogonality relation above

$$x = a_1 v_1 + a_2 v_2, \quad a_i = (x \cdot w_i)$$

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- ▶ Inner product with an adjoint eigenfunction \mathbf{B}_α , i.e.

$$\langle \mathcal{L}\mathbf{A}_\alpha, \mathbf{B}_\alpha \rangle = \left\langle E \frac{d\mathbf{A}_\alpha}{dy}, \mathbf{B}_\alpha \right\rangle - \langle H_1 \mathbf{A}_\alpha, \mathbf{B}_\alpha \rangle - \langle (i\alpha) H_2 \mathbf{A}_\alpha, \mathbf{B}_\alpha \rangle$$

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- ▶ The adjoint operator \mathcal{L}^* is defined such that $\langle \mathcal{L}\mathbf{A}_\alpha, \mathbf{B}_\alpha \rangle = \langle \mathbf{A}_\alpha, \mathcal{L}^* \mathbf{B}_\alpha \rangle$ and this operation is a continuous operation.

- ▶ It leads naturally to the definition of the adjoint problem,

$$-E \frac{d\mathbf{B}_\alpha}{dy} - H_1^* \mathbf{B}_\alpha + i\alpha H_2^* \mathbf{B}_\alpha = 0$$

$$y = 0 : \quad B_{\alpha,4} = 0$$

$$y \rightarrow \infty : \quad |B_{\alpha,j}| < \infty.$$

Leads to an associated \mathbf{B}_{α_m} for each \mathbf{A}_{α_m} .

- ▶ $\{\mathbf{A}_{\alpha_m}, \mathbf{B}_{\alpha_m}\}$ is the BES, and derive orthogonality condition:

$$\langle H_2 \mathbf{A}_{\alpha_m}(k), \mathbf{B}_{\alpha_n}(k') \rangle = Q_m(k) \delta(k - k') \delta_{m,n}$$

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- ▶ It turns out α_m , the entropy and vorticity modes will be equivalent to the branch cuts and the branch cut integrals will be equivalent to the acoustic modes:

$$\sum_{m=3,4} \int_0^{\infty} C_m(k) \mathbf{A}_{\alpha_m} e^{i\omega x} dk = \text{Res}(\mathbf{A}_s e^{sx}) \Big|_{s=i\omega}$$

- ▶ The proof boils down to the explicit evaluation of the following two integrals:

$$\int_0^{\infty} \frac{\omega k}{\omega^2 + k^2} \cos ky \sin ky' dk, \text{ \& } \int_0^{\infty} \frac{\omega^2}{\omega^2 + k^2} \cos ky \sin ky' dk$$

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¹pp. 453

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- ▶ Refer to *Table of Integrals, Series and Products*¹ by Gradshteyn and Ryzhik.
- ▶ In the case of the acoustic modes, the change of variable to the real line essentially proves the equivalence, i.e.

$$\sum_{m=1,2} C_m(k) \int_0^{\infty} \mathbf{A}_{\alpha_m} e^{i\alpha_m} dk = -\frac{1}{2\pi i} \int_{\gamma^+} + \int_{\gamma^-} \mathbf{A}_s e^{sX} ds$$

¹pp. 453

- ▶ What is the point? Mathematically showed the equivalence of textbook Laplace solution to BES approach. The physical connections were made:
 - ▶ Branch cuts \Leftrightarrow acoustic perturbations \Leftrightarrow continuous spectra.
 - ▶ Pole $s = i\omega$ \Leftrightarrow vorticity and entropy perturbations \Leftrightarrow discrete spectra.
- ▶ More importantly, developed orthogonality condition to calculate amplitude of each mode for a given flow.
- ▶ The next section deals with a specific application to numerical results.

Numerical example : receptivity problem

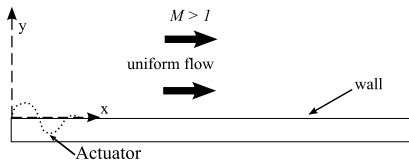


Figure: Uniform flow is in the x direction and there is periodic-in-time actuator on $y = 0$.

The boundary condition on the wall models blowing and suction (normal velocity disturbance). This problem has analytical solution²

²H. Ashley and M. Landahl, *Aerodynamics of wings and bodies*, Dover, 1965, pp. 255-257

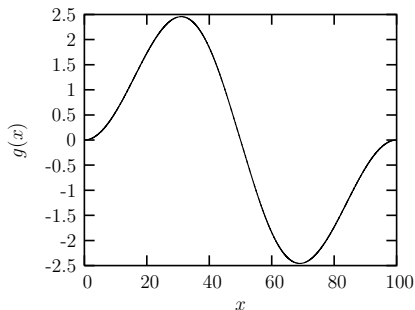


Figure: The BC on the wall goes like $v(x, y = 0, t) = \varepsilon g(x) \text{Im}(e^{-i\omega t})$.

The particular parameters are as follows: freestream Mach number $M = 6.62$; angular frequency of the perturbations $\omega = 0.1$; the actuator's length $w = 100$; and specific heat ratio $\gamma = 1.4$.

The numerical scheme gives us the entire perturbation field that is produced by the actuator.

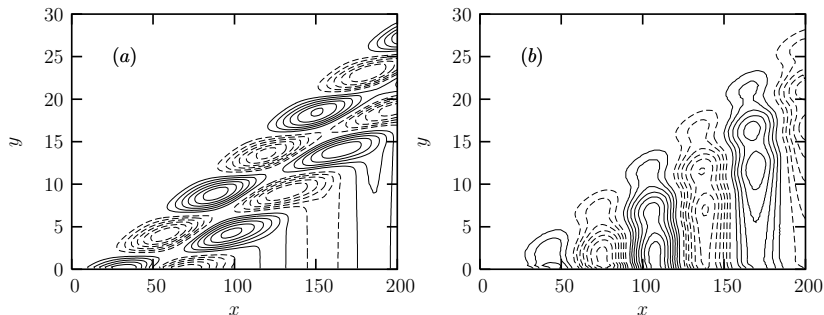


Figure: Coarse grid result for the (a) pressure perturbation (increment of $.5 \times 10^{-5}$ between the contours) and (b) streamwise velocity perturbation (increment 1×10^{-5})

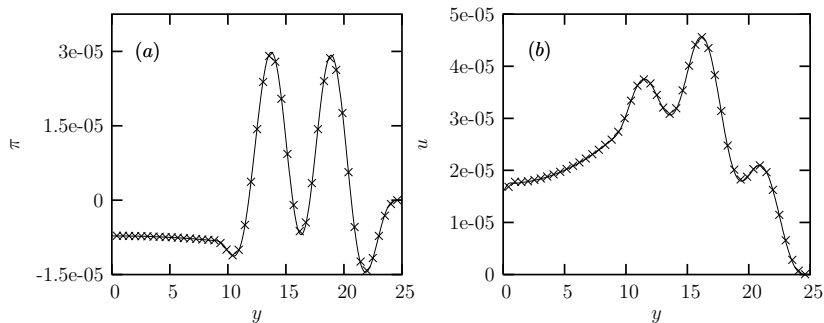


Figure: Coarse (crosses) and fine (solid-line) grid comparison for $x = 160$ and $t = 200$ for the (a) pressure perturbation and (b) streamwise velocity perturbation

The results show good agreement between coarse and fine grids.

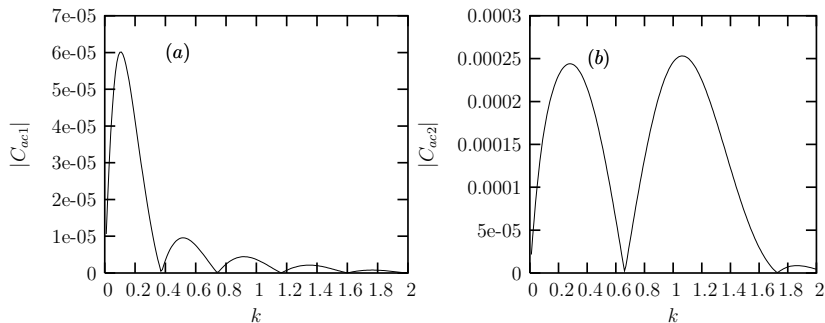


Figure: Projection result for the magnitudes of the (a) slow acoustic mode ($C_{\alpha_1}(k)$) and (b) fast acoustic mode ($C_{\alpha_2}(k)$)

The projection was obtained from the profile $x = 160$. We notice that the slow mode magnitude is smaller than the fast acoustic mode (by a factor of 10^1).

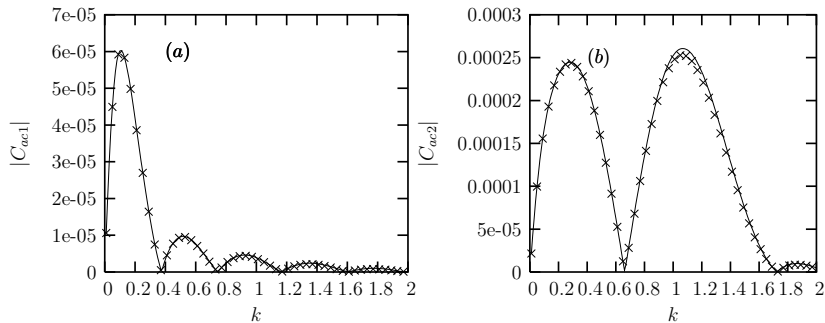


Figure: Comparison for the projection (crosses) and theoretical result (solid line) for the (a) slow acoustic mode (C_{α_1}) and (b) fast acoustic mode (C_{α_2}). The theoretical amplitude can be found

$$C_{\alpha_m}(k) = -\frac{\rho(\alpha_m(k))}{Q_m(k)} (B_{\alpha_m})_2(y=0; k).$$

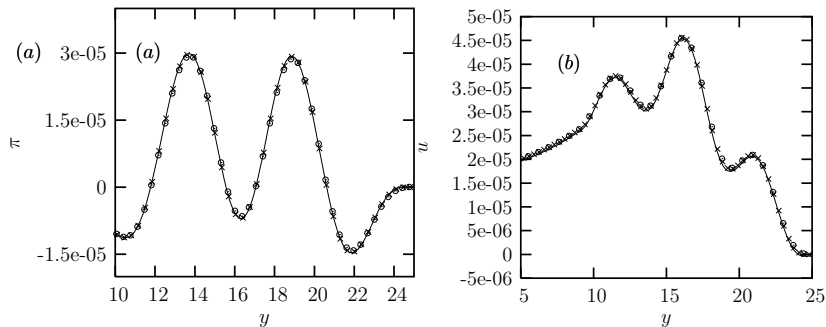


Figure: Comparison of the theory (solid-line) with the coarse (circle) and fine (cross) results for the (a) pressure and (b) streamwise velocity. Agreement is better for fine grid.

Conclusion

- ▶ The study case allows us to rigorously show the equivalence of Laplace transform (or Fourier) to the BES developed for this "simple" problem.
- ▶ The method of decomposition of data for numerical data is shown for another related problem.
- ▶ The steps in the application of the BES are all shown analytically i.e.
 1. Formulation of transformed problem, i.e. $\mathbf{A} \propto e^{i\alpha x + \lambda y}$, etc...
 2. Declare direct problem and obtain then adjoint system.
 3. Obtain orthogonality condition for decomposition.
 4. Coefficients are determined.
- ▶ Most importantly we gain physical insight into numerical and mathematical results.

Future work

- ▶ Decomposition of perturbations in idealized one-reaction detonations.

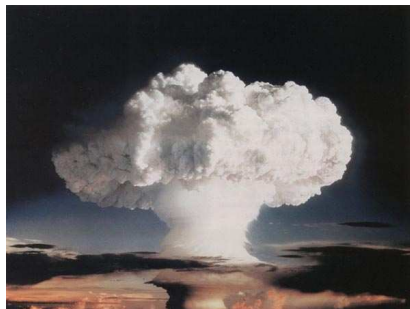


Figure: Boom!