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1 Introduction to the stress tensor

The stress vector Σ is given element wise by the relation:

$$\Sigma_i = \sum_{j=1}^{3} \sigma_{ij} n_j$$

The stress tensor is symmetric and can therefore be diagonalized since the eigenvectors are orthogonal (via similarity transform $\sigma = [a]^{-1}[\lambda][a]$). It can be made rigorous as the following:

$$\Sigma'_k = a_{ik}\sigma_{ij}a_{jl}n'_l, \; n_j = a_{jl}n'_l, \; \Sigma_i = a_{ip}\Sigma'_p$$

We can define

$$\sigma_{ij} = -\frac{1}{3}p\delta_{ij} + \underbrace{d_{ij}}_{deviatoric}$$

with $p = (1/3)\sigma_{ii}$ the pressure.

The alternating tensor is given by

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i,j,k \text{ are cyclic eg. 1,2,3...} \\ -1, & \text{if } i,j,k \text{ are a-cyclic eg. 3,2,1...} \\ 0, & \text{for repeated indices....} \end{cases}$$

The **divergence theorem** is given as

$$\int_A \mathbf{f} \cdot \mathbf{n} dA = \int_V \nabla \cdot \mathbf{f} dV$$

A counter-clockwise rotation is achieved by the matrix:

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

1.1 Static fluid case

In this case we can set

$$\mathbf{F} = -\frac{\nabla P}{\rho}$$

(assuming no deviatoric contribution).

For a conservative body force $\mathbf{F} = -\nabla\varphi$. Therefore for ρ constant $P/\rho + \varphi = f(t)$ since $\nabla(P/\rho + \varphi) = 0$.

1.2 Boundary/Matching conditions

For the interface between materials with no holes and interface is simply connected.

$$\mathbf{v}_1 \cdot \mathbf{n} = \mathbf{v}_2 \cdot \mathbf{n}$$

If viscous:

$$\mathbf{v}_1 \cdot \mathbf{t} = \mathbf{v}_2 \cdot \mathbf{t}$$

with $\mathbf{t}$ is the tangential direction at the interface. $\mathbf{n}$ is the normal vector.

1.3 Surface Tension

For the 2-D case we have the following relation

$$p_{in} = p_{out} + \frac{\gamma}{R}$$

where p_{in} refers to the pressure internal to the curvature and p_{out} is the pressure opposite this curvature. γ is the surface tension coefficient and R is the radius of curvature. The length scale of this effect is $\sqrt{\rho g/\gamma}$. For 3D case $p_{in} = p_{out} + \gamma(1/R_2 + 1/R_1)$. A particular result from mechanics tells us

$$\frac{1}{R} = \frac{\zeta''}{(1 + (\zeta')^2)^{3/2}}$$

For the case of a capillary effect at the wall we have the BC $-d\zeta/dx = \cot\theta$ at interface between wall/air/liquid.

1.4 Eulerian/Lagrangian

Lagrangian-follows fluid particles with variables $\mathbf{a}$ which is the initial condition of the fluid particle and t which is time. Eulerian is a field theory with variables $\mathbf{x}$ and t. The material derivative is given sa

$$D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$$

which is taken to be acting on a material particle, i.e. as in acceleration for the governing equations. A streamline follows the instantenious velocity field which is given by $dx/U = dy/V = dz/W$. A streamsurface is a surface of streamlines which can form a streamtube.

1.5 Governing equations

The continuity equation in integral form is given by

$$\frac{d}{dt}\oint_A \rho dV + \oint_A \rho(\mathbf{u}\cdot\mathbf{n})dA = 0$$

and also we can write the momentum equation

$$\underbrace{\frac{d}{dt}\oint_V \mathbf{u}\rho dV}_{\text{unsteady comp.}} + \oint_A \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA = \dots$$

$$\oint_V \rho\mathbf{F}dV - \oint_A P\mathbf{n}dA$$

When manipulating these result involving the pressure term we use the *gradient theorem* i.e. $\int_A f\mathbf{n}dA = \int_V \nabla f dV$. The differential equations appear in section 2.

We can form the Euler n-equation is obtained by projecting the momentum equation to the normal direction to the streamline.

$$\frac{\partial P}{\partial n} = \rho F_n + \frac{\rho V^2}{R}$$

and we obtain the Euler s-equation by doing the same along the streamwise direction s:

$$\rho\frac{DV}{Dt} = -\frac{\partial P}{\partial s} + \rho F_s$$

If we integrate this equation along two points in the streamline we obtain Bernoulli's equation:

$$\int_1^2 \left(\frac{\partial V}{\partial t}\right)ds + \left(\frac{V^2}{2} + \frac{P}{\rho} + \varphi\right)\Bigg|_1^2 = 0$$

where in the derivation we use that $\mathbf{u}\nabla\mathbf{u} = \nabla(\mathbf{u}\cdot\mathbf{u}/2) + \nabla\times(\nabla\times\mathbf{u})$.

1.6 Accelerating momentum equation

We can also write the momentum equation in an accelerating reference system, i.e.

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla P + \rho\mathbf{F} - \dots$$

$$\rho\left[\underbrace{\frac{d^2\mathbf{r}}{dt^2}}_{\text{Accl}} + \underbrace{2(\Omega\times\mathbf{v})}_{\text{Coriolis effect}} + \left(\frac{d\Omega}{dt}\times\mathbf{y}\right) + \underbrace{\Omega\times(\Omega\times\mathbf{y})}_{\text{Centrifugal}}\right]$$

where $\mathbf{y}$ are the local coordinates and $\mathbf{r}$ is the distance to the origin. See diagram. The identity $\Omega\times(\Omega\times\mathbf{y}) = -(1/2)\nabla(\Omega\times\mathbf{x})^2$.

1.7 Relative motion near a point

This is a linearized theory for velocity variation across a small scale i.e. a small change in velocity $\delta u_i = r_j\partial u_i/\partial x_j$. The velocity gradient tensor can be written as a sum of symmetric and anti-symmetric parts i.e.

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]}_{e_{ij}=e_{ji},\text{ symmetric}} + \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]}_{\xi_{ij}=-\xi_{ji},\text{anti-symmetric}}$$

The rate of strain tensor e_{ij} stretches and contracts the fluid particle along orthogonalized axes which we obtain by diagonalizing (symmetric matrix) can get this through $(a_{ij})^T e_{kl}a_{lm} = e'_{jm}$ or in matrix form as $E' = A^TEA$ where A is the matrix of orthogonal eigenvectors ($A^T = A^{-1}$). The anti-symmetric or deviatoric part is related to rotation (as of a solid body). Therefore Anyway, we can write the change in velocity as a sum of symmetric and anti-symmetric parts i.e. $\delta u_i^{(s)} + \delta u_i^{(a)} = \delta u_i$. and we write explicitly

$$\delta u_i^{(s)} = \frac{\partial}{\partial x_i}\left(\frac{1}{2}r_k r_l e_{kl}\right),\ \delta\mathbf{u}^{(s)} = \nabla\phi$$

and

$$\delta u_i^{(a)} = \frac{1}{2}\epsilon_{ijk}w_k r_j,\ \delta\mathbf{u}^{(s)} = \frac{1}{2}\omega\times\mathbf{r}.$$

1.8 Navier-Stokes Eqn. and Vorticity

The navier stokes equation appears in section 2. We mention that μ is referred to as the kinematic viscosity. Also we define the vorticity as

$$\omega = \nabla\times\mathbf{u}, \text{with } \nabla\cdot(\nabla\times\mathbf{u}) = 0$$

via a vector identity. Analogous to streamtubes and lines, we can define vortextubes and vortexlines. In particular we can define circulation κ through the use of the Stokes theorem, $\oint_A(\nabla\times\mathbf{F})\cdot\mathbf{n}dA = \oint_C \mathbf{F}\cdot d\mathbf{l}$:

$$\kappa = \oint_C \mathbf{u}\cdot d\mathbf{l} = \oint_A \omega\cdot\mathbf{n}dA$$

and in particular we have the true lemma that surfaces who share the same bounding line have the same circulation no matter the shape of the surface

$$\oint_{C(A_1)} \mathbf{u}\cdot d\mathbf{l} = \oint_{C(A_2)} \mathbf{u}\cdot d\mathbf{l}$$

or that the vorticity flux is the same through any surface bounded by C.

2 Dimensional Analysis

We define the matrix of dependence of our parameters Q_1, Q_2, Q_3 and write the columns as 3×1 vectors where we encode the dependence of the parameters on M,L and T for example (could also add Θ for temp.). Anyway the resulting matrix is usually rank 3 and we can extract dimensionless combinations by finding the null vectors of A. We can then specify a type of relation we want i.e. $Q_1/\hat{Q_1} = f(Q_2/\hat{Q_2}, \text{etc}\ldots)$.

3 Simplifications to full Navier-Stokes equations

The full Navier-Stokes equations:

$$\texttt{Continuity} \ \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0$$

where $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$.

$$\texttt{Momentum} \ \rho\frac{D\mathbf{U}}{dt} = -\nabla p + \rho\mathbf{F} + \mu\nabla^2\mathbf{u}$$

$$\texttt{Equation of state} \ \ f(\rho, S, P)$$

3.1 Simplifications

- **Incompressible Flow**

$$\frac{D\rho}{Dt} = 0, \text{ or } \rho = Const!$$

- **Inviscid flow** If $\mu|\mathbf{u}|/L^2 \ll 1, \Rightarrow Re \gg 1$ then we neglect viscous effects at large length scales.

- **Irrotational flow** We hold that $\nabla\times\mathbf{u} = \omega = 0$ which implies we can write $\mathbf{u} = \nabla\phi$, the velocity potential, or likewise as a streamfunction.

3.2 The simplified equations

$$\nabla\cdot\mathbf{u} = 0, \ \rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \rho\mathbf{F}, \text{ and } \nabla\times\mathbf{u} = 0$$

3.3 Streamfunction

Can define stream function such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ for $\rho = Const$. Can also extend to other coordinate systems.

3.4 General momentum integral

Using that we have a velocity potential for the velocity and supposing we have a conservative body force ($\mathbf{F} = \nabla\varphi$) we can integrate the momentum equation to produce:

$$\frac{\partial\phi}{\partial t} + \frac{|\mathbf{u}|^2}{2} + \frac{P}{\rho} + \varphi = f(t)$$

If we are able to solve for ϕ then all quantities are known enabling us to determine the pressure from the above equation.

3.5 Properties of the velocity potential

If we are in a singly connected region or domain, if normal velocity component is known on boundary then we know that the velocity potential solution ($\nabla^2\phi = 0$) is **unique**. Can prove through use of $\nabla\cdot(\phi\mathbf{u}) = \mathbf{u}\cdot\nabla\phi + \phi\nabla\cdot\mathbf{u}$.

The **Kelvin's minimum energy theorem** says that no incompressible flow with defined $\mathbf{u}\cdot\mathbf{n}$ on boundary can have as small Total kinetic energy as a rotational flow.

$$E(irrotational) < E(rotational)$$

The **minima** of the pressure and **maxima** of $q = |\mathbf{u}|$ occur on the boundary of the simply connected region. The opposite extrema can occur in the interior of the simply connected region.

4 How to get solutions to the equations

We can solve in three ways: **superposition of singularities** i.e. sources, sinks, vortices etc., **separation of variables**, and **conformal mapping** (only for 2-D).

4.1 Singularities method

Note: The streamfunction and velocity potential satisfy Laplace's equations in 2-D. In the follong $\partial\psi\ \partial r = -u_\theta$.

4.1.1 Fundamental singularities

- 2-D Point Source (+) or Sink (-) of strength Q for $(x, y) = r\exp i\theta$

$$\phi = \pm\frac{Q}{2\pi}\ln r, \ \psi = \pm\frac{Q}{2\pi}\theta$$

- 2-D Line vortex strength κ.

$$\phi = \frac{\kappa}{2\pi}\theta, \psi = \frac{\kappa}{2\pi}\ln(r)$$

- 3-D point source

$$\phi = -\frac{Q}{4\pi r}$$

- Streamfunction in 3-D is impractical and no point vortex in 3-D.

4.1.2 Method of Images

We place vortices or sources in the image space to satisfy some boundary condition at some wall.

4.2 Complex potential

We construct the **complex potential** $W(z) = \phi + i\psi$ such that $z = x + iy$. The Riemann conditions are satisfied for this formulation since both ψ and ϕ satisfy the Laplace equation. The Riemann conditions are that a complex function $f = u + iv$ is analytic iff $u_y = -v_x$ and $u_x = v_y$. In particular the complex potential for a point source W_s and pt. vortex W_d are as follows:

$$W_s(z) = \frac{Q}{2\pi}\ln(z - z_0),\ W_v(z) = -\frac{i\kappa}{2\pi}\ln z - z_0$$

E.g. Note that to obtain the positive x-dipole we orient the source on the right and the sink on the left.

The **complex velocity** can be had from the following fact, i.e. $dW/dz = qe^{-i\beta} = u - iv$ and so β is the local orientation of the flow. In particular stagnation points occur when $dW/dz = 0$ since $q = (dW/dz)^*(dW/dz)$.

4.3 Rankine shapes

The **Rankine half-body** is obtained by superimposing a uniform flow $W = Uz$ and a source at some coordinate (it has max-thickness $Q/2U$). The **Rankine oval** is obtained by place a source at $-a$ and a sink at a as well as uniform flow. The streamline originating from the stagnation pt. at the front of the bodies defines the shape of the body.

4.4 Higher order singularities

We can have pt. source/vortex dipoles, quadrapoles etc.. In general dipole has $q \propto 1/r^2$, quadrapole $q \propto 1/r^3$, etc.. (far away). The process for obtaining the field for any configuration of sources/sinks or vorticies and make the limiting procedure that $Q \to \infty$ and the separation $\epsilon \to 0$ so as to guarantee $Q\epsilon \to \mu$ a constant. In geneal a source dipole at angle γ is given by the complex potential

$$W(z) = -\frac{\mu e^{i\gamma}}{2\pi}\frac{1}{z - z_0}$$

5 Conformal Mapping

We use a conformal transformation $F(z)$ in order to simplify otherwise complicated geometry. In the transformed (usually $\zeta = F(z)$) we can use the method of images to solve the simple problem. Then we transform back to the real space having obtained the complex potential in terms of ζ.

5.1 Transformation of singularities

- The form of these in the transformed space are exactly the same except $z - z_0 = \zeta - \zeta_0$.
- For doublets the form is

$$W(z) = \frac{A}{F'(\zeta_0)}\frac{1}{\zeta - \zeta_0}$$

  where $A = -e^{i\gamma}\mu/2\pi$ for example.

A conformal map has the following two properties, $F'(\zeta_0) \neq 0$ and F is analytic at ζ_0.

6 Blasius Force theory

$$F_x - iF_y = \frac{i\rho}{2}\oint_{body}(dW/dz)^2 dz$$

or through changing the path integral $F_x = \rho\kappa v$ and $F_y = -\rho\kappa u$ where $\kappa = \oint \mathbf{u}\cdot d\mathbf{x} = \oint \omega \cdot \mathbf{n}dA$.