

TORSION 1-CRYSTALLINE REPRESENTATIONS
AND NÉRON COMPONENT GROUPS OF
SEMI-STABLE ABELIAN VARIETIES OVER
 p -ADIC FIELDS

by

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Abstract

Let p be a rational prime, let K be a finitely ramified extension of \mathbb{Q}_p , let \bar{K} be an algebraic closure of K , and let $\Gamma_K = \text{Gal}(\bar{K}/K)$. We provide an interpretation within p -adic Hodge theory of the Néron component groups attached to semi-stable abelian K -varieties. Let A_K be a semi-stable abelian K -variety and let $m \in \mathbb{Z}_{\geq 1}$. We construct a finite flat \mathcal{O}_K -group scheme with generic fiber isomorphic to $\text{Crys}_1(A_K[n])$, the maximal $\mathbb{Z}_p[\Gamma_K]$ -submodule of $A_K[p^m]$ that is equal to the quotient of two \mathbb{Z}_p -lattices in a crystalline $\mathbb{Q}_p[\Gamma_K]$ -module with Hodge-Tate weights in $[0, 1]$. From this construction, we deduce that the p -Sylow subgroup of the Néron component group of A_K is isomorphic to the torsion submodule of $R^1 \text{Crys}_1(T_p A_K)$, where $T_p A_K$ is the p -adic Tate module of A_K , and where $R^1 \text{Crys}_1$ is defined as in the work of Minhyong Kim Kim and Susan Marshall. This formula was proved by Kim and Marshall under the assumptions that $p > 2$ and K/\mathbb{Q}_p is unramified; we remove these restrictions. We also provide another proof of the result due to Breuil and Coleman-Iovita that A_K has good reduction if and only if $T_p A_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline.

Chapter 0

Introduction

Abelian varieties form a special but fundamental class of geometric objects. They are the proper algebraic groups. Their geometry is highly constrained (e.g., in any reasonable cohomology theory, their cohomology is determined in degree one), yet they can be employed, through the Jacobian and Albanese constructions, in the study of cycles on more general classes of varieties. They also give rise to delicate arithmetic structures. These are the focus of the present dissertation.

Let F/\mathbb{Q} be a number field and let A_F be an abelian F -variety. It is a difficult result that A_F admits a canonical smooth \mathcal{O}_F -model, its Néron model $\text{Nér}(A_F)$, which, additionally, has the structure of a commutative \mathcal{O}_F -group scheme. The formation of Néron models is sensitive to the arithmetic of \mathcal{O}_F at places of bad reduction, i.e., at places where the reduction of $\text{Nér}(A_F)$ is not proper. Suppose v is a finite place of F where A has bad reduction. Then the reduction $\text{Nér}(A_F)_{\kappa(v)}$ may not be connected, and its identity component $\text{Nér}(A_F)_{\kappa(v)}^\circ$ may have nontrivial unipotent radical. Let F'/F be a finite extension, let $\text{Nér}(A_{F'})$ be the Néron model of $A_{F'}$, let v' be a place of F' over v . There is a canonical morphism

$$\text{Nér}(A_F) \times_{\mathcal{O}_F} \mathcal{O}_{F'} \longrightarrow \text{Nér}(A_{F'})$$

extending the natural base change morphism on generic fibers, but it is often not the case that $\text{Nér}(A_{F'})$ is isomorphic to the base change $\text{Nér}(A_F) \times_{\mathcal{O}_F} \mathcal{O}_{F'}$. For instance, the number of connected components of the reduction $\text{Nér}(A_{F'})_{\kappa(v')}$ may be larger than that of $\text{Nér}(A_F)_{\kappa(v)}$, and it may be that the identity component $\text{Nér}(A_{F'})_{\kappa(v')}^\circ$ has trivial unipotent radical even if $\text{Nér}(A_F)_{\kappa(v)}^\circ$ does not.

Each of these two phenomena will feature in our work.

We will focus on the behavior of Néron models at places over a fixed arbitrary rational prime p . Let K/\mathbb{Q}_p be a finitely ramified extension over \mathbb{Q}_p , let A_K be an abelian K -variety, and write A for the Néron model of A_K . Write κ for the residue field of \mathcal{O}_K , fix an embedding $K \subset \overline{K}$ into an algebraic closure and let $\Gamma_K = \text{Gal}(\overline{K}/K)$. Assume that A_K has semi-stable reduction, i.e., that the identity component A_κ° of the reduction A_κ has trivial unipotent radical. We prove a formula for the p -primary part $\Phi_{A_K, \kappa}[p^\infty]$ of the finite abelian group of connected components $\Phi_{A_K, \kappa}$ of A_κ . This formula is proved via an in-depth study of the Hodge-theoretic properties of the (dual of) the p -adic étale cohomology of A_K with torsion coefficients. We will return to this. For now, we motivate the formula for $\Phi_{A_K, \kappa}[p^\infty]$. For this, we digress on earlier work of Ogg-Néron-Shafarevich, Serre-Tate, Grothendieck, Fontaine, Coleman-Iovita, and Kim-Marshall.

Let l be a prime distinct from p . The l -primary subgroup $\Phi_{A_K, \kappa}[l^\infty]$ has a description in terms l -adic étale cohomology due to Grothendieck. He obtained the following result.

Proposition 1 ([19] 11.3.8).

Let l be a prime distinct from p . Writing $T_l A_K$ for the l -adic Tate module of A_K , and $I_K \subset \Gamma_K$ for the inertia subgroup, there is a canonical isomorphism of unramified $\mathbb{Z}_p[\Gamma_K]$ -modules

$$\Phi_{A_K, \kappa}[l^\infty] \simeq (H_{\text{cts}}^1(\Gamma_K, T_l A_K))_{\text{tors}},$$

where $(-)_{\text{tors}}$ gives the maximal torsion submodule.

The formula requires an understanding of the maximal unramified $\mathbb{Z}_p[\Gamma_K]$ -submodules of the finite torsion¹ $\mathbb{Z}_p[\Gamma_K]$ -modules $A_K[l^m]$. Loosely speaking, the formula breaks down in the case $l = p$ because maximal unramified submodules tend to be too small in that case. A related phenomenon is that all finite flat \mathcal{O}_K -group schemes of l -power order are étale. The following idea originates in the work [31] of Kim and Marshall: to ‘repair’ Grothendieck’s formula in the case $l = p$, one should replace the condition of unramifiedness with something more general that is appropriate for understanding finite torsion $\mathbb{Z}_p[\Gamma_K]$ -modules (i.e.,

¹In this introduction, we make the following definition: a given a topological group Γ and a topological ring R , a *finite torsion $R[\Gamma]$ -module* is a finitely generated torsion R -module given the discrete topology equipped with a continuous R -linear action of Γ .

objects of the category $\text{Rep}_{\text{tors}}(\Gamma_K)$ to be defined in Situation 0.1.2) arising from finite flat \mathcal{O}_K -groups of p -power order.

The sought-for condition on representations of Γ_K on \mathbb{Q}_p -vector spaces was introduced in the 1983 paper [15] of Fontaine. However, it was not known to be “appropriate” in the above sense until Kisin’s 2006 paper [33]. The condition is that of crystallinity. Roughly, a finite-dimensional \mathbb{Q}_p -representation V of Γ_K is *crystalline* when it can be recovered from $V \otimes_{\mathbb{Q}_p} B_{\text{crys}}$, where B_{crys} is a certain field, the field of “crystalline p -adic periods” (defined in (3.1.6) below). Every crystalline representation has an associated invariant, its multi-set of Hodge-Tate weights (a collection of integers with multiplicities). When a \mathbb{Q}_p -representation is crystalline with Hodge-Tate weights in the interval $[0, r]$, we say that the representation is *r -crystalline*.

The condition of being r -crystalline has sufficient functorial properties that we may form, for any $\mathbb{Z}_p[\Gamma_K]$ -module M of finite-type over \mathbb{Z}_p , a maximal r -crystalline submodule $\text{Crys}_r(M)$. When M is torsion-free, it may be that, as r grows, $\text{Crys}_r(M)$ stabilizes to a proper submodule of M . It is a recent, perhaps surprising, result that if M is finite torsion then $\text{Crys}_r(M) = M$ for $r \gg 0$ (see [17], Theorem 3.3.2). In this sense, the crystalline condition carries complete information about all finite torsion $\mathbb{Z}_p[\Gamma_K]$ -modules.

Returning to the problem of understanding component groups, recall that we sought a condition weaker than unramifiedness that is sufficient to understand $\mathbb{Z}_p[\Gamma_K]$ -modules arising from finite flat \mathcal{O}_K -groups. With the fact that a $\mathbb{Z}_p[\Gamma_K]$ -module is unramified if and only if it is 0-crystalline, the following result shows that 1-crystallinity is the appropriate condition.

Proposition 2 (Tate [52]; Raynaud (see [4]); Kisin [33]; Kim [32]. More precise references are given in Remark 3.2.12).

Let M be a finite torsion $\mathbb{Z}_p[\Gamma_K]$ -module. Then M is the generic fiber of a finite flat \mathcal{O}_K -group if and only if M is 1-crystalline.

The proofs of this result are inexplicit, in a way, proceeding via Raynaud’s embedding theorem (see Remark 3.2.12) and abstract p -adic Hodge theory. In particular, in the case where $M = \text{Crys}_1(A_K[p^m])$ with A_K as above, one knows that M is the generic fiber of a finite flat \mathcal{O}_K -group H_{p^m} , but the above result and its proof do not reveal any apparent relationship between H_{p^m} and the geometry of the abelian variety A_K . Despite this complication, in 2001 Kim and Marshall were able to use this result in the case where $p > 2$ and K/\mathbb{Q}_p is unramified to construct a finite flat \mathcal{O}_K -group with generic fiber $\text{Crys}_1(A_K[p^m])$.

Via this description, they were able to prove the following result, which we prove for general finitely ramified K and arbitrary p .

Theorem 3 (Theorem 4.3.7; [39] for $p > 2$ and K/\mathbb{Q}_p unramified).

Suppose A_K has semi-stable reduction. Then there is an isomorphism of unramified $\mathbb{Z}_p[\Gamma_K]$ -modules

$$\Phi_{A_K, \kappa}[p^\infty] \simeq (R^1 \text{Crys}_1(T_p A_K))_{\text{tors}}.$$

Note that $\text{Crys}_0(-)$ (say, on $\mathbb{Z}_p[\Gamma_K]$ -modules that are torsion, to avoid technicalities) is the functor of inertial invariants, since being 0-crystalline is equivalent to being unramified.

Replacing Fontaine-Laffaille's result with Kisin's generalization, the proof by Kim and Marshall extends immediately to the case where $p > 2$ and the ramification index e_K of K/\mathbb{Q}_p satisfies $e_K < p - 1$. This ramification restriction is essential to their approach, which relies on the following result of Raynaud, which fails dramatically without the ramification assumption.

Proposition 4 ([46] 3.3.3 and 3.3.6).

Suppose $e_K < p - 1$. Then every finite torsion $\mathbb{Z}_p[\Gamma_K]$ -module arises as the generic fiber of at most one finite flat \mathcal{O}_K -group, up to isomorphism. Let G, H be finite flat \mathcal{O}_K -groups.

(i) *Every morphism $G_K \rightarrow H_K$ prolongs uniquely to a morphism $\pi : G \rightarrow H$. Moreover, the kernel and cokernel of π formed in the category of group schemes over \mathcal{O}_K are finite flat over \mathcal{O}_K .*

(ii) *The category of finite flat \mathcal{O}_K -groups is an abelian category.*

Using Fontaine-Laffaille theory, Kim and Marshall choose a finite flat \mathcal{O}_K -group H_{p^m} with generic fiber isomorphic to $\text{Crys}_1(A_K[p^m])$, and used Raynaud's result to produce an embedding of the identity component $H_{p^m}^\circ$ into the Néron model A . They then used Raynaud's result to deduce that this embedding gives an isomorphism of H_{p^m} with the finite flat \mathcal{O}_K -group that we will denote ${}^* \mathcal{Q}_\varpi^1[p^m]$. Their work is in terms of finite parts of quasi-finite flat \mathcal{O}_K -groups, which allows them to deduce from results of [19] that, for $m \geq 1$,

$$\Phi_{A_K, \kappa}[p^m] \simeq \frac{\text{Crys}_1(T_p A_K \otimes \mathbb{Z}/p^m \mathbb{Z})}{\text{Crys}_1(T_p A_K) \otimes \mathbb{Z}/p^m \mathbb{Z}}. \quad (5)$$

We take a different path. Rather than choosing H_{p^m} by Kisin's result and showing that it has some relationship to A , we construct a finite flat \mathcal{O}_K -group with generic fiber $\text{Crys}_1(A_K[p^m])$ directly from the Néron model A of A_K . For this, we will use a semi-abelian K -variety, the generic fiber of the *Raynaud extension* \tilde{A} attached to A_K , that uniformizes A_K in the category of rigid analytic spaces over K . From this space and the monodromy pairing that can be constructed from it, we will construct a certain log finite flat \mathcal{O}_K -group, after defining this notion in Definition 2.4.1 and developing some related foundations. An isomorphism as in (5) can then be constructed using results on the monodromy pairing proved in [19].

Theorem 6 (Theorem 4.2.23).

Fix a uniformizer ϖ of \mathcal{O}_K . Let A_K be a semi-stable abelian K -variety, and let $n \in \mathbb{Z}_{\geq 2}$, be a power of p . Let $(\mathcal{Q}_\varpi^1[p^m], N_{p^m})$ denote the the log finite flat \mathcal{O}_K -group of Proposition 2.5.10. There is an isomorphism of $\mathbb{Z}_p[\Gamma_K]$ -modules

$$A_K[p^m] \times_{\mathcal{Q}_\varpi^1[p^m]_{\text{ét}}} \ker N_{p^m, K}(-1) \simeq \text{Crys}_1(A_K[p^m]).$$

Moreover, $\text{Crys}_1(A_K[p^m])$ is the generic fiber of the finite flat \mathcal{O}_K -group

$$*\mathcal{Q}_\varpi^1[p^m] := \mathcal{Q}_\varpi^1[p^m] \times_{\mathcal{Q}_\varpi^1[p^m]_{\text{ét}}} \ker N_{p^m}(-1). \quad (7)$$

where the Tate twist is as in Proposition 2.2.23.

To show that $*\mathcal{Q}_\varpi^1[p^m]$ actually has the claimed generic fiber is nontrivial. The proof relies in an essential way on the following recent technical result.

Proposition 8 ([43] 1.2; others in special cases, e.g. $r = 1$ $p > 2$ case by [8] 3.4.3. See the introduction to [43] for a detailed history of this result.).

For $m \geq 1$, choose a system of p^m -th roots $\tilde{\varpi}^{(m)}$ of the uniformizer ϖ satisfying $(\tilde{\varpi}^{(m+1)})^p = \tilde{\varpi}^{(m)}$, and let $K_{\tilde{\varpi}}$ denote the subfield of \overline{K} generated by all of the $\tilde{\varpi}^{(m)}$. Let $\Gamma_{\tilde{\varpi}} = \text{Gal}(\overline{K}/K_{\tilde{\varpi}})$. Let $\text{Rep}_{\text{tors}}(\Gamma_{\tilde{\varpi}})$ denote the category of finite torsion $\Gamma_{\tilde{\varpi}}$ -modules, and let $\text{Rep}_{\text{tors}}^{\text{crys}, r}(\Gamma_K)$ denote the category of finite torsion r -crystalline $\mathbb{Z}_p[\Gamma_K]$ -modules.

Suppose $e_K(r-1) < p-1$. Then the restriction functor

$$\text{Res}_{\Gamma_{\tilde{\varpi}}}^{\Gamma_K} : \text{Rep}_{\text{tors}}^{\text{crys}, r}(\Gamma_K) \longrightarrow \text{Rep}_{\text{tors}}(\Gamma_{\tilde{\varpi}})$$

is fully faithful.

From Theorem 6, it is straightforward to prove the following result.

Corollary 9 ([10] 4.7; also [7] 5.3.4 when $p > 2$ (see also [7] 5.3.5 for more on the history of this result)).

With assumptions as in Theorem 6, we have

$$\mathrm{Crys}_1(T_p A_K) \simeq T_p \tilde{A}_K,$$

where \tilde{A}_K is the Raynaud extension (see Proposition 1.4.1).

We note that this can be proved using the equivalent result Corollary 4.6 of Coleman and Iovita's paper [10]. Their approach is markedly different from ours, at least in that it relies on computation of p -adic integrals of de Rham cohomology classes. We have not attempted compare their results with ours. Their Corollary 4.6 is used in [10] to prove the following theorem.

Theorem 10 (Corollary 4.3.3; proved in [10] Theorem 1, and [7] 4.7).

An abelian K -variety A_K has good reduction over \mathcal{O}_K if and only if its p -adic Tate module $T_p A_K$ is a 1-crystalline $\mathbb{Z}_p[\Gamma_K]$ -module.

Using Theorem 6 and Corollary 9, we have the expression

$$\frac{{}^* \mathcal{Q}_{\varpi}^1[p^m]_K}{\tilde{A}_K[p^m]} \simeq \frac{\mathrm{Crys}_1(T_p A_K \otimes \mathbb{Z}/p^m \mathbb{Z})}{\mathrm{Crys}_1(T_p A_K) \otimes \mathbb{Z}/p^m \mathbb{Z}}.$$

for the right-hand side of (5). From the construction of ${}^* \mathcal{Q}_{\varpi}^1[p^m]_K$ and results about the monodromy pairing that can be found in [19], it is straightforward to deduce that there is an isomorphism

$$\frac{{}^* \mathcal{Q}_{\varpi}^1[p^m]_K}{\tilde{A}_K[p^m]} \simeq \Phi_{A_K, \kappa}[p^m]$$

so we have (5) without assuming any bound on e_K . From there, we may reuse the homological algebra calculations of Grothendieck, re-purposed by Kim-Marshall in [31], to obtain Theorem 3.

We hope we have made our debt to Kim and Marshall apparent. The idea of studying derived functors of Crys_1 originates in their work [31], as does their definition of these derived functors in the spirit of Jannsen's definition of continuous étale cohomology. Marshall's thesis [39], was an invaluable resource to us in study of these derived functors and the general categories of (e.g., non-finite

type torsion) r -crystalline $\mathbb{Z}_p[\Gamma_K]$ -modules that must be used in their construction. We would like to point out three ways in which the work of Kim-Marshall goes beyond (with their ramification restrictions in place) the present work. First, in their setting ($p > 2$ and K/\mathbb{Q}_p is unramified), Kim and Marshall give a description of $\text{Crys}_r(A_K[p^m])$ for all $r = 1, \dots, p-2$ (it turns out that, in this range, the module is independent of r). We present results about Crys_1 , but we would like to better understand their work for more general r and its relation to Proposition 8. As for Marshall's thesis, it develops the homological algebra of the functors Crys_r far beyond what we do here. In particular, it constructs, for any finite torsion $\mathbb{Z}_p[\Gamma_K]$ -module M , a spectral sequence from the functors Crys_r that abuts to $H^i(\Gamma_K, M)$. Secondly, Marshall's thesis gives some calculations with non-semi-stable Jacobians of Fermat curves, with an eye toward understanding Néron component groups in this case. The component groups of non-semi-stable abelian K -varieties have a very mysterious structure. We are very interested by this case, which at present seems to be outside of the reach of our techniques.

0.1 Conventions

Following [9] and [16], we make the following definition.

Definition 0.1.1 (*p*-adic field).

Let p be any rational prime. A ***p*-adic field** is a field of characteristic zero that is complete with respect to a fixed discrete valuation that has a perfect residue field of characteristic $p > 0$.

A *p*-adic field K is, therefore, a finitely ramified extension of \mathbb{Q}_p , but it may be that K has infinite degree.

Every action of a topological group on a module will be continuous with respect to a topology on the module, which will most often be a *p*-adic topology or, in the case of a finite \mathbb{Z}_p -module, the discrete topology.

Situation 0.1.2.

We establish the following notational situation.

- (i) K is a *p*-adic field with absolute value denoted simply by $|\cdot|$; K is equipped with an embedding $K \subset \bar{K}$ into an algebraic closure which is equipped with the unique absolute value $|\cdot|$ extending that of K ; $\Gamma_K := \text{Gal}(\bar{K}/K)$;
- (ii) $v_{\bar{K}} : \bar{K}^\times \rightarrow \mathbb{Q}$ is the valuation on \bar{K} satisfying $v_{\bar{K}}(K^\times) = \mathbb{Z}$; we also write $v_L = v_{\bar{K}|L}$ for any subfield $L \subset \bar{K}$ containing K ;
- (iii) κ is the residue field of \mathcal{O}_K ; κ is equipped with an embedding $\kappa \subset \bar{\kappa}$ into an algebraic closure; $\varpi \in \mathcal{O}_K$ is a fixed uniformizer of \mathcal{O}_K ; $\Gamma_\kappa := \text{Gal}(\bar{\kappa}/\kappa)$;
- (iv) C is an algebraically closed complete extension of \bar{K} and \mathcal{O}_C is its ring of integers; $\mathcal{O}_C^b := \lim_{x \rightarrow x^p} \mathcal{O}_C$ is the tilt of \mathcal{O}_C ;
- (v) $\tilde{\varpi} \in \mathcal{O}_C^b$ is defined recursively by setting $\tilde{\varpi}^{(0)} := \varpi$ and choosing, for $m \in \mathbb{Z}_{\geq 1}$, (any) element $\varpi^{(m)} \in C^\times$ of order p^m satisfying $(\tilde{\varpi}^{(m)})^p = \tilde{\varpi}^{(m-1)}$;
- (vi) $K_{\tilde{\varpi}} := \bigcup_{m \geq 0} K(\tilde{\varpi}^{(m)}) \subset \bar{K}$ and $\Gamma_{\tilde{\varpi}} := \text{Gal}(\bar{K}/K_{\tilde{\varpi}}) \subset \Gamma_K$;
- (vii) For Γ and any one of the groups $\Gamma_K, \Gamma_\kappa, \Gamma_{\tilde{\varpi}}$, let $\text{Rep}_{\text{tors}}(\Gamma_F)$ denote the abelian category of finite abelian groups of *p*-power order having the discrete topology and equipped with a continuous action of Γ , where each Γ is equipped with its natural profinite topology.

Chapter 1

Semi-abelian varieties, models and 1-motives

In this chapter we give fundamental definitions and results around semi-abelian varieties. We discuss Néron models and torsion in Néron models. Néron models with bad reduction have quasi-finite but non-finite torsion. To fix this, we pass from the Néron model to the Raynaud extension, which is a semi-abelian scheme with the same generic fiber, but which has finite torsion. We discuss uniformization of abelian varieties over p -adic fields, log 1-motives, the monodromy pairing, and Raynaud decomposition of log 1-motives. Raynaud decomposition allows us to define log finite flat \mathcal{O}_K -groups that play a crucial role in our main results.

There are several very good references for the material in this chapter. The most recent of these is the dissertation-turned-book [35] of Lan, which extended the work on compactifications of integral models of Siegel modular varieties to construct compactifications of smooth integral models for general PEL-type Shimura varieties. The work of Faltings-Chai builds on Mumford's foundational work [41] on degeneration of abelian varieties into tori. Faltings-Chai allow degeneration into semi-abelian schemes, introducing certain important categories of degeneration data. Before Mumford's work, Raynaud had announced in [45] uniformization by semi-abelian varieties for general abelian varieties over p -adic fields, building on the earlier work of Tate on uniformization of elliptic curves. Around the time of the announcement of Raynaud's work, Grothendieck wrote Exposé VII, VIII, and IX in SGA 7 Volume 1. The first two of these develop

the theory of biextensions, which had recently been introduced by Mumford. Exposé IX is a monumental work that studies torsion in Néron models and its finer structure in relation to the Weil pairing and more general pairings arising from biextensions, and also defines and studies the monodromy pairing and its relation to Néron component groups. It also develops Raynaud’s uniformization results.

Our preference has been to use [35] whenever possible because it is very thorough, well-referenced and readily available on the author’s website. For issues relating to torsion and Néron component groups, we use [19]. For issues relating to Néron models, we use the book [6]. In the final sections, we will make essential use of the results of [3] on log finite flat group \mathcal{O}_K -groups.

Aside from the exposition, nothing in this chapter is original.

1.1 Algebro-geometric preliminaries

We record some definitions and results from algebraic geometry that will be used later.

Proposition 1.1.1 ([51, Tag 01KM]; [51, Tag 01RH]; [51, Tag 056D]).

Let S be a scheme, let X, X' be S -schemes and let $f, g : X \rightarrow X'$ be two S -morphisms.

- (i) *The category Sch_S contains an equalizer $E(f, g)$ of f and g . The scheme $E(f, g)$ is a locally closed subscheme of X which is closed if X' is separated over S*
- (ii) *Suppose that there exists an open subscheme U of X such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is all of X , then $f = g$.*
- (iii) *If X is reduced and U is an open subscheme of X that is topologically dense in X , then the scheme theoretic closure of U in X is all of X .*

Proposition 1.1.2.

Let F be a field and let F^{sep} be a separable closure.

- (i) *Let Y be a smooth F -scheme. Then $Y(F^{\text{sep}})$ is dense in Y .*
- (ii) *Let X be a geometrically reduced scheme locally of finite type over F .*
 - (a) *The scheme X contains a dense open subscheme smooth over F .*

(b) The set $X(F^{\text{sep}})$ is dense in X .

(c) Let $f, g : X \rightarrow X'$ be two morphisms of F -schemes and suppose that X' is separated. Then $f = g$ if and only if f and g agree as maps of sets $X(F^{\text{sep}}) \rightarrow X'(F^{\text{sep}})$.

Proof. Claim (i) is [51, Tag 056U]. Claim (ii)(a) is [51, Tag 056V]. To prove (ii)(b), apply (ii)(a) to X to get a dense open subscheme Y of X . Applying (i) to Y , we obtain (ii)(b), since Y has the subspace topology induced from X . To prove (iii)(c), suppose that f and g agree on F^{sep} -points, i.e., that the equalizer $E(f, g)$ satisfies $E(f, g)(F^{\text{sep}}) \supset X(F^{\text{sep}})$. By (ii)(b), $E(f, g)$ is dense in X . By Proposition 1.1.1 (i), $E(f, g)$ is closed, hence equal to X . \square

Proposition 1.1.3 ([6] 6.1/6; cf also [51, Tag 0247]).

Let S be a scheme and let $\pi : S' \rightarrow S$ be a faithfully flat and quasi-compact morphism of schemes.

(i) The functor π^* from Sch_S to the category of S' -schemes with descent data is fully faithful.

(ii) If S and S' are affine, then the essential image of π^* consists exactly of the isomorphism classes of those S' -schemes with descent data (X', ϕ) such that X' can be covered by ϕ -stable open subschemes that are quasi-affine over S' .

Remark 1.1.4. Let K be as in Situation 0.1.2. We are most interested in the case of Proposition 1.1.3 (ii) where S is the spectrum of K or \mathcal{O}_K , π is a finite étale surjection $S' \rightarrow S$ corresponding to a ring map $K \rightarrow K'$ or $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$. This is the situation of Galois descent ([6], 6.2/B), where $S' \rightarrow S$ is naturally equipped with the action of a finite group Γ which is equal to $\text{Gal}(K'/K)$ or equal to the Galois group $\text{Gal}(\kappa'/\kappa)$ of the residue field κ' of $\mathcal{O}_{K'}$. A descent datum relative to π on an S' -scheme X' is the same as an action of Γ on $X' \rightarrow S'$.

Definition 1.1.5 (F -variety).

Let F be a field. An F -variety is an integral scheme X equipped with a separated and finite type morphism $X \rightarrow \text{Spec } F$.

1.2 Some commutative group schemes

Throughout this dissertation, we will assume familiarity with the basic theory of group schemes. We informally review some of the essential concepts now.

Let S be a scheme and write $h_{(-)}$ for the Yoneda embedding sending an S -scheme to its functor of points on Sch_S . Let X be an S -scheme. By a Yoneda point of X , we mean an element of $X(Z)$ for some (possibly unnamed) S -scheme Z . An S -group scheme is a representable sheaf of groups on the fppf site $\text{Sch}_{S,\text{fppf}}$. If such a functor is represented by a group G , then fully faithfulness of h implies that to give a homomorphism of group schemes $H \rightarrow G$ is the same as to give, for each S -scheme Z a homomorphism of groups $H(Z) \rightarrow G(Z)$ functorial in Z . We will conflate an S -group scheme G with the corresponding sheaf h_G , regarding Yoneda points of G as local sections of h_G . Following others, we will sometimes write $g \in G$ to mean that g is a local section of G .

The group schemes we will study are all commutative, so will sometimes not mention this condition. We make the following definitions.

Definition 1.2.1 (CGp_S).

Let S be a scheme.

(i) Define CGp_S to be the category of commutative S -group schemes.

(ii) A **short exact sequence** of objects of CGp_S is a cochain complex

$$[H' \longrightarrow H \longrightarrow H'']$$

of objects of CGp_S placed in degrees $-1, 0$ and 1 that gives rise to a short exact sequence

$$0 \longrightarrow H' \longrightarrow H \longrightarrow H'' \longrightarrow 0$$

when regarded as a complex of objects of $\text{Sh}_{\text{Ab}}(\text{Sch}_{S,\text{fppf}})$.

We will be interested in both algebraic tori and commutative groups arising via Cartier duality (to be defined later) from finite étale S -groups. Such groups belong to a general class, the groups of multiplicative type, which we define now for uniformity and because it is a natural setting for Proposition 1.4.11.

Definition 1.2.2 (twisted constant; multiplicative type; X^* ; character group; split; torus; see [35], 3.1.1).

Let S be a scheme.

There is a natural functor C from abstract groups to S -group schemes taking an abstract group M to the associated constant S -group scheme $C(M)$. If an S -group scheme H is isomorphic to $C(M)$, then we say H is **modeled on M** .

Let TC_S be the full subcategory of CGp_S consisting of those S -groups Y that are locally on $S_{\text{ét}}$ modeled on a finitely generated abelian group. Such a group

is said to be a **twisted constant** S -group. Given Y in TC_S , define

$$X^*(Y) := \underline{\mathrm{Hom}}(Y, \mathbb{G}_{m,S}).$$

An S -group H of finite type is of **multiplicative type** if the S -group

$$X^*(H) := \underline{\mathrm{Hom}}(H, \mathbb{G}_{m,S})$$

is twisted constant. Let Mult_S denote the full subcategory of CGP_S consisting of the groups of multiplicative type. We call $X^*(H)$ the **character group** of H . We say that H is **split** if $X^*(H)$ is constant. We say that H is a **torus** if $X^*(H)$ is locally on $S_{\text{ét}}$ modeled on a free abelian group.

One has the following result generalizing the well-known duality of tori.

Proposition 1.2.3 ([35] 3.1.1.6).

The natural functor $X^* : \mathrm{Mult}_S \rightarrow \mathrm{TC}_S$ given by passage to character groups is an equivalence of categories. The natural functor $X^* : \mathrm{TC}_S \rightarrow \mathrm{Mult}_S$ is a quasi-inverse.

Remark 1.2.4. With the assumption that S is locally Noetherian and normal, one knows that every group of multiplicative type (in our sense) is iso-trivial, meaning that it is split by a finite étale surjection onto S ; see [35] 3.1.1.3 and 3.2.5.6.

The groups defined in Definition 1.2.2 are all affine. We will use such groups to study abelian varieties, which are non-affine except in the case of the trivial group. We will do this via , which says that any smooth connected affine group over a perfect field F can be decomposed in terms of a proper F -group, a group of multiplicative type, and a unipotent group.

Definition 1.2.5 (abelian variety; abelian scheme; semi-abelian variety; semi-abelian scheme).

Let F be a field and let S be a scheme.

- (i) An **abelian F -variety** is a smooth proper connected F -variety with the structure of an F -group scheme.
- (ii) A **semi-abelian F -variety** is a smooth commutative F -group variety that is an extension of an abelian F -variety by an F -torus
- (iii) An **abelian S -scheme** is an S -group scheme B that is proper, flat, and finitely presented over S such that each fiber B_s is an abelian $\kappa(s)$ -variety.

(iv) A **semi-abelian S -scheme** is a separated smooth commutative S -group G such that each fiber G_s is a semi-abelian $\kappa(s)$ -variety.

Remark 1.2.6. By definition, an abelian F -variety has a F -rational smooth point (the identity section), so it is geometrically integral ([51, Tag 0CDW]).

Remark 1.2.7. Let G be a semi-abelian S -scheme. Even though every fiber of G is an extension of an abelian variety by a torus, it is not necessarily the case that G is an extension of an abelian S -scheme by an S -torus. Indeed, any abelian variety over a p -adic field having bad semi-stable reduction gives rise to a Néron model that is semi-abelian with abelian generic fiber and non-proper special fiber.

To develop duality for abelian schemes, one needs to define Picard functors. We will not address issues of representability, but we provide some fundamental definitions for completeness.

Definition 1.2.8 (rigidification; rigidified line bundle).

Let Z be any scheme and let B be an abelian Z -scheme with identity section e_Z . Let L_Z be a line bundle on B_Z . A **rigidification** of L_Z is a trivialization $r_Z : \mathcal{O}_Z \xrightarrow{\sim} e_Z^* L$. A **rigidified line bundle** on B_Z is a pair (L_Z, r_Z) where L_Z is a line bundle on B_Z and r_Z is a rigidification of L_Z .

Proposition 1.2.9 (duality for abelian schemes, [35] 1.3.2.3 and 1.3.2.8).

Let S be a scheme and let B be an abelian S -scheme with identity section e . Let $\underline{\text{Pic}}_e^0(B/S)$ be the functor on S -schemes sending T/S to the set of isomorphism classes of rigidified line bundles (L_T, r_T) on B_T such that L_T is algebraically equivalent to zero in each fiber of $B_T \rightarrow T$. Then $\underline{\text{Pic}}_e^0(B/S)$ is representable by an abelian S -scheme B^D . Moreover, there is a canonical isomorphism $B \xrightarrow{\sim} (B^D)^D$.

By the Proposition, $B^D(T) = \underline{\text{Pic}}_e^0(T)$ for every S -scheme T . In particular, $\text{id}_B \in B^D(B^D)$ corresponds to a rigidified line bundle on $B \times_S B^D$.

Definition 1.2.10 (Poincaré bundle).

For any abelian S -scheme B , the universal line bundle (P_B, r_B) on $B \times_S B^D$ corresponding to $\text{id}_{B^D} \in B^D(B^D)$ is called the “Poincaré” bundle. By Proposition 1.2.9, P_B has the following universal property: for every S -scheme T and every rigidified line bundle (L_T, r) on B_T that is algebraically equivalent to zero in every fiber of $B_T \rightarrow T$, there is a unique morphism of S -schemes $\pi : B_T \rightarrow B \times_S B^D$ such that $\pi^*(P_B, r_B) = (L_T, r)$.

We will need a duality for (certain) semi-abelian S -schemes G that extends the duality theory for abelian schemes. The existence of such a duality requires that S be normal. This hypothesis is used in the construction of G^D , to show that G contains a certain quasi-finite flat S -group, and to show that the quotient of G by this group is representable (by the scheme that is defined to be G^D). Normality is also used again to verify uniqueness of G^D .

Proposition 1.2.11 (duality for semi-abelian schemes, [35], 3.4.3.2 (1.)).

Let S be a normal, locally Noetherian integral scheme with generic point η . Let G be a semi-abelian S -scheme such that the generic fiber G_η is an abelian variety. Then there is a unique semi-abelian S -scheme G^D such that $(G^D)_\eta = (G_\eta)^D$. Moreover, the Poincaré biextension over $G_\eta \times_\eta G_\eta^D$ extends uniquely to a $\mathbb{G}_{m,S}$ -biextension over $G \times_S G^D$.

1.3 Néron models

Definition 1.3.1 (S -models; Néron mapping property; Néron lft-model; Néron model (from [6])).

Let S be a Dedekind scheme (i.e., a Noetherian normal scheme of dimension at most 1) and let K be the ring of rational functions on S . Let X_K be a K -scheme that is smooth, separated and of finite type.

- (i) *An **S -model** X of X_K is an S -scheme such that $X \times_S \text{Spec } K$ isomorphic to X_K .*
- (ii) *An S -model X of X_K is said to satisfy **Néron mapping property** if the following holds: for any smooth S -scheme Y , every morphism $Y_K \rightarrow X_K$ over K prolongs uniquely to a morphism $Y \rightarrow X$ over S .*
- (iii) *A **Néron lft-model** of X_K is a smooth and separated S -model satisfying the Néron mapping property.*
- (iv) *A **Néron model** of X_K is a Néron lft-model that is finite type over S .*

Being smooth over S , Néron lft-models are locally of finite type, as the name suggests. Néron lft-models are unique up to canonical isomorphism, so that any Néron lft-model is a Néron model if a Néron model exists.

Proving existence of Néron models is a highly technical affair. In fact, even the proof of the following basic result of the theory requires nontrivial inputs.

Proposition 1.3.2 ([6], 1.2/8).

Let S be a Dedekind scheme and let A be an abelian S -scheme. Then A is the Néron model of $A \times_S \text{Spec } K$.

While [6] works in the context of Dedekind schemes, we focus in the rest of the section on the case where the base is a field or a discrete valuation ring, since this is all we need later, noting that this is the essential case in the proof of existence (cf [6] 10.1/9) of Néron lft-models in general.

Proposition 1.3.3 ([6] 10.1, 10.1/3, 10.2/1, 10.2/2).

Let R be a discrete valuation and let K be the field of fractions of R . Let G_K be a smooth separated commutative K -group. Write \hat{K}^{sh} for the field of fractions of the strict Henselization of the completion of R .

- (i) If the Néron lft-model of G_K exists, then it is an R -group whose identity component is of finite type over R .
- (ii) Suppose $R \rightarrow R'$ is an unramified morphism of discrete valuation rings, and let K' denote the field of fractions of R' . If G is a Néron lft-model of G_K , then $G_{R'}$ is a Néron lft-model of $G_K \otimes K'$.
- (iii) The group G_K has a Néron lft-model if and only if $G_K \times \hat{K}^{\text{sh}}$ contains no subgroup of type \mathbb{G}_a .
- (iv) The group G_K has a Néron model if and only if $G_K \times \hat{K}^{\text{sh}}$ contains no subgroup of type \mathbb{G}_a or \mathbb{G}_m .

Remark 1.3.4. In the case of a K -torus T_K , an lft-Néron model can be constructed in a simple and explicit fashion analogous to the construction of an analytic Néron model for the K -analytification T_K^{an} . This informs a construction of Néron models of Tate curves which is at the heart of Kato's theory of log abelian varieties and their torsion. For the essential case of $\mathbb{G}_{m,K}$, see [6] 10.1/5.

Corollary 1.3.5.

With notation as in Situation 0.1.2, suppose that A_K is a semi-abelian K -variety. Then A_K admits a Néron lft-model A . Moreover, if A_K is an abelian K -variety, then A is finite type over R , i.e., A is a Néron model.

Remark 1.3.6. In Situation 0.1.2, let A_K be an abelian K -variety and let A be the Néron model of A_K . Let $U \subset A$ be a non-empty and open subscheme. Since

A is flat over \mathcal{O}_K , U is flat over \mathcal{O}_K . Since flat ring morphisms satisfy the going-down property, U_K is nonempty. Hence, if A is a disjoint union of non-empty open subschemes, then so is A_K . Since A_K is connected, we conclude that A is connected. We point this out before giving Definition 1.3.7.

Definition 1.3.7 (connected Néron model A° ; cf [18] 15.6.5).

In Situation 0.1.2, let A_K be an abelian K -variety and let A be the Néron model of A_K . Define A° to be the open subgroup scheme $A^\circ \subset A$ which is the disjoint union of A_K and the identity component A_κ° of the special fiber.

However, one often abuses language to speak of the “component group of the Néron model of A_K ” when referring to the étale k -group A_κ/A_κ° of the special fiber A_κ of A , which need not be connected. By Proposition 2.1.6, A_κ/A_κ° prolongs uniquely (up to isomorphism) to a finite étale \mathcal{O}_K -group.

Definition 1.3.8 (The component group Φ_{A_K}).

*Let A_K be an abelian K -variety, and let A be the Néron model of A_K . Define the **Néron component group** of A_K (and of A) to be the unique étale \mathcal{O}_K -group Φ_{A_K} with special fiber $\Phi_{A_K, \kappa} = A_\kappa/A_\kappa^\circ$*

Remark 1.3.9. Part (ii) of Proposition 1.3.3 in the above result shows that unramified ring extensions do not change the Néron component group. As we described in Chapter 0, it is not the case that the Néron model of the base change is isomorphic to the base change of the Néron model. One obstruction is the component group. Let K be a p -adic field, let A_K is an abelian K -variety, let K'/K is a finite extension. It can happen that $\Phi_{A_{K'}}$ is larger than Φ_{A_K} , even in the case where A_K has semi-stable reduction.

The following result is fundamental.

Theorem 1.3.10 (Barsotti-Chevalley Theorem; [40] and 8.27 and 16.15).

Let F be a perfect field and let G_F be a smooth connected F -group scheme. There exists a unique smooth, connected, closed, normal, affine F -subgroup N_F of G_F such that G_F/N_F is an abelian F -variety. The group N_F is the direct product of its maximal torus T_F and its unipotent radical U_F

$$0 \longrightarrow T_F \times_F U_F \longrightarrow G_F \longrightarrow B_F \longrightarrow 0. \quad (1.3.11)$$

Returning to Situation 0.1.2, Let A_K be an abelian K -variety. By Corollary 1.3.5, we may choose a Néron model A of A_K . Then A_κ° is smooth and connected, so one can take its Chevalley decomposition.

Definition 1.3.12 (reduction types).

In Situation 0.1.2, let A_K be an abelian K -variety. Let A be the Néron model of A_K . Write

$$0 \longrightarrow T_\kappa \times_\kappa U_\kappa \longrightarrow A_\kappa^\circ \longrightarrow B_\kappa \longrightarrow 0 \quad (1.3.13)$$

for the Barsotti-Chevalley decomposition of the identity component of the special fiber A_κ . We say that A_K has

- (i) **semi-stable reduction** if U_κ is trivial;
- (ii) **good reduction** if both T_κ and U_κ are trivial.

Remark 1.3.14. Let A_K be an abelian K -variety and let A be the Néron model of A_K . It is a result of Grothendieck (see the appendix of [49]) that there exists a finite extension L/K such that A_L has semi-stable reduction.

While Grothendieck's result is very useful and important, it does not allow one to reduce every question about A_K to one about A_L . Questions about Néron component groups of abelian varieties not having semi-stable reduction are especially inaccessible.

A favorable property of abelian varieties with semi-stable reduction relates to base change. While semi-stability is not enough to make formation of Néron models commute with base change, one does have the following result.

Proposition 1.3.15 ([22] 3.3.6.6).

In Situation 0.1.2, let A_K be an abelian K -variety and let A be the Néron model. Let K' be a finite extension of K , and let A' be the Néron model of $A_{K'}$. Choose an isomorphism $A_{K'} \simeq A'_{K'}$, and write

$$h : A \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow A'$$

for the natural morphism obtained from the Néron mapping property from the chosen isomorphism of generic fibers.

If A_K has semi-stable reduction, then h is an open immersion, hence it induces an isomorphism $A^\circ \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \simeq A'^\circ$.

At present, our methods for relating Néron component groups and p -adic Hodge theory do not apply without the assumption of semi-stability. One reason for this can be seen in the following result.

Proposition 1.3.16 ([19] 2.2.1).

With notation as in Definition 1.3.12, choose $n \in \mathbb{Z}_{\geq 1}$ and write $[n]$ for the multiplication-by- n morphism on A° . The following are equivalent:

(i) $[n]$ is flat.

(ii) $[n]$ is surjective.

(iii) $[n]$ is quasi-finite.

(iv) p does not divide n or A_κ° is semi-abelian.

Remark 1.3.17. Proposition 1.3.16 is an obstruction to extending our later results to the case of a general (i.e., not necessarily semi-stable) abelian K -variety. If A_κ° contains a nontrivial unipotent subgroup, then it is not sensible to speak of the finite part of $A[p^m]$ since $A[p^m]$ is not quasi-finite. The submodule $(A[p^m]^f)_K \subset A_K[p^m]$ plays an essential role in our proofs in the semi-stable case. In [19], Grothendieck defines subgroups $A_K[p^m]^{\text{et}} \subset A_K[p^m]^{\text{ef}} \subset A_K[p^m]$, the essentially toric and essentially fixed parts. These may allow a p -adic Hodge-theoretic understanding of Néron component groups in general, though there are other be other issues, e.g., the failure of the surjectivity of the morphism φ of Lemma 1.3.18 in the case $l = p$.

Lemma 1.3.18.

With notation as in Definition 1.3.12, let l be any prime and let $m \in \mathbb{Z}_{\geq 1}$. The natural morphism

$$\varphi : A_\kappa[l^m]/A_\kappa^\circ[l^m] \longrightarrow \Phi_{A_K, \kappa}[l^m]$$

is injective. If A_κ° is l -divisible, then φ is surjective.

Proof. Let $\varphi : A_\kappa[l^m]/A_\kappa^\circ[l^m] \longrightarrow \Phi_{A_K, \kappa}[l^m]$ be the morphism that sends the class of $x \in A_\kappa[l^m](\bar{\kappa}) \subset A_\kappa$ to its class in $(A_\kappa/A_\kappa^\circ)(\bar{\kappa}) = \Phi_{A_K, \kappa}(\bar{\kappa})$. It is immediate that φ is injective.

Assume now that A_κ° is l -divisible. We show that φ is surjective. Let $x \in A_\kappa(\bar{\kappa})$ satisfy $l^m x \in A_\kappa^\circ$. Since A_κ° is l -divisible, we may write $l^m x = l^m x'$ for some $x' \in A_\kappa^\circ(\bar{\kappa})$. Then $x - x' \in A_\kappa[l^m](\bar{\kappa})$, and $x - x'$ has image in $\Phi_{A_K, \kappa}[l^m]$ equal to the class represented by x . This shows that φ is surjective. \square

1.4 Raynaud extensions

We will only need Raynaud extensions over complete discrete valuation rings, which were first described by Raynaud in [45]. We give a more general existence result developed by Lan in [35] using the more modern framework of degeneration data. Lan's result generalizes results of Mumford [41] and Faltings-Chai [13].

Proposition 1.4.1 ([35] 3.1.1.2, 3.3.3.6, 3.3.3.9, 3.4.3.2, 3.4.4.1).

Let R be a Noetherian integral domain complete with respect to the I -adic topology for some radical ideal I , let $S = \text{Spec } R$, and let $\eta \in S$ be the generic point. Let G be a semi-abelian S -scheme. For each $i \in \mathbb{Z}_{\geq 0}$, let $S_i = \text{Spec}(R/I^{i+1})$ and let $G_i = G \times_S S_i$. Write $\widehat{S} = \text{Spf}(R, I)$ and write \widehat{G} for the formal \widehat{S} -scheme given by the inductive system of all of the G_i 's. Assume that G_0 lies in an extension

$$0 \longrightarrow T_0 \longrightarrow G_0 \longrightarrow B_0 \longrightarrow 0 \quad (1.4.2)$$

of an abelian S_0 -scheme B_0 by an iso-trivial S_0 -torus T_0 .

(i) For each $i \in \mathbb{Z}_{\geq 1}$, the short exact sequence (1.4.2) lifts to an extension of S_i -group schemes

$$0 \longrightarrow T_i \longrightarrow G_i \longrightarrow B_i \longrightarrow 0 \quad (1.4.3)$$

where B_i is an abelian S_i -scheme and T_i is an S_i -torus. That is, (1.4.2) lifts an extension of formal \widehat{S} -group schemes

$$0 \longrightarrow \widehat{T} \longrightarrow \widehat{G} \longrightarrow \widehat{B} \longrightarrow 0 \quad (1.4.4)$$

where \widehat{B} is an abelian \widehat{S} -formal scheme and \widehat{T} is an \widehat{S} -torus.

(ii) Suppose that there is an ample cubical invertible sheaf¹ on G . Then (1.4.4) algebraizes to a short exact sequences of semi-abelian S -group schemes

$$0 \longrightarrow T \longrightarrow \widetilde{G} \longrightarrow B \longrightarrow 0 \quad (1.4.5)$$

where B is an abelian S -scheme and T is an S -torus. The group \widetilde{G} , which

¹Existence of such a sheaf will be no condition in our primary applications because of Remark 1.4.8, but see [35] 3.2.2.5 for the definition of a "cubical structure".

is unique up to unique isomorphism and is independent of the invertible sheaf, is called the **Raynaud extension associated to G** .

(iii) Suppose that G_η is an abelian variety, so that the dual semi-abelian scheme G^D is defined. Then G^D has the property that $(G^D)_0$ is also an extension of an abelian S_0 -scheme by an iso-trivial S_0 -torus. Therefore, parts (i) and (ii) give a dual Raynaud extension $\widetilde{G^D}$. In this case, the abelian part of $\widetilde{G^D}$ is B^D . By (ii), we have a short exact sequence

$$0 \longrightarrow T^D \longrightarrow \widetilde{G^D} \longrightarrow B^D \longrightarrow 0 \quad (1.4.6)$$

Remark 1.4.7. The notation \widetilde{G} for Raynaud extensions is used by Faltings-Chai. Lan uses the notation G^\natural . This is what Grothendieck would call $G^{\natural\circ}$, reserving G^\natural for an associated $G^{\natural\circ}$ -torsor over the component group Φ_{A_K} .

Remark 1.4.8. As explained in [35] 3.3.3.9, the requirement in (ii) of existence of an ample cubical invertible sheaf is always met when R is assumed normal.

Remark 1.4.9. We will apply Proposition 1.4.1, where G is the connected Néron model A° of an abelian K -variety with semi-stable reduction, where K is as in Situation 0.1.2. Assume the notation of Proposition 1.3.15. In particular, we have an isomorphism $h : A^\circ \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \xrightarrow{\sim} A'^\circ$. This gives isomorphisms

$$(A^\circ \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}) \times_{\mathcal{O}_{K'}} (\mathcal{O}_{K'}/\mathfrak{m}_{K'}^{i+1}) = A'^\circ \otimes_{\mathcal{O}_K} (\mathcal{O}_{K'}/\mathfrak{m}_{K'}^{i+1}) \quad (i \in \mathbb{Z}_{\geq 0})$$

which, in turn, give isomorphisms

$$(\widetilde{A}^\circ \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}) \times_{\mathcal{O}_{K'}} (\mathcal{O}_{K'}/\mathfrak{m}_{K'}^{i+1}) = \widetilde{A}'^\circ \otimes_{\mathcal{O}_K} (\mathcal{O}_{K'}/\mathfrak{m}_{K'}^{i+1}) \quad (i \in \mathbb{Z}_{\geq 0}),$$

since the Raynaud extension \widetilde{G} associated to G has, by definition, the same formal neighborhood of the identity as G . By the uniqueness assertion of Proposition 1.4.1 (ii), we have that

$$\widetilde{A}^\circ \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} = \widetilde{A}'^\circ.$$

This has shown that formation of Raynaud extensions attached to semi-stable abelian K -varieties commutes with base change to finite extensions of K .

We will use the following abbreviation of notation.

Definition 1.4.10 (\tilde{A}).

In Situation 0.1.2, let A_K be an abelian variety with semi-stable reduction, and let A be the Néron model of A_K . In this situation, we will abuse language by referring to \tilde{A} as the **Raynaud extension** of A_K , and we define

$$\tilde{A} := \widetilde{A^\circ},$$

where $\widetilde{A^\circ}$ is constructed by applying Proposition 1.4.1 to the semi-abelian \mathcal{O}_K -scheme A° . We extend this use of notation by writing

$$\tilde{A}^D = (\widetilde{(A^\circ)^D})$$

for the Raynaud extension of the dual of the semi-abelian \mathcal{O}_K -scheme A° .

Raynaud extensions are torsors for groups of multiplicative type over abelian schemes. Such objects can be reinterpreted in terms of homomorphisms from twisted constant S -groups. Given a semi-stable abelian variety A_K over a p -adic field K , the morphisms furnished by the definition that follows are used to construct the monodromy pairing attached to A_K . This is a \mathbb{Z}_K -valued bilinear pairing of twisted constant \mathcal{O}_K -group which will be used to decompose the torsion $\mathbb{Z}_p[\Gamma_K]$ -modules $A_K[p^m]$ as Baer sums.

Proposition 1.4.11 ([35] 3.1.5.1).

Let S be a Noetherian normal scheme. Let H be an S -group of multiplicative type. Let B be an abelian S -scheme. Let $\text{Ext}(B, H)$ denote the category of H -torsors on B . Let $\text{Hom}(X^*(H), B^D)$ denote the category whose objects are morphisms of S -schemes $X^*(H) \rightarrow B^D$, and where a morphism $\pi^\bullet : c_1 \rightarrow c_2$ of objects of $c_1, c_2 \in \text{Hom}(X^*(H), B^D)$ is the information of a commutative diagram

$$\begin{array}{ccc} X^*(H) & \xrightarrow{c_1} & B^D \\ \downarrow \pi^0 & & \downarrow \pi^1 \\ X^*(H) & \xrightarrow{c_2} & B^D \end{array}$$

There is an anti-equivalence of categories $\text{Ext}(B, H) \rightarrow \text{Hom}(X^*(H), B^D)$.

Remark 1.4.12. Omitting certain non-trivial details, we describe the equivalence and a quasi-inverse. We assume that $X^*(H)$ is constant, say modeled on the finitely generated abelian group X , noting that the case of general H can be handled by Galois descent (cf Remark 1.2.4). Being constant, to give

a morphism over S from $X^*(H)$ is the same as to give a morphism from each of its connected components, each of which is a copy of S . A homomorphism $X^*(H) \rightarrow B^D$ is, therefore, a collection of S -points of B^D indexed by elements of X .

Suppose we have an object of $\text{Ext}(H, B)$ and a homomorphism $\chi \in X^*(H)(S)$. We obtain a $\mathbb{G}_{m,S}(S)$ -torsor $c(-\chi)$ over B by pushout as in

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow -\chi & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & c(-\chi) & \longrightarrow & B \longrightarrow 0 \end{array} .$$

Identifying $\mathbb{G}_{m,S}$ -torsors on B with line bundles, we have an S -point of the S -scheme $\underline{\text{Pic}}_e^0(B/S) = B^D$. Being that $X^*(H)$ is locally constant on $S_{\text{ét}}$, a morphism $X^*(H) \rightarrow B^D$ is given by a collection of S -points of B^D parameterized by the connected components of $X^*(H)$. In this way we obtain a morphism $c_G : X^*(H) \rightarrow B^D$ that can be shown to be a homomorphism.

Suppose now that we are given a homomorphism $d : X^*(H) \rightarrow B^D$. Then each $\chi \in X^*(H)(S)$ gives $d(\chi) \in B^D(S)$. Using the structure sheaves $\mathcal{O}_{d(\chi)}$ of these $\mathbb{G}_{m,S}$ -torsors, one obtains an H -torsor $G(d)$ on B , a relatively affine B -scheme, by setting

$$G(d) := \underline{\text{Spec}}_{\mathcal{O}_B} \left(\bigoplus_{\chi \in X^*(H)(S)} \mathcal{O}_{d(\chi)} \right). \quad (1.4.13)$$

The functors $G \mapsto c_G$ and $d \mapsto G(d)$ are quasi-inverse equivalences.

In the setting of Proposition 1.4.1, where T and T^D are S -tori, let

$$Y := X^*(T^D) \quad (1.4.14)$$

$$Y^D := X^*(T). \quad (1.4.15)$$

Via Proposition 1.4.11, the Raynaud extensions (1.4.6) and (1.4.5) are equivalent to the information of two S -homomorphisms

$$c : Y \longrightarrow B \quad (1.4.16)$$

$$c^D : Y^D \longrightarrow B^D. \quad (1.4.17)$$

We will use these morphisms to construct the monodromy pairing attached to A_K . We first review some additional structure carried by the Poincaré sheaf.

As described in [35], 3.2.1, the Poincaré sheaf has additional structure coming from the theorem of the square, namely, that its associated $\mathbb{G}_{m,S}$ -torsor (also denoted P_B) has a canonical $\mathbb{G}_{m,S}$ -biextension structure. Roughly, this means that P_B comes equipped with isomorphisms (writing $m_{(-)}$ for group laws) of $\mathbb{G}_{m,S}$ -torsors

$$\begin{aligned} \mathrm{pr}_{13}^* P_B \otimes \mathrm{pr}_{23}^* P_B &\xrightarrow{\sim} (m_B \times \mathrm{id}_{B^D})^* P_B && \text{on } B \times_S B \times_S B^D \\ \mathrm{pr}_{12}^* P_B \otimes \mathrm{pr}_{13}^* P_B &\xrightarrow{\sim} (\mathrm{id}_B \times m_{B^D})^* P_B && \text{on } B \times_S B^D \times_S B^D \end{aligned}$$

satisfying certain identities reflecting the structures of B and B^D as commutative S -groups. Given local sections (b_1, b') and (b_2, b') of $B \times_S B^D$, the first of these isomorphisms lets us compare the values $s(b_i, b')$ of a section s of P_B in the fibers over the (b_i, b') with the value $s(m_B(b_1, b_2), b')$ in the fiber over $(m_B(b_1, b_2), b')$.

The trivial $\mathbb{G}_{m,S}$ -bundle on any product of two S -groups has a canonical biextension structure. A trivialization of a $\mathbb{G}_{m,S}$ -biextension is a trivialization of the associated line bundle that respects the biextension structures. As we will see in Example 1.4.19 below, a trivialization of a \mathbb{G}_m -biextension is sometimes determined by much less data than an arbitrary trivialization of a \mathbb{G}_m -torsor.

To give a precise account of the theory of biextensions is an unnecessary diversion given our objectives; for more information, see [35] or [20], Exposés VII and VIII. We content ourselves with an example, which we give after stating the result that is ultimately the root of our (mostly implicit) use of the theory of biextensions.

Proposition 1.4.18 ([35] 4.2.1.7).

Let notation be as in Proposition 1.4.1, with $c : Y \rightarrow B$ defined as in (1.4.16). Then the datum of a morphism $Y_\eta \rightarrow \tilde{A}_\eta$ lifting c determines and is determined by a trivialization of $(c_\eta \times c_\eta^D)^ P_{B,\eta}$ as a $\mathbb{G}_{m,R}$ -biextension on $Y_\eta \times Y_\eta^D$. Symbolically, we have*

$$\left\{ \begin{array}{ccc} & & Y_\eta \\ & \swarrow \iota_\eta & \downarrow c_\eta \\ \tilde{A}_\eta & \longrightarrow & B_\eta \end{array} \right\}_{\iota_\eta} \longleftrightarrow \left\{ \begin{array}{ccc} (c_\eta \times c_\eta^D)^* P_{B,\eta} & \longrightarrow & P_{B,\eta} \\ s_\eta \uparrow \downarrow & & \downarrow \\ Y_\eta \times Y_\eta^D & \xrightarrow{c_\eta \times c_\eta^D} & B_\eta \times B_\eta^D \end{array} \right\}_{s_\eta} .$$

Example 1.4.19. In Situation 0.1.2 (for concreteness), suppose $S = \mathrm{Spec} \mathcal{O}_K$ or $S = \mathrm{Spec} K$. Let $t \in \mathbb{Z}_{\geq 1}$ and let Y and Y' be copies of the constant group

scheme $\mathbb{Z}_{\mathcal{O}_K}^t$. Then Y (resp. Y' , resp. $Y \times_{\mathcal{O}_K} Y'$) is a disjoint union of copies of S indexed by $y \in \mathbb{Z}^t$ (resp. $y \in \mathbb{Z}^t$, resp. $(y, y') \in \mathbb{Z}^t \times \mathbb{Z}^t$); write $S(y)$, (resp. $S(y')$, resp. $S(y, y')$) for the component indexed by y , (resp. y' , resp. (y, y')). The trivial $\mathbb{G}_{m,S}$ -biextension $\mathbf{1}_{Y, Y'; S}$ over S is the trivial $\mathbb{G}_{m,S}$ -torsor $\mathbb{G}_{m,S} \times_S Y \times_S Y'$ on $Y \times_S Y'$ equipped with the obvious isomorphisms given by multiplication

$$\begin{aligned} \mathrm{pr}_{13}^* \mathbf{1}_{Y, Y'; S} \otimes \mathrm{pr}_{23}^* \mathbf{1}_{Y, Y'; S} &\simeq (m_Y \times \mathrm{id}_{Y'})^* \mathbf{1}_{Y, Y'; S} \text{ on } Y \times_S Y \times_S Y' \\ \mathrm{pr}_{12}^* \mathbf{1}_{Y, Y'; S} \otimes \mathrm{pr}_{13}^* \mathbf{1}_{Y, Y'; S} &\simeq (\mathrm{id}_Y \times m_{Y'})^* \mathbf{1}_{Y, Y'; S} \text{ on } Y \times_S Y' \times_S Y'. \end{aligned}$$

For instance, for $i = 1, 2$, let s_i be a global section of $\mathrm{pr}_{i3}^* \mathbf{1}_{Y, Y'; S}$. Over the connected component $S(y_1, y_2, y')$ of $Y \times_S Y \times_S Y'$ indexed by (y_1, y_2, y') , s_i is the same as a section $s_i(y_i, y')$ of $\mathbf{1}_{Y, Y'; S}$ over the component $S(y_i, y')$ of $Y \times_S Y'$, which is just to say that $s_i(y_i, y')$ is an element of K^\times (and maybe even of \mathcal{O}_K^\times , depending on the base). A section of $(m_Y \times \mathrm{id}_{Y'})^* \mathbf{1}_{Y, Y'; S}$ over $S(y_1, y_2, y')$ is the same as a section of $\mathbf{1}_{Y, Y'; S}$ over $S(y_1 + y_2, y')$. The first isomorphism is defined to take $s_1 \otimes s_2$ to the section of $(m_Y \times \mathrm{id}_{Y'})^* \mathbf{1}_{Y, Y'; S}$ which over $S(y_1 + y_2, y')$ is simply $s_1 s_2$.

Definition 1.4.20 (period homomorphism).

*In the setting of Proposition 1.4.18, an injective homomorphism $Y_\eta \rightarrow \tilde{A}_\eta$ lifting c is called a **period homomorphism**.*

1.5 1-motives and log 1-motives

The concept of a 1-motive was introduced by Deligne in [12], though examples over fields were studied earlier in Raynaud's work [45] on uniformization.

Definition 1.5.1 ([3], 1-motive; log 1-motive; generic fiber of a log 1-motive). *Let S be a scheme. An S -1-motive is a two-term complex $\mathcal{M} = [\iota : Y \rightarrow G]$ in degrees $-1, 0$ of commutative S -group schemes such that*

- (i) Y is a twisted constant group which is étale-locally modeled on a free abelian group of finite rank;
- (ii) G is an extension of an abelian S -scheme by an S -torus;
- (iii) ι is an homomorphism of S -groups.

Suppose now $S = \text{Spec } R$ with R a complete discrete valuation ring with positive residue characteristic and field of fraction K . A log 1-motive \mathcal{Q} over R is a triple (Y, G, ι_K) where Y and G are commutative group schemes over S satisfying (i) and (ii) above and ι_K is a homomorphism $\iota_K : Y_K \rightarrow G_K$. In this case $\mathcal{Q}_K := [\iota_K : Y_K \rightarrow G_K]$ is a K -1-motive.

Remark 1.5.2. The sort of semi-abelian S -schemes G that can give a term in a 1-motive over S have constant toric rank. In particular, in Situation 0.1.2, if A is the Néron model of an abelian K -variety A_K with bad reduction, then A is not a term in any 1-motive over \mathcal{O}_K .

Remark 1.5.3. The definition of a log 1-motive does not involve any log schemes. However, from such a thing it is possible to build a complex of logarithmic group schemes defined over $\text{Spec } R$ equipped with any fine saturated log structure by taking a “logarithmic enhancement” of the semi-abelian scheme G . For more on this, see [26] 1.4.

Example 1.5.4 (abelian varieties with good reduction). In Situation 0.1.2, an abelian variety A_K with good reduction gives rise to a 1-motive over \mathcal{O}_K as follows. Let A be the Néron model of A_K . Then the complex $[0 \rightarrow A]$ is a 1-motive over \mathcal{O}_K .

Example 1.5.5 (Tate 1-motive (cf [50], V §5)). In Situation 0.1.2, an elliptic curve E_K with bad semi-stable reduction gives rise to a log 1-motive over K as follows. There is a unique $q \in K^\times$ with $|q| < 1$ such that the Tate curve $E_{q,K}$ has $j(E_{q,K}) = j(E_K)$. This implies that $E_{q,\bar{K}} = E_{\bar{K}}$. With reference to any Weierstrass equation describing E_K , one associates a class $\gamma \in K^\times / K^{\times 2}$ that is independent of the chosen Weierstrass equation. One proves by calculations with Weierstrass equations that $\gamma = 1$ if and only if E_K has split reduction, and that $L := K(\sqrt{\gamma})$ is unramified quadratic if E_K has non-split reduction.

Recall that, letting x be a coordinate on $\mathbb{G}_{m,K}$ and letting $\{e_i : i \in \mathbb{Z}\}$ be the standard basis in $K^\mathbb{Z}$, any homomorphism $\mathbb{Z}_K \rightarrow \mathbb{G}_{m,K}$ is determined by sending x to $\sum_i v^i e_i$, with $v \in K^\times$. Then, if Z is a connected K -scheme, $\iota_{q,K}$ gives the homomorphism $\mathbb{Z}_K(Z) = \mathbb{Z} \rightarrow \mathbb{G}_{m,K}(Z) = \Gamma(Z, \mathcal{O}_Z^\times)$ sending 1 to q .

Suppose E_K has non-split reduction. It can be shown that there is an isomorphism $E_K \simeq E_{q,K}$. In this case, we associate to E_K the log 1-motive

$$\mathcal{T}_{q,K} = [\mathbb{Z}_K \xrightarrow{\iota_{q,K}} \mathbb{G}_{m,K}] \quad (1.5.6)$$

where $\iota_{q,K}$ is the homomorphism of groups sending 1 to q . Then E_K^{an} can

be recovered (up to isomorphism) as $\text{coker}(\iota_{q,K}^{\text{an}})$, where $\iota_{q,K}^{\text{an}}$ is the induced morphism on rigid analytic spaces. This is done explicitly, following Tate's pioneering work, in [50] Chapter V. There it is shown that, for $u \in \overline{K}^\times \setminus q^\mathbb{Z}$, the series

$$s_1(q) := \sum_{n \geq 1} \frac{nq^n}{1 - q^n}, \quad (1.5.7)$$

$$X(u, q) := \sum_{n \in \mathbb{Z}} \frac{q^n u}{(1 - q^n u)^2} - 2s_1(q) \quad (1.5.8)$$

$$Y(u, q) := \sum_{n \in \mathbb{Z}} \frac{(q^n u)^2}{(1 - q^n u)^3} + s_1(q)$$

converge p -adically. It is proved in Theorem 3 of loc. cit. (leaving out details relating to rigid analytic geometry) that the uniformization map is the one induced by the Γ_K -equivariant group homomorphism $\psi_q : \mathbb{G}_{m, \mathcal{O}_K}(\overline{K}) \rightarrow E_q(\overline{K})$ given by

$$u \mapsto \begin{cases} (X(u, q), Y(u, q)) & u \notin q^\mathbb{Z} \\ O & u \in q^\mathbb{Z} \end{cases}. \quad (1.5.9)$$

The log 1-motive ι_q does not extend to a 1-motive over \mathcal{O}_K . If it did, there would be an invertible global function $\sum_{i \in \mathbb{Z}} c_i e_i$ on $\mathbb{Z}_{\mathcal{O}_K}$ mapping to $\sum_{i \in \mathbb{Z}} q^i e_i$ under base change to K . But the existence of such a function is equivalent to invertibility of $q \in \mathcal{O}_K$, which fails because $|q| < 1$ by assumption.

Suppose now that E_K has non-split reduction. Recall that L/K is the unramified quadratic extension. It can be shown that E_L has split reduction. Let T_K be the norm torus relative to the extension L/K , i.e., T_K is the kernel of natural norm map $\text{Res}_{L/K} \mathbb{G}_{m,L} \rightarrow \mathbb{G}_{m,K}$. Since L/K is quadratic, $\text{Res}_{L/K} \mathbb{G}_{m,L}$ is two-dimensional, so the hypersurface T in $\text{Res}_{L/K} \mathbb{G}_{m,L}$ is a non-split one-dimensional torus. The log 1-motive $\mathcal{T}_{q,L}$ over L gives rise, by Galois descent, to a log 1-motive $[Y_K \rightarrow T_K]$, where Y_K is a form of the constant group \mathbb{Z}_L .

Remark 1.5.10. The log 1-motives of Example 1.5.5 have the property that the rigid analytic cokernel is algebraizable to an elliptic curve over K . This, of course, will not be the case for every log 1-motive; the trivial homomorphisms $0 \rightarrow 0$ and $\mathbb{Z}_K \rightarrow \mathbb{G}_{m,K}$ give log 1-motives, as do the morphisms $\mathbb{Z}_K \rightarrow \mathbb{G}_{m,K}$ given by sending 1 to an element of \mathcal{O}_K^\times . In order that a log 1-motive over K $\mathcal{Q}_K = [Y_K \rightarrow G_K]$ have rigid analytic cokernel that algebraizes to an abelian K -

variety, one needs to impose a p -adic analogue of the Riemann bilinear relations. This is addressed by studying categories of degeneration data. For more on this and on relevant history, see the introduction to [5].

Since the work of Faltings-Chai, much work around uniformization has been phrased in terms of categories of “degenerations of abelian schemes” and categories of “degeneration data”. We define the simplest of two such categories now. We note that $DD'(R, I)$, to be defined next, is not defined in [35], but that $DD(R, I)$ as described below agrees with the definition 4.4.10 of [35].

Definition 1.5.11 ($DEG(R, I)$ cf [35] 4.4.1; $DD'(R, I)$).

Let R be a Noetherian integral domain complete with respect to the I -adic topology for some radical ideal I , let $\eta \in \text{Spec } R$ be the generic point, and let $R_0 = R/I$.

In this setting, define $DEG(R, I)$ to be the category of semi-abelian R -schemes G such that G_η is an abelian variety and G_{R_0} is an extension of an abelian R_0 -scheme by an R_0 -torus.

Define also $DD'(R, I)$ to be category consisting of pairs (G, ι) where G is a semi-abelian R -scheme which is an extension of an abelian R -scheme B by an R -torus T , and ι is a period homomorphism (in the sense of Definition 1.4.20 $\iota : Y_\eta \rightarrow G_\eta$.

To study $DEG(R, I)$ and $DD'(R, I)$, one defines enrichments $DEG_*(R, I)$ and $DD_*(R, I)$ of these categories, where $*$ \in {ample, pol, IS}. One then defines

$DD(R, I) :=$ essential image of the forgetful functor $DD_{\text{pol}}(R, I) \rightarrow DD'(R, I)$.

Using the auxiliary information in these enriched categories, one can define functors $M_* : DD_*(R, I) \rightarrow DEG_*(R, I)$ and $F_* : DEG_*(R, I) \rightarrow DD_*(R, I)$. The construction and study of these functors, and of their dependence on the auxiliary data in the enriched categories, occupies a difficult part of the theory of compactifications of integral models of Shimura varieties. Ultimately, one has the following result which is the culmination of work of Tate, Raynaud, Grothendieck, Mumford, and Faltings-Chai. As we mentioned above, this result asserts that certain period relations (encoded in a degeneration datum) are necessary and sufficient for rigid uniformization of an abelian K -variety by its Raynaud extension. However, we do not make use of rigid analytic spaces. The reader interested in this perspective should see [5].

Proposition 1.5.12 ([35] 4.5.5.5 and [5] 6.17; [35] 4.5.5.3).

Let (R, I) be a pair as in Definition 1.5.11.

(i) There exist quasi-inverse equivalences

$$M(R, I) : \text{DD}(R, I) \rightarrow \text{DEG}(R, I)$$

and

$$F(R, I) : \text{DEG}(R, I) \rightarrow \text{DD}(R, I).$$

(ii) Assume the pair (R, I) is $(\mathcal{O}_K, \mathfrak{m}_K)$ as in Situation 0.1.2. Let A_K be an abelian K -variety with semi-stable reduction. Let A be the Néron model of A_K . Then A° is an object of $\text{DEG}(\mathcal{O}_K, \mathfrak{m}_K)$. Then $F(\mathcal{O}_K, \mathfrak{m}_K)(A^\circ)$ is a pair (\tilde{A}, ι_K) , where \tilde{A} is the Raynaud extension attached to A and $\iota_K : Y_K \rightarrow \tilde{A}_K$ is a period homomorphism. There exists an isomorphism

$$A(K) \simeq \tilde{A}(K)/\iota_K(Y(K)). \quad (1.5.13)$$

Since \tilde{A}_K and Y_K both prolong over \mathcal{O}_K , the degeneration datum (\tilde{A}, ι_K) is the same information as a log 1-motive

$$\mathcal{Q}_K := [Y_K \xrightarrow{\iota_K} \tilde{A}_K]. \quad (1.5.14)$$

Remark 1.5.15. The construction $F(R, I)$ proceeds by first constructing a functor F_{ample} , then another functor F_{pol} . The functor F_{pol} attaches a degeneration datum (which includes, in particular, a period homomorphism) to a pair consisting of an object G of $\text{DEG}(R, I)$ and an invertible sheaf L with additional structure. One then shows (cf [35] 4.5.5.1) that the period homomorphism is independent of L .

We are interested in the torsion in the K -1-motives arising via Proposition 1.5.12. We define this notion now.

Proposition 1.5.16 ([1], §1.3).

Let S be a scheme. Let $\mathcal{M} = [\iota : Y \rightarrow G]$ be a 1-motive over S . Each positive integer n gives a morphism of complexes of S -groups $[n] : \mathcal{M} \rightarrow \mathcal{M}$ which is multiplication by n on each term. Let $c(\mathcal{M}, n)$ be the cone of this morphism, and let $\mathcal{M}[n] := H^{-1}(c(\mathcal{M}, n))$. Then $\mathcal{M}[n]$ is a finite locally free S -group scheme.

Remark 1.5.17. Note that if \mathcal{Q} is a log 1-motive over a discrete valuation ring R , then the above proposition does not give a group “ $\mathcal{Q}[n]$ ” because the datum

of \mathcal{Q} does not include a morphism of R -schemes. On the other hand, the generic fiber \mathcal{Q}_K is a genuine 1-motive over K , so we do have finite K -groups $\mathcal{Q}_K[n]$.

1.6 Log 1-motives over p -adic fields and Raynaud decomposition

In this section, adopt the notation of Situation 0.1.2. Let $n \in \mathbb{Z}_{\geq 2}$ be a power of a prime l that may be equal to p . Suppose that $R = K$ or $R = \mathcal{O}_K$. Let $\mathcal{M} = [\iota : Y \rightarrow G]$ be an R -1-motive. Recall that \mathcal{M} is a cochain complex concentrated in degrees -1 and 0 . Write $\mathcal{M}[1]$ for shift of the cochain complex as in [51, Tag 011G], and write $d_{\mathcal{M}}$ and $d_{\mathcal{M}[1]}$ for the differentials on \mathcal{M} and $\mathcal{M}[1]$, respectively.

By definition, the cone $c(\mathcal{M}, n)$ of multiplication by n on \mathcal{M} is the complex

$$\mathcal{M}[1] \oplus \mathcal{M} = [Y \oplus 0 \longrightarrow G \oplus Y \longrightarrow 0 \oplus G] \quad d_{c(\mathcal{M}, n)} = \begin{pmatrix} d_{\mathcal{M}[1]} & 0 \\ n[1] & d_{\mathcal{M}} \end{pmatrix}$$

concentrated in degrees -2 , -1 and 0 with differential $d_{c(\mathcal{M}, n)}$ acting on columns of sections. These morphisms are described on local sections as follows. The first morphism sends $(y, 0)$ to $(-\iota(y), ny)$. The second morphism sends (g, y) to $(0, ng + \iota(y))$. We therefore have that (in local sections)

$$\mathcal{M}[n] = \frac{\{(g, y) \in G \oplus Y : ng = -\iota(y)\}}{\{(-\iota(y), ny) : y \in Y\}} \quad (1.6.1)$$

Consider the case where $\mathcal{M} = \mathcal{Q}_K$ where \mathcal{Q}_K is as in Proposition 1.5.12. By (1.5.13), we have that for any finite L/K

$$A_K(L) \simeq \tilde{A}_K(L) / \iota_L(Y_K(L)).$$

One has the following related result about the torsion in $A_K(\overline{K})$.

Proposition 1.6.2 (Corollary of [13] III.7.3; cf [37] 1.2.2.1).

Assume the setting of Proposition 1.5.12 (ii), so that A_K is an abelian variety with semi-stable reduction and $\mathcal{Q}_K = F(\mathcal{O}_K, \mathfrak{m}_K)(A^\circ)$ is the associated log 1-motive over K . Then, for each $n \in \mathbb{Z}_{\geq 1}$, there is a canonical isomorphism of finite K -groups

$$A_K[n] \simeq \mathcal{Q}_K[n]. \quad (1.6.3)$$

The isomorphism (1.6.3) plays an important role in our study of 1-crystalline Γ_K -submodules of $A_K[n]$. It will allow us to realize $A_K[n]$ as the torsion in a log finite flat \mathcal{O}_K -group, which amounts to giving a certain decomposition of $A_K[n]$ (or, more accurately, a decomposition of short exact sequence containing $A_K[n]$) as the Baer sum. That decomposition ultimately arises from Raynaud decomposition of \mathcal{Q}_K , which we now describe.

Proposition 1.6.4 (Raynaud decomposition; special case of [47] 4.5.1).

Let $\mathcal{Q}_K = [Y_K \xrightarrow{\iota_K} \tilde{A}_K]$ be as in (1.5.14). Attached to our choice of uniformizer ϖ is a pair of homomorphisms $\iota_{K,\varpi}^1, \iota_{K,\varpi}^2 : Y_K \rightarrow \tilde{A}_K$ such that, using addition in the common target \tilde{A}_K , we have

$$\iota_K = \iota_{K,\varpi}^1 + \iota_{K,\varpi}^2. \quad (1.6.5)$$

The homomorphisms $\iota_{K,\varpi}^1, \iota_{K,\varpi}^2$ satisfy

- (i) $\iota_{K,\varpi}^1$ is the generic fiber of an \mathcal{O}_K -1-motive $\mathcal{Q}_{\varpi}^1 := [Y \xrightarrow{\iota_{\varpi}^1} \tilde{A}]$; and
- (ii) $\iota_{K,\varpi}^2$ factors through the inclusion $T_K \rightarrow \tilde{A}_K$.

In this setting, we will write $\mathcal{Q}_K = \mathcal{Q}_{\varpi,K}^1 + \mathcal{Q}_{\varpi,K}^2$ and say that \mathcal{Q}_K is the sum of its Raynaud constituents.

Proposition 1.6.4 is proved using the monodromy pairing, which we now define. References for this construction are [47], Section 4.3 and [13] Chapter II, Remark 6.3, where the pairing is called B .

Recall Proposition 1.4.11, which says that the structure of \tilde{A} and \tilde{A}^D as semi-abelian \mathcal{O}_K schemes is encoded in two homomorphisms

$$\begin{aligned} c : Y &\longrightarrow B \\ c^D : Y^D &\longrightarrow B^D. \end{aligned}$$

Recall also that the Poincaré torsor P_B on $B \times_{\mathcal{O}_K} B^D$ has the structure of a $\mathbb{G}_{m,\mathcal{O}_K}$ -biextension. The pullback $(c \times c^D)^* P_B$ to $Y \times_{\mathcal{O}_K} Y^D$ need not be trivial, but it is trivial over $Y_K \times_K Y_K^D$, which is the disjoint union of copies of $\text{Spec } K$ (one for each Γ_K -orbit of points of $Y_K(\bar{K}) \times Y_K^D(\bar{K})$). By [35] 4.2.1.7, the information of the morphism ι_K is equivalent to the information of a morphism of K -schemes ²

$$s_K : Y_K \times Y_K^D \longrightarrow (c \times c^D)^* P_{B_K}$$

²This morphism is called τ^{-1} in [35].

or, equivalently, of a section of the $\mathbb{G}_{m,K}$ -biextension

$$(c \times c^D)^* P_{B_K}^{\otimes -1}$$

on $Y_K \times_K Y_K^D$. Since T and T^D are tori over \mathcal{O}_K , they are iso-trivial (see Remark 1.2.4). Hence we may choose finite unramified L/K inside of \overline{K} to split the tori T_K and T_K^D , or equivalently, so that Y and Y^D become isomorphic to $\mathbb{Z}_{\mathcal{O}_L}^t$, ($t = \text{rk } T_K$). Then the groups $Y_L \times Y_L^D$ and $(c \times c^D)^* P_{B_L}^{\otimes -1}$ carry actions of $\text{Gal}(L/K)$, and s_K gives rise to a morphism s_L that is equivariant for these actions. As in Example 1.4.19, s_L is determined by a collection of elements $s_L(y, \chi) \in L^\times$ indexed by $(y, \chi) \in Y_K(L) \times_K Y_K^D(L)$ that satisfy

$$s_L(\gamma \cdot y, \gamma \cdot \chi) = \gamma \cdot s_L(y, \chi).$$

By (cf [35] 4.3.1.9), these elements also satisfy the relations

$$\begin{aligned} s_L(y_1 + y_2, \chi_1 + \chi_2) &= s_L(y_1, \chi_1) s_L(y_2, \chi_1) s_L(y_1, \chi_2) s_L(y_2, \chi_2) \\ s_L(-y, \chi) &= s_L(y, \chi)^{-1} = s_L(y, -\chi), \end{aligned} \quad (1.6.6)$$

so that, using the valuation $v_{\overline{K}}$ of Situation 0.1.2, we can define a bilinear map by

$$\mu_{\overline{K}} : \begin{cases} Y_K(L) \times Y_K(L) & \longrightarrow & \mathbb{Z} \\ (y, \chi) & \longmapsto & v_{\overline{K}}(s_L(y, \chi)) \end{cases}$$

Since s_L is Γ_K -equivariant, $\mu_{\overline{K}}$ is Γ_K -equivariant.

Write $\{e_n : n \in \mathbb{Z}\}$ (resp., $\{e_y \otimes e_\chi : (y, \chi) \in \mathbb{Z}^t \times \mathbb{Z}^t\}$) for the standard idempotents of $\Gamma(\mathbb{Z}_{\mathcal{O}_L}, \mathcal{O})$ (resp., of $\Gamma(Y_{\mathcal{O}_L} \times_{\mathcal{O}_L} Y_{\mathcal{O}_L}^D, \mathcal{O})$). Use these symbols for the standard idempotent functions on the generic fibers \mathbb{Z}_L and $Y_L \times_L Y_L^D$, as well. The bilinear map $\mu_{\overline{K}}$ determines a unique morphism of affine group schemes

$$\mu_L : Y_L \times_L Y_L^D \longrightarrow \mathbb{Z}_L$$

which is determined on global sections by

$$\mu_L^\# : \begin{cases} \Gamma(Y_L \times_L Y_L^D, \mathcal{O}) & \longrightarrow & \Gamma(\mathbb{Z}_{\mathcal{O}_L}, \mathcal{O}) \\ e_n & \longmapsto & \sum_{(y, \chi) \in \mu_{\overline{K}}^{-1}(n)} e_y \otimes e_\chi \end{cases}$$

It is clear from the form of μ_L^\sharp that μ_L has a natural prolongation to a homomorphism of \mathcal{O}_L -groups

$$\mu_{\mathcal{O}_L} : Y_{\mathcal{O}_L} \times_{\mathcal{O}_L} Y_{\mathcal{O}_L}^D \longrightarrow \mathbb{Z}_{\mathcal{O}_L}. \quad (1.6.7)$$

The morphism $\mathcal{O}_K \rightarrow \mathcal{O}_L$ being finite étale since L/K is unramified, we are in the situation Remark 1.1.4 of Galois descent, where the $\mathrm{Gal}(L/K)$ -equivariant morphism is the same as a morphism of schemes with descent data relative to the covering $\mathrm{Spec} \mathcal{O}_L \rightarrow \mathrm{Spec} \mathcal{O}_K$. By Remark 1.1.4, the homomorphism $\mu_{\mathcal{O}_L}$ descends uniquely to a morphism

$$\mu : Y \times_{\mathcal{O}_K} Y^D \longrightarrow \mathbb{Z}_{\mathcal{O}_K}. \quad (1.6.8)$$

In fact, all of the diagrams exhibiting $\mu_{\mathcal{O}_L}$ as a homomorphism of \mathcal{O}_L -groups descend uniquely to diagrams of \mathcal{O}_K -schemes, so μ is a morphism of \mathcal{O}_K -groups.

Definition 1.6.9 (monodromy pairing).

*In Situation 0.1.2, let A_K be an abelian variety with semi-stable reduction, and let \mathcal{Q}_K be the associated log 1-motive. The pairing (1.6.8) defined above is called the **monodromy pairing** associated to A_K .*

The pairing μ_K is the obstruction to trivializing $(c \times c^D)^* P_B$; if μ were equal to zero, then all of the $s_L(y, \chi) \in L^\times$ would have valuation zero, so s_L would prolong over $Y_{\mathcal{O}_L} \times_{\mathcal{O}_L} Y_{\mathcal{O}_L}^D$, and descent would give a prolongation of s_K over $Y_{\mathcal{O}_K} \times_{\mathcal{O}_K} Y_{\mathcal{O}_K}^D$, hence a section of the $\mathbb{G}_{m, \mathcal{O}_K}$ -biextension $(c \times c^D)^* P_B$. Returning to Proposition 1.6.4, Raynaud decomposition with respect to the uniformizer ϖ arises from the identity

$$s_L(y, \chi) = (s_L(y, \chi) \varpi^{-\mu_{\overline{K}}(y, \chi)}) \varpi^{\mu_{\overline{K}}(y, \chi)}.$$

We illustrate this in the example of the q -Tate log 1-motive.

Example 1.6.10. We describe Raynaud decomposition and the monodromy pairing in the case of the q -Tate log 1-motive

$$\mathcal{T}_{q, K} = [\mathbb{Z}_K \xrightarrow{\iota_{q, K}} \mathbb{G}_{m, K}]$$

described in Example 1.5.5. Here, the Raynaud extension \tilde{E}_q of the Néron model E_q of $E_{q, K}$ is simply the torus $\mathbb{G}_{m, \mathcal{O}_K}$. Therefore, the abelian part of \tilde{E}_q is the trivial \mathcal{O}_K -group $B = \mathrm{Spec} \mathcal{O}_K$. A trivialization s_K of the pullback

of the $\mathbb{G}_{m,K}$ -torsor $P_{B,K}$ to $Y_K \times_K Y_K^D = \mathbb{Z}_K \times_K \mathbb{Z}_K$ is determined (via the relations (1.6.6)) by a single element $s_K(1,1) \in K^\times$. For the q -Tate curve, we have $s_K(1,1) = q$. The monodromy map $\mu_{\overline{K}} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ sends $(y, \chi) \in \mathbb{Z} \times \mathbb{Z}$ to

$$\mu_{\overline{K}}(y, \chi) = v_{\overline{K}}(s_K(y, \chi)) = v_{\overline{K}}(s_K(1,1)^{y\chi}) = v_{\overline{K}}(q^{y\chi}) = y\chi v_{\overline{K}}(q).$$

To get the Raynaud decomposition, we write $q = u\varpi^v$ with $u \in \mathcal{O}_K^\times$ and $v \in \mathbb{Z}_{\geq 1}$. This allows us to write

$$\mathcal{J}_{q,K} = \mathcal{J}_{q,K,\varpi}^1 + \mathcal{J}_{q,K,\varpi}^2$$

with

$$\mathcal{J}_{q,K,\varpi}^1 = [\mathbb{Z}_K \xrightarrow{\iota_{q,K,\varpi}^1} \mathbb{G}_{m,K}] \quad \iota_{q,K,\varpi}^1(1) = u$$

and

$$\mathcal{J}_{q,K,\varpi}^2 = [\mathbb{Z}_K \xrightarrow{\iota_{q,K,\varpi}^2} \mathbb{G}_{m,K}] \quad \iota_{q,K,\varpi}^2(1) = \varpi^v.$$

To recap the present chapter: after uniformizing A_K by its Raynaud extension and choosing a uniformizer ϖ of K , Raynaud decomposition gives an \mathcal{O}_K -1-motive \mathcal{Q}_{ϖ}^1 and another K -1-motive which is determined by the system of elements $\mu_{\overline{K}}(y, \chi)$ in a finite extension L/K indexed by the geometric points of the character lattices of two \mathcal{O}_K -tori. Our interest in this is that the torsion $\mathbb{Z}_p[\Gamma_K]$ -module $A_K[n]$ ($n \in \mathbb{Z}_{\geq 1}$) can be decomposed in terms of the finite flat \mathcal{O}_K -group $\mathcal{Q}_{\varpi}^1[n]$ and the reduction of $\mu_{\overline{K}} \bmod n$.

We end this chapter by establishing a new notational situation extending that of Situation 0.1.2.

Situation 1.6.11.

For the convenience of the reader and to simplify certain statements later, we establish a notational situation.

- (i) *All of the notation of Situation 0.1.2 is adopted;*
- (ii) *A_K is an abelian K variety;*
- (iii) *A is the Néron model of A_K over \mathcal{O}_K ;*
- (iv) *Φ_{A_K} is the component group of A over \mathcal{O}_K ;*
- (v) *A° is the subgroup scheme of A consisting of A_K and A_K° ;*

If A_K is assumed to have semi-stable reduction, then we follow Proposition 1.4.1 in using the notation below.

(vi) \tilde{A} is the Raynaud extension of A_K . It fits in a canonical short exact sequence of commutative \mathcal{O}_K -group schemes

$$0 \longrightarrow T \longrightarrow \tilde{A} \longrightarrow B \longrightarrow 0 \quad (1.6.12)$$

where T is an \mathcal{O}_K -torus and B is an abelian \mathcal{O}_K -scheme;

(vii) \tilde{A}^D is the Raynaud extension of A_K^D . It fits in a canonical short exact sequence of commutative \mathcal{O}_K -group schemes

$$0 \longrightarrow T^D \longrightarrow \tilde{A}^D \longrightarrow B^D \longrightarrow 0 \quad (1.6.13)$$

where T^D is an \mathcal{O}_K -torus and B is an abelian \mathcal{O}_K -scheme.

(viii) Y is the character group of T^D and Y^D is the character group of T ;

(ix) via Proposition 1.4.11, the short exact sequences (1.6.12) and (1.6.13) determine and are determined by two homomorphisms of \mathcal{O}_K -groups

$$\begin{aligned} c : Y &\longrightarrow B \\ c^D : Y^D &\longrightarrow B^D. \end{aligned}$$

(x) $\mathcal{Q}_K = F(\mathcal{O}_K, \mathfrak{m}_K)(A^\circ)$ is the log 1-motive of Proposition 1.5.12;

(xi) $\mu : Y \times_{\mathcal{O}_K} Y^D \rightarrow \mathbb{Z}_{\mathcal{O}_K}$ is the monodromy pairing (see Definition 1.6.9).

Chapter 2

Finite flat group schemes and monodromy

In the chapter that follows this one, we record a result of Liu (Theorem 3.2.4) that has the following consequence: if A_K is an abelian K -variety over a p -adic field with bad reduction, then, for m sufficiently large, $A_K[p^m]$ is not the generic fiber of a finite flat \mathcal{O}_K -group scheme. There is, hope, however, to realize every $A_K[p^m]$ as the generic fiber of a log finite flat \mathcal{O}_K -group in the case where A_K has semi-stable reduction. This is the primary objective of this chapter.

The theory of log finite flat group schemes is, to a large extent, due to Kato. Unfortunately, the foundational material on log finite flat group schemes has not all been published, and since the aspects that have been published are scattered, we take additional care with foundational issues. We say more about the history of log finite flat \mathcal{O}_K -groups in the introduction to Section 2.4. For now, we mention that we introduce a category $\text{Ext}'_{\text{fin}^p \mathcal{O}_K}$, which is only slightly more general than an analogous category introduced by Kato, and use this as our category of log finite flat \mathcal{O}_K -groups. We define a “generic fiber functor” on this category and show that it is exact. Since our primary technical concern is to remove the assumption $e_K < p - 1$ that appears in Raynaud’s work, we digress a bit to illustrate the effects of that assumption.

The proof that $A_K[p^m]$ is the torsion in a log finite flat \mathcal{O}_K -group is not original to our work, but is due to Bertapelle, Candilera and Cristante in [3] (see Proposition 2.5.3). We introduce a modification of their construction by defining a monodromy homomorphism that will be called N_{p^m} in order to pro-

duce an object of Kato’s category $\text{fin}_{\mathcal{O}_K}^{p,N}$. Most of our original work in this chapter is in checking certain compatibilities between objects of the category $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\vee$, and in and carefully defining the generic fiber functor on $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\vee$ and proving that it is exact.

Conventions

Throughout this chapter, we place ourselves in Situation 0.1.2. We point out here that many of our results have natural generalizations to discrete valuation rings more general than \mathcal{O}_K , and that statements hold for finite flat group schemes and $\mathbb{Z}_p[\Gamma_K]$ -modules of orders not just a power of p . The statements for orders coprime to p tend to be easier to prove.

2.1 Categories of finite flat R -groups

Definition 2.1.1 (fin_R^p).

Let R be a field or a complete discrete valuation ring. Let fin_R^p denote the category of commutative $\text{Spec } R$ -group schemes of p -power order that are finite and flat over $\text{Spec } R$. We will call objects of this category “finite flat R -groups,” suppressing mention of the commutativity and the assumption on the order.

Remark 2.1.2. In general, given any scheme S , one obtains a reasonable category by letting fin_S be the category of S -group schemes that are finite and locally free over S . Hence an S -group $\pi : H \rightarrow S$ is an object of fin_S if and only if $\pi_* \mathcal{O}_H$ is a quasi-coherent, finite locally free \mathcal{O}_S -module and $H = \text{Spec}_{\mathcal{O}_S}(\pi_* \mathcal{O}_H)$. For $S = \text{Spec } R$ with R as in the definition, the local freeness is equivalent to being without torsion, which is equivalent to flatness. We use the term “finite flat” because this seems to be more common.

Definition 2.1.3 (FÉt_R ; $\text{fin}_R^{p,\text{ét}}$).

For any affine scheme $\text{Spec } R$, let FÉt_R denote the full subcategory of Sch_R consisting of objects that are finite étale over $\text{Spec } R$. In the case where R is a field or a complete discrete valuation ring, write $\text{fin}_R^{p,\text{ét}}$ for the full subcategory of fin_R^p consisting of objects whose underlying scheme is in FÉt_R .

With R as in the above definition, the objects of fin_R^p that are étale over R form a simple but very important class. We can study such objects using the following result.

Proposition 2.1.4 (See [51, Tag OBND]). (i) Let $\text{Spec } R$ be a connected affine scheme. Let \bar{x} be a geometric point of $\text{Spec } R$, and let $\text{Fib}_{\bar{x}}$ denote the natural functor taking an object $Y \rightarrow \text{Spec } R$ of $\text{F}\acute{\text{E}}\text{t}_R$ to the finite $\pi_1(\text{Spec } R, \bar{x})$ -set $|Y \times_X \bar{x}|$, where $| - |$ denotes the underlying set of the scheme. Then $\text{Fib}_{\bar{x}}$ is an equivalence of categories.

(ii) Let F be a perfect field, let \bar{F} be an algebraic closure, let $F \subset \bar{F}$ be an embedding corresponding to a geometric point $\bar{x} : \text{Spec } \bar{F} \rightarrow \text{Spec } F$, and let $\Gamma_F = \text{Gal}(\bar{F}/F)$. Then there is a canonical isomorphism $\Gamma_F \rightarrow \pi_1(\text{Spec } F, \bar{x})$, and the fiber functor $\text{Fib}_{\bar{x}}$ induces an equivalence of categories between the category of finite étale F -schemes and the category of finite abelian groups with an action of Γ_F that is continuous for the discrete topology.

(iii) Every object of fin_K^p is étale since K has characteristic zero. Therefore we have an equivalence of categories between fin_K^p and $\text{Rep}_{\text{tors}}(\Gamma_K)$

Remark 2.1.5. For $F = K$ or κ , Proposition 2.1.4 shows that $\text{fin}_F^{p,\acute{\text{e}}\text{t}}$ inherits the structure of an abelian category from the category $\text{Rep}_{\text{tors}}(\Gamma_F)$. We will freely identify objects of $\text{fin}_F^{p,\acute{\text{e}}\text{t}}$ with their corresponding Γ_F -modules. For example, we will often conflate the difference between an object H_K of fin_K^p and the group $H_K(\bar{K})$ equipped with its Γ_K action.

The ring \mathcal{O}_K is Henselian, so one has the following result.

Proposition 2.1.6 (Special case of [51, Tag 0A48]).

The functor $\text{F}\acute{\text{E}}\text{t}_{\mathcal{O}_K} \rightarrow \text{F}\acute{\text{E}}\text{t}_{\kappa}$ induced by base change is an equivalence of categories. By Proposition 2.1.4, $\text{F}\acute{\text{E}}\text{t}_{\mathcal{O}_K}$ is therefore equivalent to $\text{Rep}_{\text{tors}}(\Gamma_{\kappa})$.

The objects of fin_K^p that admit models over \mathcal{O}_K form a small but important class. We introduce the following standard term.

Definition 2.1.7 (prolongation).

A **prolongation** of an object V of fin_K^p is an object H of $\text{fin}_{\mathcal{O}_K}^p$ such that there exists an isomorphism $H_K \simeq V$ of objects of fin_K^p .

As we will see below in Example 2.2.4, it may happen even in simple situations that a finite K -group admits non-isomorphic prolongations. We will see in that example, and in Proposition 2.2.1, that this complication is related to ramification of K/\mathbb{Q}_p . Before addressing these more subtle issues, we first describe the simpler situation of finite K -groups with unramified generic fiber.

Example 2.1.8. Suppose that H_K is an object of fin_K^p such that $H_K(\bar{K})$ is unramified as a $\mathbb{Z}_p[\Gamma_K]$ -module. We claim that H_K admits an étale prolongation.

By assumption, $H_K(\bar{K})$ is a finite abelian group with a continuous action of Γ_K . Via Proposition 2.1.6, then, H_K uniquely determines a finite étale \mathcal{O}_K -group J with $J(\bar{\kappa})$ isomorphic to $H(\bar{K})$ as Γ_K -modules. There is a finite unramified L/K such that $J_{\mathcal{O}_L}$ is a finite constant \mathcal{O}_L -group. This field therefore satisfies $J(L) = J(\bar{K})$ and $J(\kappa_L) = J(\bar{\kappa})$, where κ_L is the residue field of L . Since $J_{\mathcal{O}_L}$ is constant, specialization gives a canonical isomorphism of finite abelian groups

$$\text{sp} : J(L) \xrightarrow{\sim} J(\kappa_L).$$

If we show that the map sp is Γ_K -equivariant, where $J(\kappa_L)$ is given an action of Γ_K via the map $\Gamma_K \rightarrow \Gamma_{\kappa}$, then we have shown that J is a prolongation of H_K , since $J(\bar{\kappa}) \simeq H(\bar{K})$. But the action of Γ_K on L preserves \mathcal{O}_L and is even determined by its restriction to \mathcal{O}_L , so $\sigma \in \Gamma_K$ determines and is determined by an automorphism of $J(\mathcal{O}_L)$, which determines an automorphism of $J(\kappa_L)$. By the fully faithfulness assertion of Proposition 2.1.6 along with Proposition 2.1.4 for the field κ_L , an automorphism of $J(\kappa_L)$ is the same as an automorphism of the finite étale \mathcal{O}_L -group J . This shows that sp is an isomorphism. This has also shown that if H is an object of $\text{fin}_{\mathcal{O}_K}^{p,\text{ét}}$ then H_K is unramified.

Example 2.1.9. Let $n \in \mathbb{Z}_{\geq 1}$ and write $[n]$ for the multiplication-by- n endomorphism on any object of $\text{CGp}_{\mathcal{O}_K}$.

Let X be a finitely generated abelian group, and let $X_{\mathcal{O}_K}$ denote the constant \mathcal{O}_K -group. The group X/nX gives a constant \mathcal{O}_K -group $(X/nX)_{\mathcal{O}_K}$. It can be shown that $(X/nX)_{\mathcal{O}_K}$ is the cokernel of $[n]$ on $X_{\mathcal{O}_K}$. The \mathcal{O}_K -group $(X/nX)_{\mathcal{O}_K}$, which we will sometimes denote by $X/nX_{\mathcal{O}_K}$, is finite étale.

More generally, let Y be any twisted constant \mathcal{O}_K -group, and let $S' \rightarrow \text{Spec } \mathcal{O}_K$ be a finite étale cover such that $Y_{S'}$ is constant. Then the cokernel of $[n]$ on $Y_{S'}$ is $Y_{S'}/nY_{S'}$. This descends uniquely to an \mathcal{O}_K -scheme Y/nY . To check that Y/nY is, in fact, the cokernel of $[n]$ on Y , one checks that certain diagrams in $\text{CGp}_{\mathcal{O}_K}$ commute. But these diagrams give diagrams in $\text{CGp}_{S'}$ involving $Y_{S'}/nY_{S'}$. Since $Y_{S'}/nY_{S'}$ is the cokernel of $[n]$ on $Y_{S'}$, these diagrams, commute. By uniqueness of descent of these diagrams, Y/nY is the cokernel of $[n]$ on Y .

Example 2.1.10. Let H be an \mathcal{O}_K -group of multiplicative type (for instance, H could be a torus), and let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . Then the n -torsion $H[n]$ is an object of $\text{fin}_{\mathcal{O}_K}^p$ which is multiplicative, hence connected by Remark 2.2.21.

Example 2.1.11. Let G be a Yoneda extension of an abelian \mathcal{O}_K -scheme by an \mathcal{O}_K -torus T . Let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . Then ([21], Example III 6.4) $G[n]$ is a finite flat \mathcal{O}_K -group.

Remark 2.1.12. Let A_K be an abelian K -variety with semi-stable reduction, and let A be the Néron model. Let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . One knows Proposition 1.3.16 that the semi-abelian \mathcal{O}_K -scheme A° has quasi-finite and flat n -torsion $A^\circ[n]$ over \mathcal{O}_K , but it may be that $A^\circ[n]$ is not finite over \mathcal{O}_K . By Example 2.1.11, this can only happen if the toric rank of A° is non-constant.

Quasi-finite flat but non-finite \mathcal{O}_K -groups arise naturally from torsion in Néron models of abelian varieties. For the study of such groups, the following definition is useful.

Definition 2.1.13 (finite part).

Let R be a henselian discrete valuation ring. For any quasi-finite flat R -group H , define the finite part H^f of H to be the largest finite flat \mathcal{O}_K -subgroup of H .

Remark 2.1.14. Existence of H^f is given in [19], 2.2.3 (but see also [51, Tag 04GG], part (13)). In fact, one defines H^f canonically as the locus where $H \rightarrow \text{Spec } R$ is finite. With our assumptions, a morphism is finite if and only if it is quasi-finite and proper. By the valuative criterion for properness, H^f is the union in H of H_κ and the open subset of H_K consisting of those points of that specialize into H_κ . The fact that H^f is a subgroup follows from functoriality of formation of H^f .

Remark 2.1.15. Let A be as in Situation 1.6.11, so that A is the Néron model of its generic fiber. Let $n \in \mathbb{Z}_{\geq 2}$. In general the \mathcal{O}_K -group $A[n]$ of n -torsion need not be quasi-finite or flat (see Proposition 1.3.16). If A° is semi-abelian, then both $A^\circ[n]$ and $A[n]$ are quasi-finite and flat over \mathcal{O}_K , so we can form $A[n]^f$ and $A^\circ[n]^f$. As we saw above, formation of the finite part amounts to discarding points of the generic fiber, so in the case where $A^\circ[n]$ is quasi-finite but non-finite, we have proper K -subgroups

$$A^\circ[n]_K^f \subset A[n]_K^f \subset A[n]_K = A_K[n].$$

Note that every quasi-finite K -group is necessarily finite flat, so that there is no useful notion of a maximal finite flat subgroup of such a thing. Therefore, writing $A^\circ[n]_K^f$ for $(A^\circ[n]^f)_K$ should cause no confusion.

By Theorem 3.2.11, our study in Chapter 3 and Chapter 4 of the functor Crys_1 on torsion in semi-stable abelian K -varieties is the same as the study of $\mathbb{Z}_p[\Gamma_K]$ -submodules arising as generic fibers of finite flat \mathcal{O}_K -groups. Néron models of abelian K -varieties with semi-stable reduction have quasi-finite torsion, and thus we could hope to phrase all of our work in terms of maximal finite flat \mathcal{O}_K -subgroups; indeed, this is the perspective taken in the work Marshall and Kim-Marshall. Their work borrows an idea from the proof of Lemma 6.2 of Ribet's paper [48], which is phrased in terms of finite parts. The connection with our work is via the following result.

Proposition 2.1.16 ([35] 3.4.2.1).

Let A_K be a semi-stable abelian K -variety. Let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . The Raynaud extension \tilde{A} satisfies

$$\tilde{A}[n] \simeq A^\circ[n]^f. \quad (2.1.17)$$

2.2 Exactness in $\text{fin}_{\mathcal{O}_K}^p$

Still in Situation 0.1.2, passing to generic fibers defines an exact functor $\text{fin}_{\mathcal{O}_K}^p \rightarrow \text{fin}_K^p$. This functor is very far from being essentially surjective. It is also, in general, not fully faithful. As we have seen in Remark 2.1.5, fin_K^p has the structure of an abelian category. The following result shows that $\text{fin}_{\mathcal{O}_K}^p$ sometimes inherits this structure when the ramification index e_K of K/\mathbb{Q}_p is small.

Proposition 2.2.1 ([46] 3.3.3 and 3.3.6).

Suppose $e_K < p - 1$. Then every finite torsion $\mathbb{Z}_p[\Gamma_K]$ -module arises as the generic fiber of at most one finite flat \mathcal{O}_K -group, up to isomorphism. Let G, H be finite flat \mathcal{O}_K -groups.

- (i) *Every morphism $G_K \rightarrow H_K$ prolongs uniquely to a morphism $\pi : G \rightarrow H$. Said differently, the generic fiber functor $\text{fin}_{\mathcal{O}_K}^p \rightarrow \text{fin}_K^p$ is fully faithful.*
- (ii) *The kernel and cokernel of π formed in the category of group schemes over \mathcal{O}_K are finite flat over \mathcal{O}_K .*
- (iii) *The category of finite flat \mathcal{O}_K -groups is an abelian category.*

Example 2.2.2. We saw in Example 2.1.8 that an object J_K of fin_K^p which is unramified as a $\mathbb{Z}_p[\Gamma_K]$ -module admits an étale prolongation J over \mathcal{O}_K . Under the hypothesis $e_K < p - 1$, this is the only prolongation of J_K . We see that,

when $e_K < p - 1$, a finite flat group scheme J is étale if and only if J_K is unramified. This statement is equivalent to the statement that, for H an object of $\text{fin}_{\mathcal{O}_K}^p$ and $H^{\text{ét}}$ is defined as in Proposition 2.2.14, $(H^{\text{ét}})_K$ is the maximal unramified quotient of H_K .

We illustrate the nature of the ramification assumption in Raynaud's result Proposition 2.2.1 by giving a detailed example. For this we recall a criterion of Grothendieck for existence of certain quotients.

Proposition 2.2.3 (Special case of [51, Tag 03BM]; see also [51, Tag 03LK]). *Let $H = \text{Spec } A$ be an object of $\text{fin}_{\mathcal{O}_K}^p$ and let $H' = \text{Spec } A'$ be a closed \mathcal{O}_K -subgroup. Then the quotient H/H' in $\text{Sh}_{\text{Ab}}(\text{Sch}_{\mathcal{O}_K, \text{fpf}})$ is representable by an object of $\text{fin}_{\mathcal{O}_K}^p$. The representing object has the form $\text{Spec } C$, where C is the equalizer of the two morphisms $A \rightarrow A' \times_{\mathcal{O}_K} A$ corresponding (by taking global sections) to pr_2 and the morphism $H' \times_{\mathcal{O}_K} H \rightarrow H$ giving the action of H' on H . The natural quotient morphism $H \rightarrow H/H'$ is faithfully flat.*

Example 2.2.4. Let $K = \mathbb{Q}_p(\zeta)$ with ζ a primitive p -th root of unity (e.g., we could take $K = \mathbb{Q}_2$). Then $\mathcal{O}_K = \mathbb{Z}_p[\zeta]$ and $\varpi = 1 - \zeta$ is a uniformizer of \mathcal{O}_K .

Since K contains a p -th root of 1, $K[t]/(t^p - 1)K[t] \simeq \prod_{i=0}^{p-1} K$ as K -algebras. Hence $\mu_{p,K}$ is (equivalent to) a free K -vector space of dimension p . Every geometric point of $\mu_{p,K}$ is defined over K , so the action of Γ_K on $\mu_{p,K}$ is trivial. We see that $\mu_{p,K}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}_K$. By comparing special fibers we see immediately that μ_{p,\mathcal{O}_K} and $\mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K}$ are not isomorphic. We conclude that the uniqueness of prolongations asserted in Proposition 2.2.1 can fail without the ramification assumption.

Let $\pi : \mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K} \rightarrow \mu_{p,\mathcal{O}_K}$ be the morphism determined on global sections by

$$\pi^\# : \begin{cases} \mathcal{O}_K[t]/(t^p - 1)\mathcal{O}_K[t] & \longrightarrow & \prod_{x \in \mathbb{Z}/p\mathbb{Z}} \mathcal{O}_K \\ t & \longmapsto & (1, \zeta, \dots, \zeta^{p-1}). \end{cases} \quad (2.2.5)$$

This gives a homomorphism of group schemes which is clearly trivial on special fibers and is injective on generic fibers. The kernel in \mathcal{O}_K -groups, then, is not finite flat, since its rank is not locally constant. More precisely, let J be the kernel of π formed in the category of affine \mathcal{O}_K -groups. Then J is represented by the ring

$$A = \prod_{x \in \mathbb{Z}/p\mathbb{Z}} \mathcal{O}_K \otimes_{\pi^\#, c} \mathcal{O}_K,$$

where the tensor product is taken over $\mathcal{O}_K[t]/(t^p - 1)\mathcal{O}_K$ with respect to π^\sharp and the counit c of μ_{p, \mathcal{O}_K} , which sends t to 1. Hence we have

$$A \simeq \prod_{x \in \mathbb{Z}/p\mathbb{Z}} \mathcal{O}_K/(0, 1 - \zeta, \dots, 1 - \zeta^{p-1})$$

Of course, each $1 - \zeta^i$ is also a uniformizer of \mathcal{O}_K , so we see that

$$(0, 1 - \zeta, \dots, 1 - \zeta^{p-1}) = \varpi(0, 1, u_2, \dots, u_{p-1})$$

for $u_2, \dots, u_{p-1} \in \mathcal{O}_K^\times$. Hence we see that A has torsion as an \mathcal{O}_K -module, which is to say that A is not flat over \mathcal{O}_K . Hence J is not an object of $\text{fin}_{\mathcal{O}_K}^p$.

As a replacement, it is natural to consider the maximal torsion-free quotient of A . From our above work, we see that the closed scheme $J' = \text{Spec}(A/\text{tors})$ of J is simply the trivial subgroup $\text{Spec } \mathcal{O}_K \subset J$. We claim that J' is a kernel in $\text{fin}_{\mathcal{O}_K}^p$. Indeed, suppose that $\psi : H \rightarrow \mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K}$ has the property that $\pi \circ \psi : H \rightarrow \mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K} \rightarrow \mu_p$ is zero. Then ψ factors through $J \rightarrow \mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K}$. Suppose $H = \text{Spec } A'$, so that $\pi : H \rightarrow J$ is equivalent to a morphism $\pi^\sharp : A \rightarrow A'$. Since H is finite flat, A' is torsion-free over \mathcal{O}_K , so π^\sharp factors through A/tors . This shows that ψ factors through J' , so that J' is a kernel in $\text{fin}_{\mathcal{O}_K}^p$. Noting that the torsion in A is exactly the kernel of the morphism $J_K \hookrightarrow J$, we see that J' is the scheme-theoretic closure of J_K in J .

We now construct a cokernel of π . We would like to construct the quotient of μ_{p, \mathcal{O}_K} under the action of $\mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K}$ via π . Let a be the morphism

$$a : \mu_p \times_{\mathcal{O}_K} \mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K} \xrightarrow{\text{id} \times \pi} \mu_{p, \mathcal{O}_K} \times_{\mathcal{O}_K} \mu_{p, \mathcal{O}_K} \xrightarrow{\text{mult}} \mu_{p, \mathcal{O}_K}$$

which, on global sections, is the morphism of \mathcal{O}_K -algebras

$$\begin{aligned} \mathcal{O}_K[t]/(t^p - 1)\mathcal{O}_K[t] &\xrightarrow{a^\sharp} \mathcal{O}_K[t]/(t^p - 1)\mathcal{O}_K[t] \otimes_{\mathcal{O}_K} \prod_{x \in \mathbb{Z}/p\mathbb{Z}} \mathcal{O}_K \\ t &\longmapsto t \otimes (1, \zeta, \dots, \zeta^{p-1}). \end{aligned}$$

Write pr_1 for the projection onto the first factor of $\mu_p \times_{\mathcal{O}_K} \mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K}$, which on global sections is determined by $t \mapsto t \otimes 1$. By Proposition 2.2.3, the quotient of the action a is represented by the fiber product of rings (see Proposition 2.2.3)

$$B = \{f \in \mathcal{O}_K[t]/(t^p - 1)\mathcal{O}_K[t] : a^\sharp(f) = \text{pr}_1^\sharp(f)\}.$$

Considering the natural basis of $\mathcal{O}_K[t]/(t^p - 1)\mathcal{O}_K[t] \otimes_{\mathcal{O}_K} \prod_{x \in \mathbb{Z}/p\mathbb{Z}} \mathcal{O}_K$ as a free \mathcal{O}_K -module, we see that B is simply $\mathcal{O}_K \subset \mathcal{O}_K[t]/(t^p - 1)\mathcal{O}_K[t]$. We conclude that the cokernel of π in affine \mathcal{O}_K -groups is the trivial group. Since this is finite flat, this is a cokernel in $\text{fin}_{\mathcal{O}_K}^p$, so we need not worry about killing torsion as we did when forming the kernel of π .

The image of π is defined to be kernel of the natural morphism $\mu_{p, \mathcal{O}_K} \rightarrow \text{coker}(\pi)$, which is of course all of μ_{p, \mathcal{O}_K} . The coimage of π is the cokernel of $J' \rightarrow \mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K}$, which is all of $\mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K}$. Hence we see that the coimage of π is not isomorphic to the image of π . It is an axiom of abelian categories that the natural map from the coimage of π to the image of π is an isomorphism, so we conclude that $\text{fin}_{\mathcal{O}_K}^p$ is not an abelian category.

To be clear, Example 2.2.4 shows that $\text{fin}_{\mathcal{O}_K}^p$ is not, in general, abelian, but it leaves open the possibility that kernels and cokernels (at least of certain morphisms) exist, and illustrates issues that arise in their construction. We address this construction now.

There are two approaches to the construction of kernels and cokernels in $\text{fin}_{\mathcal{O}_K}^p$. One approach is to show that there is a fully faithful embedding of $\text{fin}_{\mathcal{O}_K}^p$ into the abelian category $\text{Sh}_{\text{Ab}}(\text{Sch}_{\mathcal{O}_K, \text{fppf}})$ and show that the essential image of this embedding is stable under formation of kernels and cokernels in $\text{Sh}_{\text{Ab}}(\text{Sch}_{\mathcal{O}_K, \text{fppf}})$. On the other hand, following heuristics illustrated in Example 2.2.4, one can simply form kernels and cokernels in the category of affine \mathcal{O}_K -groups and then kill the torsion global sections to get group schemes flat over \mathcal{O}_K . We follow the first approach. We give Quillen's original definition of exact categories, taken from [44].

Definition 2.2.6 (exact category; exactness structure; admissible monomorphism; admissible epimorphism; exact functor).

An **exact category** is an additive category \mathcal{C} equipped with a set $E(\mathcal{C})$ (an **exactness structure**) of three-term cochain complexes placed in degrees $-1, 0$ and 1 , called **short exact sequences**, satisfying the conditions (a)-(c) below. Given $E(\mathcal{C})$, one has two distinguished classes of morphisms of objects of \mathcal{C} ;

- (i) A morphism $d^0 : C^{-1} \rightarrow C^0$ of objects of \mathcal{C} is an **admissible monomorphism** if there exists a sequence of the form $[C^{-1} \xrightarrow{d^0} C^0 \rightarrow \eta^1]$ in $E(\mathcal{C})$.
- (ii) A morphism $d^1 : C^0 \rightarrow C^1$ of objects of \mathcal{C} is an **admissible epimorphism** if there exists a sequence of the form $[\eta^{-1} \rightarrow C^0 \xrightarrow{d^1} C^1]$ in $E(\mathcal{C})$.

The objects of $E(\mathcal{C})$ are required to satisfy the following conditions.

- (a) Any three-term cochain complex of objects of \mathcal{C} placed in degrees $-1, 0$ and 1 , that is isomorphic to a sequence in $E(\mathcal{C})$ is itself in $E(\mathcal{C})$. If η^{-1} and η^1 are objects of \mathcal{C} , then any sequence of the form

$$\eta^{-1} \xrightarrow{(\text{id}, 0)} \eta^{-1} \oplus \eta^1 \xrightarrow{\text{Pr}_2} \eta^1$$

is in $E(\mathcal{C})$. For any diagram in $E(\mathcal{C})$ of the form

$$\eta^\bullet = [\eta^{-1} \xrightarrow{d^0} \eta^0 \xrightarrow{d^1} \eta^1]$$

d^1 is a cokernel for d^0 in \mathcal{C} and d^0 is a kernel for d^1 in \mathcal{C} .

- (b) A composition of admissible monomorphisms (resp., epimorphisms) is an admissible monomorphism (resp., epimorphism). The pushout (resp., pullback) of an admissible monomorphism (resp., epimorphism) is an admissible monomorphism (resp., epimorphism).
- (c) Let $d^0 : C^{-1} \rightarrow C^0$ be a morphism in \mathcal{C} possessing a cokernel in \mathcal{C} . If there exists a morphism $d' : C^0 \rightarrow C'^0$ such that the composition $d' \circ d^0 : C^{-1} \rightarrow C'^0$ is an admissible monomorphism, then d^0 is an admissible monomorphism. Dually, let $d^1 : C^0 \rightarrow C^1$ be a morphism in \mathcal{C} possessing a kernel in \mathcal{C} . If there exists a morphism $d' : C'^0 \rightarrow C^0$ such that the composition $d^1 \circ d' : C'^0 \rightarrow C^1$ is an admissible epimorphism, then d^1 is an admissible epimorphism.

Let $(\mathcal{C}_1, E(\mathcal{C}_1))$ and $(\mathcal{C}_2, E(\mathcal{C}_2))$ be two exact categories. An **exact functor** $F : (\mathcal{C}_1, E(\mathcal{C}_1)) \rightarrow (\mathcal{C}_2, E(\mathcal{C}_2))$ is an additive functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ sending sequences in $E(\mathcal{C}_1)$ to sequences in $E(\mathcal{C}_2)$.

Example 2.2.7. An abelian category has a canonical exactness structure, the one in which every monomorphism (resp., epimorphism) is an admissible monomorphism (resp., admissible epimorphism). We will always tacitly equip any abelian category with this exactness structure.

Lemma 2.2.8 ([44], §2).

Let \mathcal{A}' be a full subcategory of an abelian category \mathcal{A} , and write $E(\mathcal{A})$ for the canonical set of short exact sequences in \mathcal{A} . Let

$$E(\mathcal{A}') := \{\eta^\bullet \in E(\mathcal{A}) : \eta^{-1}, \eta^1 \in \mathcal{A}'\}.$$

If \mathcal{A}' is closed under Yoneda extensions in the sense that $\eta^0 \in \mathcal{A}'$ for every $\eta^\bullet \in E(\mathcal{A}')$, then $(\mathcal{A}', E(\mathcal{A}'))$ is an exact category.

It turns out that every monomorphism of the category $\text{fin}_{\mathcal{O}_K}^p$ can be put into a short exact sequence of objects of $\text{fin}_{\mathcal{O}_K}^p$.

Proposition 2.2.9 (Corollary of Proposition 1.1.3, since η^{-1}/\mathcal{O}_K is affine).
Let $[\eta^{-1} \rightarrow \eta \rightarrow \eta^1]$ be a short exact sequence of objects of $\text{Sh}_{\text{Ab}}(\text{Sch}_{\mathcal{O}_K, \text{fppf}})$. Suppose that η^{-1} and η^1 are representable by objects of $\text{fin}_{\mathcal{O}_K}^p$. Then η is representable by an object of $\text{fin}_{\mathcal{O}_K}^p$. Therefore, the functor

$$h : \text{fin}_{\mathcal{O}_K}^p \rightarrow \text{Sh}_{\text{Ab}}(\text{Sch}_{\mathcal{O}_K, \text{fppf}})$$

realizes $\text{fin}_{\mathcal{O}_K}^p$ as an exact subcategory of the abelian category $\text{Sh}_{\text{Ab}}(\text{Sch}_{\mathcal{O}_K, \text{fppf}})$.

Proposition 2.2.10.

The category $\text{fin}_{\mathcal{O}_K}^p$ is exact. Its short exact sequences are of the form

$$0 \longrightarrow \eta^{-1} \xrightarrow{i} \eta^0 \xrightarrow{\pi} \eta^1 \longrightarrow 0$$

with i a closed immersion and π faithfully flat.

Proof. Exactness of $\text{fin}_{\mathcal{O}_K}^p$ follows from combining Lemma 2.2.8 and Proposition 2.2.9. The exactness structure $E(\text{fin}_{\mathcal{O}_K}^p)$ consists of all possible Yoneda extensions of objects of $\text{fin}_{\mathcal{O}_K}^p$, which are formed in $\text{Sh}_{\text{Ab}}(\text{Sch}_{\mathcal{O}_K, \text{fppf}})$ and turn out to be in $\text{fin}_{\mathcal{O}_K}^p$ by Proposition 2.2.9.

By definition, the inclusion of an \mathcal{O}_K -subgroup $i : \eta^{-1} \rightarrow \eta^0$ into another is a monomorphism of fppf sheaves. Since finite \mathcal{O}_K -schemes are separated, [51, Tag 035D] shows that i is finite. Since i is a monomorphism, [51, Tag 03BB] shows that i is a closed immersion. Conversely, if i is a closed immersion, then i is a monomorphism by [51, Tag 01L1]. This shows that whenever we have a short exact sequence of objects $[\eta^{-1} \rightarrow \eta^0 \rightarrow \eta^1]$ of $\text{fin}_{\mathcal{O}_K}^p$, the morphism $\eta^{-1} \rightarrow \eta^0$ is a closed immersion. By Proposition 2.2.3, $\eta^0 \rightarrow \eta^1$ is faithfully flat. \square

Example 2.2.11. Suppose G is a semi-abelian \mathcal{O}_K -scheme that sits in a short exact sequence

$$0 \longrightarrow T \longrightarrow G \longrightarrow B \longrightarrow 0$$

where T is an \mathcal{O}_K -torus and B is an abelian \mathcal{O}_K -scheme. It can be shown ([21]

Example III 6.4) there is a short exact sequence in $\text{fin}_{\mathcal{O}_K}^p$

$$0 \longrightarrow T[n] \longrightarrow G[n] \longrightarrow B[n] \longrightarrow 0,$$

and that, in fact, the $G[n]$ fit into a p -divisible group. Note that this is not true for a general semi-abelian \mathcal{O}_K -scheme, since the torsion $G[n]$ is not in general an object of $\text{fin}_{\mathcal{O}_K}^p$.

Since the $G[n]$ fit into a p -divisible group $G[p^\infty]$, we may define the Tate module by

$$T_p G_K := T_p(G[p^\infty]).$$

In the case where G is the Raynaud extension $G = \tilde{A}$ attached to a K -variety with semi-stable reduction, we have the Tate module $T_p \tilde{A}_K$. This will play an important role in Chapter 4.

Example 2.2.12. Suppose $R = K$ or $R = \mathcal{O}_K$ and that we have an R -1-motive $\mathcal{Q} = [\iota : Y \rightarrow G]$. Let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . Recall (1.6.1), which says that $\mathcal{M}[n]$, regarded as a sheaf on $(\text{Sch}_R)_{\text{fppf}}$, is described in local sections by

$$\mathcal{M}[n] = \frac{\{(g, y) \in G \oplus Y : ng = -\iota(y)\}}{\{(-\iota(y), ny) : y \in Y\}}$$

Write $[g, y]$ for the class of a pair (g, y) of local sections of $G \oplus Y$ in the group $\mathcal{M}[n]$. There are homomorphisms of R -groups described in local sections by

$$i : \begin{cases} G[n] & \longrightarrow & \mathcal{M}[n] \\ g & \longmapsto & [g, 0] \end{cases}$$

and

$$\pi : \begin{cases} \mathcal{M}[n] & \longrightarrow & Y/nY \\ [g, y] & \longmapsto & \bar{y} \end{cases}.$$

To see that π is well-defined, note that $\pi([-\iota(y), ny]) = \bar{ny} = 0$ in Y/nY . The morphism i is injective, and $\ker \pi = \text{im } i$ as morphisms on local sections, hence as morphisms of sheaves. By divisibility of semi-abelian R -schemes ([6], page 180), π is surjective. Hence we have a short exact sequence

$$\eta(\mathcal{M}, n) : \quad 0 \longrightarrow G[n] \xrightarrow{i} \mathcal{M}[n] \xrightarrow{\pi} Y/nY \longrightarrow 0. \quad (2.2.13)$$

Aside from the short exact sequences of the form (2.2.13), the short exact sequences of finite flat R -groups of greatest interest to us are those described by the following proposition

Proposition 2.2.14 (connected-étale sequences; see [21], II 3.2).

Let H be an object of $\text{fin}_{\mathcal{O}_K}^p$. Then the connected component H° through which the identity section factors is an open subgroup. The quotient $H^{\text{ét}} := H/H^\circ$ is the maximal étale quotient of H . The canonical short exact sequence

$$\text{cé}(H) : \quad 0 \longrightarrow H^\circ \longrightarrow H \longrightarrow H^{\text{ét}} \longrightarrow 0$$

is functorial in H .

Remark 2.2.15. Let H be an object of $\text{fin}_{\mathcal{O}_K}^p$. We define

$$H_K^\circ := (H^\circ)_K \text{ and } H_K^{\text{ét}} := (H^{\text{ét}})_K.$$

Example 2.2.16. Continue the notation of Example 2.2.12. Though Y/nY is étale, the sequence $\eta(\mathcal{M}, n)$ need not be the connected-étale sequence of $\mathcal{M}[n]$. For instance, in the extreme case of Example 1.5.4, we have $G = A$, i.e. G is the Néron model of A_K , and $Y = 0$. Let us work out the relationship between $\eta(\mathcal{M}, n)$ and $\text{cé}(\mathcal{M}[n])$.

Since Y/nY is étale, the morphism $\mathcal{M}[n]^\circ \rightarrow \mathcal{M}[n]$ factors through $G[n]$, hence through $G[n]^\circ$ since $\mathcal{M}[n]^\circ$ is connected. Since $G[n]^\circ \rightarrow G[n] \rightarrow \mathcal{M}[n]$ factors through $\mathcal{M}[n]^\circ$, we see that

$$\mathcal{M}[n]^\circ = G[n]^\circ.$$

We obtain a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}[n]^\circ = G[n]^\circ & \longrightarrow & \mathcal{M}[n] & \longrightarrow & \mathcal{M}[n]^{\text{ét}} \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & G[n] & \longrightarrow & \mathcal{M}[n] & \longrightarrow & Y/nY \longrightarrow 0 \end{array}$$

where the dashed map is induced via the fact that $\mathcal{M}[n]^{\text{ét}}$ is the maximal étale quotient of $\mathcal{M}[n]$. By the snake lemma in the abelian category $\text{Sh}_{\text{Ab}}(\text{Sch}_{R, \text{fppf}})$, the kernel of this morphism is $G[n]^{\text{ét}}$, giving a short exact sequence

$$0 \longrightarrow G[n]^{\text{ét}} \longrightarrow \mathcal{M}[n]^{\text{ét}} \longrightarrow Y/nY \longrightarrow 0. \quad (2.2.17)$$

We will need a few different duality theories for (certain) objects of $\text{fin}_{\mathcal{O}_K}^p$.

Proposition 2.2.18 (Cartier duality [21] II 1.4).

Let H be an object of $\text{fin}_{\mathcal{O}_K}^p$. Let

$$H^* := \underline{\text{Hom}}(H, \mathbb{G}_m).$$

Then $H \mapsto H^*$ is an exact functor from the category $\text{fin}_{\mathcal{O}_K}^p$ to itself which is self quasi-inverse.

Example 2.2.19. Let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . The Cartier dual of the constant group $\mathbb{Z}/n\mathbb{Z}_{\mathcal{O}_K}$ is μ_{n, \mathcal{O}_K} , a group of multiplicative type in the sense of Definition 1.2.2. In general, Cartier duality agrees with the duality functor X^* of Proposition 1.2.3 when X^* is restricted to torsion objects.

Example 2.2.20. Let E be an elliptic curve over \mathcal{O}_K with ordinary reduction. Let $n \in \mathbb{Z}_{\geq 2}$ be an power of p . The torsion $E[n]$ lies in a short exact sequence

$$0 \longrightarrow \mu_{n, \mathcal{O}_K} \longrightarrow E[n] \longrightarrow \mathbb{Z}/n\mathbb{Z}_{\mathcal{O}_K} \longrightarrow 0.$$

Cartier duality being an exact contravariant functor, Example 2.2.19 shows that we have a short exact sequence

$$0 \longrightarrow \mu_{n, \mathcal{O}_K} \longrightarrow E[n]^* \longrightarrow \mathbb{Z}/n\mathbb{Z}_{\mathcal{O}_K} \longrightarrow 0.$$

In fact, $E[n]^* = E^D[n]$.

Remark 2.2.21. Let H be an object of $\text{fin}_{\mathcal{O}_K}^p$ of multiplicative type. We claim that H is connected. This can be checked after a finite (unramified) extension L/K such that $H_{\mathcal{O}_L}^*$ is constant. We may, therefore, assume $H_{\mathcal{O}_K}^*$ has the form $\prod_{i=1}^N \mathbb{Z}/p^{s_i}\mathbb{Z}_{\mathcal{O}_K}$, so that $H = H^{**}$ has the form $\prod_{i=1}^N \mu_{p^{s_i}, \mathcal{O}_K}$. The claim is clear from the fact that every μ_{p^s, \mathcal{O}_K} is connected.

Definition 2.2.22 (Pontryagin duality).

Let J be an object of $\text{fin}_{\mathcal{O}_K}^{\text{ét}}$. Define J^\wedge to be the object of $\text{fin}_{\mathcal{O}_K}^{\text{ét}}$ such that

$$(J^\wedge)_\kappa := (J_\kappa)^\wedge,$$

where $(J_\kappa)^\wedge$ is the finite abelian group $\text{Hom}(J_\kappa, \mathbb{Q}/\mathbb{Z})$ equipped with the action

of $\pi_1(\mathrm{Spec} \mathcal{O}_K, \bar{s})$ defined by

$$\begin{aligned} \pi_1(\mathrm{Spec} \mathcal{O}_K, \bar{s}) \times (J_{\bar{s}})^\wedge &\longrightarrow (J_{\bar{s}})^\wedge \\ (\gamma, f) &\longmapsto (x \xrightarrow{\gamma \cdot f} \gamma^{-1} \cdot f(\gamma \cdot x)) \end{aligned}$$

Then $J \mapsto J^\wedge$ is an exact functor from the category $\mathrm{fin}_{\mathcal{O}_K}^{\acute{e}t}$ to itself which is self quasi-inverse.

The image of the restriction of the Cartier duality functor to $\mathrm{fin}_{\mathcal{O}_K}^{\acute{e}t}$ is defined to be the “groups of multiplicative type”. Such groups arise naturally, e.g., the torsion in any torus is of multiplicative type. To study such groups, it is helpful to introduce a covariant surrogate for Cartier duality. We obtain such a duality between étale and multiplicative-type objects of $\mathrm{fin}_{\mathcal{O}_K}^p$ by composing with Pontryagin duality.

Proposition 2.2.23 (Tate twisting).

Let $R = \mathcal{O}_K$ or K . Let H be an object of fin_R^p and suppose that n is a power of p such that $nH = 0$.

(i) Suppose H is étale. Define

$$H(1) := (H^\wedge)^*.$$

Then $H(1)$ is a multiplicative-type object of $\mathrm{fin}_{\mathcal{O}_K}^p$ and there is a canonical isomorphism of sheaves of $\mathbb{Z}/n\mathbb{Z}_R$ -modules on $\mathrm{Sh}_{\mathrm{Ab}}((\mathrm{Sch}_R)_{\mathrm{fppf}})$

$$H(1) = H \otimes \mu_n. \tag{2.2.24}$$

(ii) Suppose H is multiplicative. Define

$$H(-1) := (H^*)^\wedge.$$

Then $H(-1)$ is an étale object of $\mathrm{fin}_{\mathcal{O}_K}^p$ and there is a canonical isomorphism of sheaves of $\mathbb{Z}/n\mathbb{Z}_R$ -modules on $\mathrm{Sh}_{\mathrm{Ab}}((\mathrm{Sch}_R)_{\mathrm{fppf}})$

$$H(-1) = \underline{\mathrm{Hom}}(\mu_n, H). \tag{2.2.25}$$

Proof. Given (2.2.24) and (2.2.25), the claims that $H(1)$ and $H(-1)$ are objects of $\mathrm{fin}_{\mathcal{O}_K}^p$ (in particular, representability of the sheaves $H \otimes \mu_n$ and $\mathrm{Hom}(\mu_n, H)$)

follows from the fact that Cartier duality and Pontryagin duality (when it is defined) each produce objects of $\text{fin}_{\mathcal{O}_K}^p$. The claimed isomorphisms are formal. When H is étale, we have

$$H \otimes \mu_{n,R} = (H^\wedge)^\wedge \otimes \mu_{n,R} = \underline{\text{Hom}}(H^\wedge, \mu_{n,R}) = (H^\wedge)^* = H(1),$$

which gives (2.2.24). When H is multiplicative, using Cartier duality,

$$\underline{\text{Hom}}(\mu_{n,R}, H) = \underline{\text{Hom}}(H^*, \mathbb{Z}/n\mathbb{Z}_R) = (H^*)^\wedge = H(-1),$$

which gives (2.2.25). □

Remark 2.2.26. If $R = K$, then the assumptions in the above proposition are always satisfied; H is always étale, hence H is also multiplicative, being the dual of the étale group H^* . In that case, one can always define $H(i) := H \otimes \mu_{n,K}^{\otimes i}$ for all $i \in \mathbb{Z}$. In the case $R = \mathcal{O}_K$, the sheaf $H \otimes \mu_{n,\mathcal{O}_K}^{\otimes i}$ may not be representable by a finite flat \mathcal{O}_K -group.

2.3 Component groups and monodromy

Let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . In Situation 1.6.11, recall that we have a morphism

$$\mu : Y \times_{\mathcal{O}_K} Y^D \longrightarrow \mathbb{Z}_{\mathcal{O}_K}.$$

Let $Y^{D,\vee} := \text{Hom}(Y^D, \mathbb{Z}_{\mathcal{O}_K})$. This is a twisted constant \mathcal{O}_K -group. We define a homomorphism

$$\mu_Y : Y \longrightarrow Y^{D,\vee} \tag{2.3.1}$$

on local sections by $\mu(z) : \chi \mapsto \mu(z, \chi)$. Twisting gives a homomorphism of finite \mathcal{O}_K -groups of multiplicative type

$$Y/nY(1) \longrightarrow Y^{D,\vee}/nY^{D,\vee}(1). \tag{2.3.2}$$

Recall that, in Situation 1.6.11, Y^D is defined so that the torus T in \tilde{A} has the form $T = \text{Hom}(Y^D, \mathbb{G}_m)$. Using the isomorphism of abelian sheaves

$$T = \underline{\text{Hom}}(Y^D, \mathbb{Z}) \otimes \mathbb{G}_m = Y^{D,\vee} \otimes \mathbb{G}_m$$

and writing $T[n] = \ker(Y^{D,\vee} \otimes \mathbb{G}_m \xrightarrow{n} Y^{D,\vee} \otimes \mathbb{G}_m)$, we see that

$$Y^{D,\vee}/nY^{D,\vee}(1) = Y^{D,\vee}/nY^{D,\vee} \otimes \mu_n \simeq Y^{D,\vee} \otimes \mu_n = T[n]. \quad (2.3.3)$$

Composing (2.3.2) and the identification (2.3.3) with the canonical closed immersion $T[n] \hookrightarrow \tilde{A}[n]$ gives a homomorphism.

$$\nu_n : Y/nY(1) \longrightarrow \tilde{A}[n]. \quad (2.3.4)$$

Remark 2.3.5. Let $m \in \mathbb{Z}_{\geq 1}$. It is immediate from the definition of the ν_{p^m} (see (2.3.2) that the ν_{p^m} are compatible in m in the sense that the diagram

$$\begin{array}{ccc} Y/p^{m+1}Y & \xrightarrow{\nu_{p^{m+1}}} & \tilde{A}[p^{m+1}] \\ \downarrow & & \downarrow \\ Y/p^m Y & \xrightarrow{\nu_{p^m}} & \tilde{A}[p^m] \end{array}$$

where the vertical morphisms are the canonical surjections, commutes.

Example 2.3.6. In the example of the q -Tate log 1-motive $\mathcal{T}_{q,K}$, where $Y = \mathbb{Z}_{\mathcal{O}_K}$, the monodromy homomorphism is a map $\mu : \mathbb{Z}_{\mathcal{O}_K} \times_{\mathcal{O}_K} \mathbb{Z}_{\mathcal{O}_K} \rightarrow \mathbb{Z}_{\mathcal{O}_K}$. We saw in Example 1.6.10 that this is determined by the bilinear map $\mu_K : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ given by

$$\mu(y, \chi) = y\chi v_K(q).$$

In this example, the morphism μ_Y arises from the morphisms

$$\mu_Y \bmod n : \mathbb{Z}/n\mathbb{Z}_{\mathcal{O}_K} \longrightarrow (\mathrm{Hom}(\mathbb{Z}_{\mathcal{O}_K}, \mathbb{Z}_{\mathcal{O}_K}))/n(\mathrm{Hom}(\mathbb{Z}_{\mathcal{O}_K}, \mathbb{Z}_{\mathcal{O}_K}))$$

of constant groups given by $y \mapsto (\chi \mapsto y\chi v_K(q))$. After identifying the \mathcal{O}_K -groups $\underline{\mathrm{Hom}}(\mathbb{Z}_{\mathcal{O}_K}, \mathbb{Z}_{\mathcal{O}_K})/n \cdot \mathrm{Hom}(\mathbb{Z}_{\mathcal{O}_K}, \mathbb{Z}_{\mathcal{O}_K})$ and $\mathbb{Z}/n\mathbb{Z}_{\mathcal{O}_K}$, the morphism $\mu_Y \bmod n$ is simply multiplication by $v_K(q)$. It follows that the the level- n monodromy homomorphism

$$\nu_n : \mu_n = \mathbb{Z}/n\mathbb{Z}(1) \longrightarrow \tilde{E}_q[n] = \mathbb{G}_{m, \mathcal{O}_K}[n] = \mu_n,$$

is the $v_K(q)$ -th power map.

The following result is needed for the proof of Theorem 4.2.23.

Proposition 2.3.7.

There is a canonical isomorphism

$$\ker \nu_n(-1) \simeq \Phi_{A_K}[n].$$

Proof. Since Tate twisting is exact, any morphism φ of groups of multiplicative type satisfies $(\ker \varphi)(-1) = \ker(\varphi(-1))$. Since ν_n factors through the torus $T \subset \tilde{A}$, we write $\ker \nu_n(-1) = (\ker \nu_n)(-1)$ without risk of confusion. By [13] Corollary 8.2, $\Phi_{A_K} \simeq \text{coker } \mu_Y$, where $\mu_Y : Y \rightarrow Y^{D^\vee}$ is the morphism of (2.3.1). By Proposition 1.1.1 and Proposition 1.1.2 (iv), μ_Y is determined by the Γ_K -equivariant homomorphism of abelian groups $\mu_Y : Y_K(\bar{K}) \rightarrow Y_K^{D^\vee}(\bar{K})$. The groups $Y_K(\bar{K}), Y_K^{D^\vee}(\bar{K})$ are free of rank $\text{rk } T = \text{rk } T^D$. Since the homomorphism $\mu_Y : Y_K(\bar{K}) \rightarrow Y_K^{D^\vee}(\bar{K})$ has finite cokernel $\Phi_{A_K}(\bar{K})$, it is injective, and it becomes an isomorphism after inverting $N := \#\Phi_{A_K}(\bar{K})$. Therefore, μ_Y gives rise to a surjection $(1/N)Y(\bar{K}) \rightarrow Y^{D^\vee}(\bar{K})$. Post-composing with the quotient map $Y^{D^\vee}(\bar{K}) \rightarrow Y^{D^\vee}(\bar{K})/NY^{D^\vee}(\bar{K})$, we obtain a surjection $\mu_{Y,N}(-1) : (1/N)Y(\bar{K})/Y(\bar{K}) \rightarrow Y^{D^\vee}(\bar{K})/NY^{D^\vee}(\bar{K})$. Consider the diagram of Γ_K -equivariant homomorphisms of abelian groups

$$\begin{array}{ccccccc} Y_K(\bar{K}) & \longrightarrow & \frac{1}{N}Y_K(\bar{K}) & \longrightarrow & \frac{1}{N}Y_K(\bar{K})/Y_K(\bar{K}) & \longrightarrow & 0 \\ & & \downarrow \mu_Y & & \downarrow N\mu_Y & & \downarrow \mu_{Y,N}(-1) \\ 0 & \longrightarrow & Y_K^{D^\vee}(\bar{K}) & \xrightarrow{N} & Y_K^{D^\vee}(\bar{K}) & \longrightarrow & Y_K^{D^\vee}(\bar{K})/NY_K^{D^\vee}(\bar{K}) \end{array} .$$

As noted above, the vertical map μ_Y is injective with cokernel $\Phi_{A_K}(\bar{K})$, hence also that the vertical map $N\mu_Y$ also is injective with cokernel $\Phi_{A_K}(\bar{K})$. The snake lemma gives an exact sequence

$$0 \longrightarrow 0 \longrightarrow \ker \mu_{Y,N}(-1) \longrightarrow \Phi_{A_K}(\bar{K}) \xrightarrow{N} \Phi_{A_K}(\bar{K}) \longrightarrow 0.$$

Since N is the order of $\Phi_{A_K}(\bar{K})$, the map N in this sequence is 0. Hence we have that $\ker \mu_{Y,N}(-1) = \Phi_{A_K}(\bar{K})$. The proposition follows upon passing to the n -torsion subgroups. Specifically, restricting $\mu_{Y,N}(-1)$ to $\frac{1}{n}Y_K(\bar{K})/Y_K(\bar{K})$, we obtain a morphism whose image is killed by n . This induces a morphism $(1/n)Y/Y \rightarrow Y^{D^\vee}/nY^{D^\vee}$. Unwinding the definitions, and making the identification $(1/n)Y/Y = Y/nY$, we see that this is exactly the morphism $\nu_n(-1)$. \square

Example 2.3.8. It is well known (see [6], page 26) that the Néron component

group of the Tate curve $E_{q,K}$ is cyclic of order $v_K(q)$. We have, therefore that

$$\Phi_{E_{q,K}}[p^\infty] = \Phi_{E_{q,K}}[p^{v_p(v_K(q))}] = \mathbb{Z}/p^{v_p(v_K(q))}\mathbb{Z}_{\mathcal{O}_K},$$

where $v_p(v_K(q))$ is the p -adic valuation of the integer $v_K(q)$. We saw in Example 2.3.6 that the morphism ν_n attached to $E_{q,K}$ (via $\mathcal{T}_{q,K}$) is the $v_K(q)$ -th power map on μ_n . This morphism kills sections of μ_n of order $p^{v_p(v_K(q))}$. We see that

$$\ker \nu_n = \begin{cases} \mu_n & \text{if } n \leq p^{v_p(v_K(q))} \\ \mu_{p^{v_p(v_K(q))}} & \text{otherwise} \end{cases},$$

hence that

$$\ker \nu_n(-1) = \begin{cases} \mathbb{Z}/n\mathbb{Z}_{\mathcal{O}_K} & \text{if } n \leq p^{v_p(v_K(q))} \\ \mathbb{Z}/p^{v_p(v_K(q))}\mathbb{Z}_{\mathcal{O}_K} & \text{otherwise} \end{cases}.$$

This has verified explicitly that Proposition 2.3.7 holds for Tate curves.

2.4 Finite flat \mathcal{O}_K -groups with monodromy

Throughout this section, adopt the notation of Situation 1.6.11, so that, in particular, K is an extension of \mathbb{Q}_p and A_K is an abelian K -variety.

Recall (Proposition 2.2.14) that a finite flat \mathcal{O}_K -group H has an associated connected-étale sequence

$$0 \longrightarrow H^\circ \longrightarrow H \longrightarrow H^{\text{ét}} \longrightarrow 0.$$

The generic fiber functor $\text{fin}_{\mathcal{O}_K}^p \rightarrow \text{fin}_K^p$ is exact, but its essential image (which we will see in Chapter 3 is equivalent to a category called $\text{Rep}_{\text{tors}}^{1,\text{crys}}(\Gamma_K)$) is not closed under extensions. Given objects H'_K, H''_K in the essential image and an extension V of H''_K by H'_K , it is natural to wonder whether one can construct an object H (in some appropriate category) out of H' and H'' from which V can be recovered. It turns out that for some such V , we can write V as the generic fiber of a log finite flat \mathcal{O}_K -group.

A log structure on a scheme is an additional structure to which one can attach a sheaf of differentials “with logarithmic poles”. Such structures were used under a different guise at least as early as Deligne’s 1974 work [11]. On the other hand, log structures provide a natural setting for toric geometry.

Though it is useful in a variety of contexts, it seems that the formalism of log structures was first developed with applications to arithmetic in mind. According to [24], log structures were first introduced in a 1988 IHES seminar lead by Fontaine during which Hyodo’s paper [23] on p -adic étale cohomology of semi-stable varieties was discussed. Following the seminar, a large body of foundational work was produced by Kato ([30], [28], [29], [27]). Some of these pre-prints remain unpublished, but many of the results in them have been published by others on an as-needed basis. An exposition of the theory was given by Illusie in [24], but detailed proofs were not provided. Keerthi Madapusi-Pera has written a detailed account [38] of Kato’s work that relies on published sources (notably [42]), but this document also is unpublished. Recently, [26] developed the theory of weak log abelian varieties over arbitrary fs log schemes S . They show that the torsion in such a thing is representable by an fs log scheme which is finite over the underlying scheme of S and representable by a log finite flat S -group. Moreover, locally in the Kummer log flat topology on S , this representing object has the induced log structure. It seems that we could have used this result in the present work, but we are content to work with log 1-motives and use the older results of [3].

We will work with a category of pairs $\text{fin}_{\mathcal{O}_K}^{p,N}$ consisting of an object of $\text{fin}_{\mathcal{O}_K}^p$ and a monodromy morphism (to be defined below). Kato’s unpublished works show that $\text{fin}_{\mathcal{O}_K}^{p,N}$ is equivalent to a category $\text{lfin}_{\mathcal{O}_K}$ of log schemes, but we will not make use of this equivalence. Rather, we will refer to objects of $\text{fin}_{\mathcal{O}_K}^{p,N}$ as log finite flat group schemes, and define a functor on this category which we will regard as producing the “generic fiber” of an object of $\text{fin}_{\mathcal{O}_K}^{p,N}$. From this perspective, we can still use the result Proposition 2.5.3 of [3], whose proof takes place in a category $\text{Ext}_{\text{fin}_R^p}^\nu(H'', H')$ which we will describe below.

Recall that we saw in Proposition 2.2.9 that the exact category $\text{fin}_{\mathcal{O}_K}^p$ is an exact subcategory of the abelian category $\text{Sh}_{\text{Ab}}(\text{Sch}_{\mathcal{O}_K, \text{fppf}})$. The essential image of the generic fiber functor $\text{fin}_{\mathcal{O}_K}^p \rightarrow \text{fin}_K^p$ is, however, not an exact subcategory of fin_K^p , as we will see in Remark 3.1.21. We now enlarge $\text{fin}_{\mathcal{O}_K}^p$ and extend the generic fiber functor so that its essential image will be closer to being an exact subcategory of fin_K^p in the sense of containing more extensions of objects in the essential image of $\text{fin}_{\mathcal{O}_K}^p$.

Definition 2.4.1 ($\text{Ext}_{\text{fin}_R^p}^\nu; \text{Ext}_{\text{fin}_R^p}^\nu(H'', H')$).

Let $R = K$ or $R = \mathcal{O}_K$.

Let $\text{Ext}_{\text{fin}_R^p}^\nu$ denote the category of pairs (η^\bullet, ν) such that

- (i) $\eta^\bullet = [\eta^{-1} \rightarrow \eta^0 \rightarrow \eta^1]$ is a short exact sequence of objects of fin_R^p regarded as a cochain complex concentrated in degrees $-1, 0$ and 1 ;
- (ii) the term η^1 is an étale R -group (this is, of course, no condition if $R = K$);
- (iii) $\nu \in \text{Hom}_{\text{fin}_R^p}(\eta^1(1), \eta^{-1})$, where $\eta^1(1)$ is the Tate twist of the étale R -group η^1 .

A morphism $\varphi : (\eta_1^\bullet, \nu_1) \rightarrow (\eta_2^\bullet, \nu_2)$ of objects of $\text{Ext}_{\text{fin}_R^p}^\nu$ is a morphism of cochain complexes $\varphi : \eta_1^\bullet \rightarrow \eta_2^\bullet$ such that

$$\begin{array}{ccc} \eta_1^1(1) & \xrightarrow{\nu_1} & \eta_1^{-1} \\ \downarrow \varphi^1(1) & & \downarrow \varphi^{-1} \\ \eta_2^1(1) & \xrightarrow{\nu_2} & \eta_2^{-1} \end{array}$$

commutes; here φ^{-1} is the morphism in degree -1 , not an inverse.

For any two objects H'', H' of fin_R^p with H'' étale over R , let $\text{Ext}_{\text{fin}_R^p}^\nu(H'', H')$ denote the full subcategory of $\text{Ext}_{\text{fin}_R^p}^\nu$ consisting of those objects (η^\bullet, ν) such that $\eta^1 \simeq H''$ and $\eta^{-1} \simeq H'$.

Remark 2.4.2. We make the restriction that the degree-one terms of objects of $\text{Ext}_{\text{fin}_R^p}^\nu$ be étale so that the Tate twist exists; this is, of course, no restriction when $R = K$.

Remark 2.4.3. Let $(\eta^\bullet, \nu) \in \text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu(H'', H')$. Taking stalks of sheaves on any site is exact ([51, Tag 04EN]) so (η^\bullet, ν) gives an object

$$(\eta_K^\bullet, \nu_K) \in \text{Ext}_{\text{fin}_K^p}^\nu(H''_K, H'_K).$$

Because of the functorial properties of connected-étale sequences, an important role is played by the following full subcategory of $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu$.

Proposition 2.4.4.

In Situation 0.1.2 with $R = K$ or $R = \mathcal{O}_K$, the category $\text{Ext}_{\text{fin}_R^p}^\nu$ inherits an exactness structure from that of fin_R^p .

Proof. Recall that fin_R^p is a full, exact subcategory of the abelian category $\text{Sh}_{\text{Ab}}(\text{Sch}_{R, \text{fppf}})$. It is straightforward to check (see [51, Tag 0114]) that, for

any abelian category \mathcal{A} , category $\text{CoCh}(\mathcal{A})$ of cochain complexes with objects in \mathcal{A} is an abelian category. By [51, Tag 0114] again, a sequence of cochain complexes is short exact if and only if the induced sequence in each degree is short exact. It follows from this that $\text{Ext}_{\text{fin}_R^p}^\nu$ is closed under Yoneda extensions in $\text{CoCh}(\text{Sh}_{\text{Ab}}(\text{Sch}_{R, \text{fppf}}))$. By Lemma 2.2.8, $\text{Ext}_{\text{fin}_R^p}^\nu$ is an exact category. \square

Remark 2.4.5. Let (η_1^\bullet, ν_1) be a subobject of (η_2^\bullet, ν_2) in the category $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu$. This means that there is morphism $\varphi_2 : (\eta_1^\bullet, \nu_1) \rightarrow (\eta_2^\bullet, \nu_2)$ which is a closed immersion in every degree, and for which the diagram

$$\begin{array}{ccc} \eta_1^1(1) & \xrightarrow{\nu_1} & \eta_1^{-1} \\ \downarrow \varphi^1(1) & & \downarrow \varphi^{-1} \\ \eta_2^1(1) & \xrightarrow{\nu_2} & \eta_2^{-1} \end{array}$$

commutes. There is a maximal subobject of (η_2^\bullet, ν_2) on which ν_2 is zero, namely, the object (η_1^\bullet, ν_1) with $\eta_1^1 = \ker \nu_2$ and with η_1^0 the pullback

$$\eta_1^0 = \eta_2^0 \times_{\eta_2^1} \ker \nu_2.$$

In this case, since the inclusion $\ker \nu_2$ is an open immersion, the morphism $\eta_1^0 \rightarrow \eta_2^1$, which is the base change of $\ker \nu_2 \hookrightarrow \eta_2^1$, is an open immersion.

Definition 2.4.6 ($\text{fin}_{\mathcal{O}_K}^{p,N}$; log finite flat \mathcal{O}_K -groups).

In Situation 0.1.2, let $\text{fin}_{\mathcal{O}_K}^{p,N}$ denote the full subcategory of $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu$ consisting of those objects (η^\bullet, N) such that

$$\eta^{-1} = \eta^{0,\circ} \text{ and } \eta^1 = \eta^{0,\text{ét}}$$

i.e., η^\bullet is the connected-étale sequence of its degree-zero term. In this setting, we will identify (η^\bullet, N) with the pair (η^0, N) . This is reasonable, as the connected-étale sequence is canonically associated to η^0 , since a morphism of objects of $\text{fin}_{\mathcal{O}_K}^p$ is the same as a morphism of their connected-étale sequence regarded as cochain complexes. Thus, we will regard a pair (H, N) , where H is an object of $\text{fin}_{\mathcal{O}_K}^p$ and N is a homomorphism $N : H^{\text{ét}}(1) \rightarrow H^\circ$, as an object of $\text{fin}_{\mathcal{O}_K}^{p,N}$, and we will call such a pair a **log finite flat \mathcal{O}_K -group**.

Remark 2.4.7. There is a fully faithful exact functor $\text{fin}_{\mathcal{O}_K}^p \rightarrow \text{fin}_{\mathcal{O}_K}^{p,N}$ sending an object H to $(\text{cé}(H), 0)$, where $\text{cé}(H)$ is the connected-étale sequence of H .

Remark 2.4.8. Let (η^\bullet, ν) be an element of $\text{Ext}_{\text{fin}_R^p}^\nu(H'', H')$, and set $H = \eta^0$ for the middle term in the complex η . As H'' is étale, H° is equal to H'° . Furthermore, there is a canonical surjection $H^{\text{ét}} \rightarrow H''$, hence also canonical surjection $H^{\text{ét}}(1) \rightarrow H''(1)$. Pulling $\nu : H''(1) \rightarrow H'$ back to $H^{\text{ét}}(1)$ gives a morphism $N : H^{\text{ét}}(1) \rightarrow H'$. Since H'' is étale and of p -power order, $H''(1)$ is multiplicative of p -power order, hence connected. It follows that ν (hence also N) factors through $H'^\circ = H^\circ$. Thus we obtain from (η, ν) a class in $\text{fin}_{\mathcal{O}_K}^{p,N}$. It is therefore reasonable to call an object of $\text{Ext}_{\text{fin}_R^p}^\nu(H'', H')$ a log finite flat group scheme.

Lemma 2.4.9.

The category $\text{fin}_{\mathcal{O}_K}^{p,N}$ has the structure of an exact category in which a morphism $(H_1, N_1) \rightarrow (H_2, N_2)$ is an admissible monomorphism (resp., admissible epimorphism) if and only if the underlying morphism $H_1 \rightarrow H_2$ is an admissible monomorphism (resp., admissible epimorphism).

Proof. This is immediate from Proposition 2.4.4. □

Proposition 2.4.10 ([51, Tag 010I], [51, Tag 05PJ]).

For any site \mathcal{C} , the category $\text{Sh}_{\text{Ab}}(\mathcal{C})$ of sheaves of abelian groups on \mathcal{C} is an abelian category. Therefore, if η^1 and η^{-1} are objects of $\text{Sh}_{\text{Ab}}(\mathcal{C})$ the set $\text{Ext}_{\text{Sh}_{\text{Ab}}(\mathcal{C})}(\eta^1, \eta^{-1})$ of Yoneda extensions of η^1 by η^{-1} (i.e., short exact exact sequences of the form $0 \rightarrow \eta^{-1} \rightarrow E \rightarrow \eta^1 \rightarrow 0$) is an abelian group under Baer sum. Explicitly, the Baer sum of two classes

$$\begin{aligned} \eta_1^\bullet : \quad & 0 \longrightarrow \eta^{-1} \xrightarrow{d_1^0} \eta_1^0 \xrightarrow{d_1^1} \eta^1 \longrightarrow 0 \\ \eta_2^\bullet : \quad & 0 \longrightarrow \eta^{-1} \xrightarrow{d_2^0} \eta_2^0 \xrightarrow{d_2^1} \eta^1 \longrightarrow 0 \end{aligned}$$

in $\text{Ext}_{\text{Sh}_{\text{Ab}}(\mathcal{C})}(\eta^1, \eta^{-1})$ is the extension

$$\eta_1^\bullet +_{\text{B}} \eta_2^\bullet : \quad 0 \longrightarrow \eta^{-1} \xrightarrow{d_+^0} \eta_1^0 +_{\text{B}} \eta_2^0 \xrightarrow{d_+^1} \eta^1 \longrightarrow 0$$

defined as follows. The sheaf $\eta_1^0 +_{\text{B}} \eta_2^0 := (\eta_1^\bullet +_{\text{B}} \eta_2^\bullet)^0$ is defined by

$$\eta_1^0 +_{\text{B}} \eta_2^0 = \frac{\eta_1^0 \times_{\eta^1} \eta_2^0}{\text{im}((d_1^0, -d_2^0) : \eta^{-1} \rightarrow \eta_1^0 \times_{\eta^1} \eta_2^0)}. \quad (2.4.11)$$

The morphism $d_+^0 : \eta^{-1} \rightarrow \eta_1^0 +_{\text{B}} \eta_2^0$ is the natural morphism induced by

$$d_1^0 \times 0 : \eta^{-1} \rightarrow \eta_1^0 \times_{\eta^1} \eta_2^0.$$

Therefore, d_+^0 is given in local sections (i.e., on an arbitrary object of the category \mathcal{C}), using $[-]$ to represent classes in the fiber product $\eta_1^0 \times_{\eta^1} \eta_2^0$, by

$$d_+^0 : \begin{cases} \eta^{-1} & \longrightarrow & \eta_1^0 +_{\text{B}} \eta_2^0 \\ a & \longmapsto & [d_1^0(a), 0] = [0, d_2^0(a)]. \end{cases} \quad (2.4.12)$$

The morphism $d_+^1 : \eta_1^0 +_{\text{B}} \eta_2^0 \rightarrow \eta^1$ is the natural morphism induced by $d_1^1 \circ \text{pr}_1 : \eta_1^0 \times_{\eta^1} \eta_2^0 \rightarrow \eta^1$. This morphism is given on local sections by

$$d_+^1 : \begin{cases} \eta_1^0 +_{\text{B}} \eta_2^0 & \longrightarrow & \eta^1 \\ [e_1, e_2] & \longmapsto & d_1^1(e_1) = d_2^1(e_2). \end{cases} \quad (2.4.13)$$

The Baer inverse of a class

$$\eta^\bullet : \quad 0 \longrightarrow \eta^{-1} \xrightarrow{d^0} \eta^0 \xrightarrow{d^1} \eta^1 \longrightarrow 0$$

in $\text{Ext}_{\text{ShAb}(\mathcal{C})}(\eta^1, \eta^{-1})$ is given by the extension

$$-_{\text{B}}\eta^\bullet : \quad 0 \longrightarrow \eta^{-1} \xrightarrow{d^0} \eta^0 \xrightarrow{-d^1} \eta^1 \longrightarrow 0$$

For any two objects H'', H' of $\text{fin}_{\mathcal{O}_K}^p$, every class in

$$\text{Ext}_{\text{ShAb}(\text{Sch}_{\mathcal{O}_K, \text{fppf}})}(H'', H')$$

is representable by an object of $\text{fin}_{\mathcal{O}_K}^p$ (see Proposition 2.2.9). Hence we can speak of Baer sums of extensions in the category $\text{fin}_{\mathcal{O}_K}^p$.

Let H'' and H' be objects of $\text{fin}_{\mathcal{O}_K}^p$. Since $\text{fin}_{\mathcal{O}_K}^p$ is an abelian category, the set $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}(H''_K, H'_K)$ is a group under Baer sum. Following Kato, we build from an object $(\eta^\bullet, \nu) \in \text{Ext}'_{\text{fin}_{\mathcal{O}_K}^p}(H'', H')$ an object $\nu_* \eta_K^\bullet$ of $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}(H''_K, H'_K)$ which may not be the generic fiber of any class in $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}(H'', H')$. Finite K -groups produced by this construction will be crucial to our understanding of the torsion subgroups $A_K[p^m]$ of abelian K -varieties with semi-stable reduction; such finite K -groups determine whether or not $A_K[p^m]$ is 1-crystalline.

Definition 2.4.14.

Let $(\eta^\bullet, \nu) \in \text{Ext}_{\text{fin}_{\mathcal{O}_K}^\nu}^p$. Let n be a power of p large enough to kill η^1 , so that η^1 is a $\mathbb{Z}/n\mathbb{Z}$ -module. Recall from Example 3.1.17 that for any Tate curve $E_{q,K}$, the finite K -group of n -torsion points $E_{q,K}[n]$ lies in a short exact sequence

$$\theta_{n,K}^q : \quad 0 \longrightarrow \mu_{n,K} \longrightarrow E_{q,K}[n] \longrightarrow \mathbb{Z}/n\mathbb{Z}_K \longrightarrow 0 \quad (2.4.15)$$

Consider the special case where q is our chosen uniformizer ϖ . Tensor the sequence $\theta_{n,K}^\varpi$ of free $\mathbb{Z}/n\mathbb{Z}$ -modules with η_K^1 and push out by ν as in

$$\begin{array}{ccccccccc} \theta_{n,K}^\varpi \otimes \eta_K^1 & 0 & \longrightarrow & \eta_K^1(1) & \longrightarrow & \eta_K^1 \otimes E_n & \longrightarrow & \eta_K^1 & \longrightarrow & 0 \\ & & & \downarrow \nu & & \downarrow & & \parallel & & \\ \nu_*(\theta_{n,K}^\varpi \otimes \eta_K^1) & 0 & \longrightarrow & \eta_K^{-1} & \longrightarrow & \nu_*\eta_K^0 & \longrightarrow & \eta_K^1 & \longrightarrow & 0 \end{array}$$

(this diagram defines $\nu_*\eta_K^0$) to obtain an extension

$$\nu_*\eta_K^\bullet := \nu_*(\theta_{n,K}^\varpi \otimes \eta_K^1) \in \text{Ext}_{\text{fin}_K^p}(\eta_K^1, \eta_K^{-1}). \quad (2.4.16)$$

Explicitly,

$$\nu_*\eta_K^0 = \frac{(\eta_K^1 \otimes E_n) \oplus \eta_K^{-1}}{(y, 0) - (0, \nu(y)) : y \in \eta_K^1(1)}. \quad (2.4.17)$$

Remark 2.4.18. Our definition of $\nu_*\eta_K^\bullet$ is an abuse of notation, but it should not cause confusion because η_K^{-1} is not, in general, identified with $\eta_K^1(1)$, so that pushing out the exact sequence η_K^\bullet by a morphism with source $\eta_K^1(1)$ does not make sense.

Remark 2.4.19. Let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . Kato has constructed a log finite flat \mathcal{O}_K -group that has generic fiber (in an appropriate and natural sense) isomorphic to $E_{\varpi,K}[n]$. Using this, one can mimic the construction in Definition 2.4.14 integrally, i.e., in a certain category of fs log schemes.

We will define a functor on $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^\nu}^p$, which we will call the “generic fiber functor on that category”, and we will show that this functor is exact. For this, we will use the following lemma.

Lemma 2.4.20.

For $i = 1, 2, 3$, let H_i^\bullet be objects of $\text{CoCh}(\text{fin}_K^p)$ concentrated in degrees $-1, 0$ and 1 . Suppose we have two morphisms

$$H_3^\bullet \xrightarrow{f_2^\bullet} H_2^\bullet \xrightarrow{f_1^\bullet} H_1^\bullet \quad (2.4.21)$$

giving short exact sequences in degrees -1 and 1 .

(i) Then f_2^0 is injective and f_1^0 is surjective.

(ii) If $\ker f_1^0 \subset \text{im } f_2^0$, then in fact $\ker f_1^0 = \text{im } f_2^0$, i.e., (2.4.21) give a short exact sequence in degree 0 .

Proof. The morphisms of (2.4.21) fit into the following commutative diagram with exact rows and where the outer two columns are short exact:

$$\begin{array}{ccccccccc} H_3^\bullet & 0 & \longrightarrow & H_3^{-1} & \xrightarrow{d_3^0} & H_3^0 & \xrightarrow{d_3^1} & H_3^1 & \longrightarrow & 0 \\ \downarrow f_2^\bullet & & & \downarrow & & \downarrow & & \downarrow & & \\ H_2^\bullet & 0 & \longrightarrow & H_2^{-1} & \xrightarrow{d_2^0} & H_2^0 & \xrightarrow{d_2^1} & H_2^1 & \longrightarrow & 0 \\ \downarrow f_1^\bullet & & & \downarrow & & \downarrow & & \downarrow & & \\ H_1^\bullet & 0 & \longrightarrow & H_1^{-1} & \xrightarrow{d_1^0} & H_1^0 & \xrightarrow{d_1^1} & H_1^1 & \longrightarrow & 0 \end{array} \quad (2.4.22)$$

Applying the Snake lemma to the top two rows gives an exact sequence

$$0 \longrightarrow \ker f_2^0 \longrightarrow 0 \longrightarrow \text{coker } f_2^{-1} \longrightarrow \text{coker } f_2^0 \longrightarrow \text{coker } f_2^1. \quad (2.4.23)$$

This shows that $\ker f_2^0 = 0$. Applying the Snake lemma to the bottom two rows gives an exact sequence

$$\ker f_1^{-1} \longrightarrow \ker f_1^0 \longrightarrow \ker f_1^1 \longrightarrow 0 \longrightarrow \text{coker } f_1^0 \longrightarrow 0, \quad (2.4.24)$$

which shows that $\text{coker } f_1^0 = 0$. This has proven (i).

We prove (ii). Note that the morphism $d_2^0 : \ker f_1^{-1} \longrightarrow \ker f_1^0$ is injective by assumption, so (2.4.24) gives a short exact sequence

$$0 \longrightarrow \ker f_1^{-1} \longrightarrow \ker f_1^0 \longrightarrow \ker f_1^1 \longrightarrow 0. \quad (2.4.25)$$

Since the outer columns of (2.4.22) are short exact by assumption, $\ker f_1^{-1}$ is isomorphic to H_3^{-1} via f_2^{-1} and $\ker f_1^1$ is isomorphic to H_3^1 via f_2^1 . We see from

(2.4.25) and the top row of (2.4.22) that we have an equality of orders of finite abelian groups

$$\# \ker f_1^0 = (\# \ker f_1^1)(\# \ker f_1^{-1}) = (\# H_3^1)(\# H_3^{-1}) = \# H_3^0.$$

By (i), f_2^0 is injective, so $\# \operatorname{im} f_2^0 = \# H_3^0$. We see that $\ker f_1^0$ and $\operatorname{im} f_2^0$ are both of order $\# H_3^0$. Therefore, if $\ker f_1^0 \subset \operatorname{im} f_2^0$, then $\ker f_1^0 = \operatorname{im} f_2^0$. This completes the proof. \square

The construction of the generic fiber functor on $\operatorname{Ext}_{\operatorname{fin}^p \mathcal{O}_K}^\nu$ involves a certain pushout functor that we define next.

Proposition 2.4.26.

There exists an exact functor

$$\operatorname{Ext}_{\operatorname{fin}^p \mathcal{O}_K}^\nu \longrightarrow \operatorname{Ext}_{\operatorname{fin}^p K} \quad (2.4.27)$$

that sends an object (η^\bullet, ν) to $\nu_ \eta_K^\bullet$ as defined in (3.2.15).*

Proof. We first show define the functor on morphisms. Let

$$\varphi_2^\bullet : (\eta_1^\bullet, \nu_1) \longrightarrow (\eta_2^\bullet, \nu_2)$$

be a morphism of objects of $\operatorname{Ext}_{\operatorname{fin}^p \mathcal{O}_K}^\nu$. We define a morphism $\nu_* \varphi_2^0$ that fits into the diagram

$$\begin{array}{ccccc} \eta_{1,K}^1(1) & \xrightarrow{\quad} & \eta_{1,K}^1 \otimes E_n & & \\ \downarrow \nu_1 & \searrow \varphi_2^1(1) & \downarrow & \searrow \varphi_2^1 \otimes \operatorname{id} & \\ & \eta_{2,K}^1(1) & \xrightarrow{\quad} & \eta_{2,K}^1 \otimes E_n & \\ & \downarrow \nu_2 & \downarrow & \downarrow & \\ \eta_{1,K}^{-1} & \xrightarrow{\quad} & \nu_{1,*} \eta_{1,K}^0 & \xrightarrow{\quad} & \nu_{2,*} \eta_{2,K}^0 \\ & \searrow \varphi_2^{-1} & \downarrow & \searrow \nu_* \varphi_2^0 & \\ & \eta_{2,K}^{-1} & \xrightarrow{\quad} & \nu_{2,*} \eta_{2,K}^0 & \end{array} \quad (2.4.28)$$

where all faces commute. To construct the diagram, begin with the front and back faces, which are the cofiber squares defining the two K -groups $\nu_* \eta_{i,K}^0$ of Definition 2.4.14. The left face commutes by the assumption that φ_2 respects the monodromy maps. The top face commutes by functoriality of tensor products.

The morphisms

$$\begin{aligned} \eta_{1,K}^1 \otimes E_n &\longrightarrow \nu_{2,*}\eta_{2,K}^0 \\ \eta_{1,K}^{-1} &\longrightarrow \nu_{2,*}\eta_{2,K}^0 \end{aligned} \quad (2.4.29)$$

obtained by composition will induce a morphism

$$\nu_*\varphi_2^0 : \nu_*\eta_{1,K}^0 \longrightarrow \nu_*\eta_{2,K}^0$$

by the universal property of cofiber product $\nu_{1,*}\eta_{1,K}^0$ if their restrictions to $\eta_{1,K}^1(1)$ agree. But, by commutativity of the top and left faces, the restrictions of the morphisms of (2.4.29) to $\eta_{1,K}^1(1)$ factor through $\eta_{2,K}^1(1)$, hence they agree as morphisms $\eta_{1,K}^1(1) \rightarrow \nu_{2,*}\eta_{2,K}^0$. Thus we have a morphism $\nu_*\varphi_2^0$. This gives rise to the morphism of complexes

$$\begin{array}{ccccccc} \nu_{1,*}\eta_{1,K}^\bullet & 0 & \longrightarrow & \eta_{1,K}^{-1} & \longrightarrow & \nu_{1,*}\eta_{1,K}^0 & \longrightarrow & \eta_{1,K}^1 & \longrightarrow & 0 \\ \downarrow \nu_*\varphi_2^\bullet & & & \downarrow \varphi_2^{-1} & & \downarrow \nu_*\varphi_2^0 & & \downarrow \varphi_2^1 & & \\ \nu_{2,*}\eta_{2,K}^\bullet & 0 & \longrightarrow & \eta_{2,K}^{-1} & \longrightarrow & \nu_{2,*}\eta_{2,K}^0 & \longrightarrow & \eta_{2,K}^1 & \longrightarrow & 0 \end{array} \quad (2.4.30)$$

This has defined the functor (2.4.27). We now show this functor is exact. Let

$$0 \longrightarrow (\eta_1^\bullet, \nu_1) \xrightarrow{\varphi_2^\bullet} (\eta_2^\bullet, \nu_2) \xrightarrow{\varphi_3^\bullet} (\eta_3^\bullet, \nu_3) \longrightarrow 0 \quad (2.4.31)$$

be a short exact sequence of objects of $\text{Ext}_{\text{fin } \mathcal{O}_K}^\nu$. Recall that this means that (2.4.31) is a short exact sequence of cochain complexes

$$0 \longrightarrow \eta_{1,K}^\bullet \xrightarrow{\varphi_2^\bullet} \eta_{2,K}^\bullet \xrightarrow{\varphi_3^\bullet} \eta_{3,K}^\bullet \longrightarrow 0 \quad (2.4.32)$$

that is compatible with the monodromy maps. We must show that the sequence

$$\nu_{1,*}\eta_{1,K}^\bullet \xrightarrow{\nu_*\varphi_2^\bullet} \nu_{2,*}\eta_{2,K}^\bullet \xrightarrow{\nu_*\varphi_3^\bullet} \nu_{3,*}\eta_{3,K}^\bullet \quad (2.4.33)$$

is short exact in each degree $\bullet \in \{-1, 0, 1\}$. For degrees $\bullet \in \{-1, 1\}$, (2.4.33) agrees with (2.4.32) which is exact by the assumption. Therefore, (2.4.33) is short exact in degrees -1 and 1 . To prove that this sequence is short exact in degree 0 , we will apply Lemma 2.4.20 to (2.4.33). By the lemma, to show (2.4.33) is short exact, it suffices to show that $\ker \nu_*\varphi_3^0 \subset \text{im } \nu_*\varphi_2^0$. We prove this now.

Recall from (2.4.17) that, for $i \in \{1, 2, 3\}$, $\nu_{i,*}\eta_{i,K}^0$ satisfies

$$\nu_{i,*}\eta_{i,K}^0 := \frac{(\eta_{i,K}^1 \otimes E_n) \oplus \eta_{i,K}^{-1}}{(y, 0) - (0, \nu_i(y)) : y \in \eta_{i,K}^1(1)}. \quad (2.4.34)$$

We will represent the class of a pair $(a, b) \in (\eta_{i,K}^1 \otimes E_n) \oplus \eta_{i,K}^{-1}$ using brackets as in $[a, b]$. Let $[a_2, b_2] \in \nu_{2,*}\eta_{2,K}^0$. From (2.4.34), we see that $\nu_*\varphi_3([a_2, b_2]) = 0$ if and only if

$$((\varphi_3^1 \otimes \text{id})(a_2), \varphi_3^{-1}(b_2)) = (y_3, -\nu_3(y_3)) \text{ for some } y_3 \in \eta_{3,K}^1(1). \quad (2.4.35)$$

Suppose (2.4.35) holds. We wish to show that $[a_2, b_2] \in \text{im } \nu_*\varphi_2$, i.e., we wish to show that there exists a pair $(a_1, b_1) \in (\eta_{1,K}^1 \otimes E_n) \oplus \eta_{1,K}^{-1}$ and a $y_2 \in \eta_{2,K}^1(1)$ such that

$$(a_2, b_2) = ((\varphi_{2,K}^1 \otimes \text{id})(a_1), \varphi_{2,K}^{-1}(b_1)) + (y_2, -\nu_2(y_2)). \quad (2.4.36)$$

By the assumption that (2.4.32) is exact in every degree coupled with the exactness of the Tate twisting functor, we know that $\varphi_{3,K}^1(1) : \eta_{2,K}^1(1) \rightarrow \eta_{3,K}^1(1)$ is surjective. Let $y_2 \in \eta_{2,K}^1(1)$ be an element satisfying $\varphi_{3,K}^1(1)(y_2) = y_3$. Then, by eq. (2.4.35), we have

$$((\varphi_{3,K}^1 \otimes \text{id}) \times \varphi_{3,K}^{-1})((a_2, b_2) - (y_2, -\nu_2(y_2))) = (0, 0)$$

in $(\eta_{3,K}^1 \otimes E_n) \oplus \eta_{3,K}^{-1}$. By exactness of (2.4.32) in degrees -1 and 1 and exactness of tensoring against the free $\mathbb{Z}/n\mathbb{Z}$ -module E_n , there exists

$$(a_1, b_1) \in (\eta_{1,K}^1 \otimes E_n) \oplus \eta_{1,K}^{-1}$$

such that we have equalities

$$\begin{aligned} (\varphi_{2,K}^1 \otimes \text{id})(a_1) &= a_2 - y_2 \\ \varphi_{2,K}^{-1}(b_1) &= b_2 + \nu_2(y_2). \end{aligned}$$

This has shown that we have an equation of the form (2.4.36). As we remarked before, this means that $\nu_*\varphi_2([a_1, b_1]) = [a_2, b_2]$. Since $[a_2, b_2]$ was an arbitrary element of $\nu_*\eta_{2,K}^0$, we have shown that $\ker \nu_*\varphi_3^0 \subset \text{im } \nu_*\varphi_2^0$. Using Lemma 2.4.20, we conclude that the functor (2.4.27) is exact. \square

We now define the generic fiber functor on $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu \longrightarrow \text{Ext}_{\text{fin}_K^p}$ and show that it is exact.

Proposition 2.4.37.

There exists an exact functor

$$(-)_K : \text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu \longrightarrow \text{Ext}_{\text{fin}_K^p} \quad (2.4.38)$$

which is given on objects by

$$(\eta^\bullet, \nu)_K := \eta_K^\bullet +_{\text{B}} \nu_* \eta_K^\bullet.$$

Proof. As we have seen in Proposition 2.4.10, if (η^\bullet, ν) is an object of $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu$ then the term in degree zero in $(\eta^\bullet, \nu)_K$ can be written as

$$((\eta^\bullet, \nu)_K)^0 = \frac{\eta_K^0 \times_{\eta_K^1} \nu_* \eta_K^0}{\text{im}(\Delta^-)}$$

where Δ^- is the morphism

$$\Delta^- : \begin{cases} \eta_K^{-1} & \longrightarrow & \eta_K^0 \times_{\eta_K^1} \nu_* \eta_K^0 \\ x & \longmapsto & (d_{\eta_K}^0(x), -d_{\nu_* \eta_K}^0(x)), \end{cases}$$

the morphisms $d_{(-)}^0$ being the morphisms from degree -1 to degree 0 in the complexes η_K^\bullet and $\nu_* \eta_K^\bullet$.

Let

$$\varphi_2^\bullet : (\eta_1^\bullet, \nu_1) \longrightarrow (\eta_2^\bullet, \nu_2)$$

be a morphism of objects of $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu$. We wish to define a morphism

$$(\eta_1^\bullet, \nu_1)_K \longrightarrow (\eta_2^\bullet, \nu_2)_K.$$

Since φ_2 already includes the information of a morphism on the terms in -1 and 1 , the main task at hand is to give a morphism on the degree-zero term. By the above description of the degree-zero terms in the Baer sums, to define a morphism on the degree-zero terms

$$((\eta_1^\bullet, \nu_1)_K)^0 \longrightarrow ((\eta_2^\bullet, \nu_2)_K)^0$$

it suffices to give a morphism $\eta_{1,K}^0 \times_{\eta_{1,K}^1} \nu_{1,K} \eta_{1,K}^0 \rightarrow \eta_{2,K}^0 \times_{\eta_{2,K}^1} \nu_{2,K} \eta_{2,K}^0$ that respects

the images of homomorphisms Δ_1^-, Δ_2^- which are defined like Δ^- above. We claim that

$$\tilde{\varphi}_{2,K,\nu}^0 := (\varphi_{2,K}^0 \times \nu_* \varphi_2^0) : \eta_{1,K}^0 \times_{\eta_{1,K}^1} \nu_* \eta_{1,K}^0 \longrightarrow \eta_{2,K}^0 \times_{\eta_{2,K}^1} \nu_* \eta_{2,K}^0$$

is such a morphism. This follows immediately from the fact that each square in the diagram

$$\begin{array}{ccccc} \eta_{1,K}^0 & \xleftarrow{d_{\eta_{1,K}}^0} & \eta_{1,K}^{-1} & \xrightarrow{-d_{\nu_* \eta_{1,K}}^0} & \nu_{1,*} \eta_{1,K}^0 \\ \downarrow \varphi_{2,K}^0 & & \downarrow \varphi_{2,K}^{-1} & & \downarrow \nu_* \varphi_2^0 \\ \eta_{2,K}^0 & \xleftarrow{d_{\eta_{2,K}}^0} & \eta_{2,K}^{-1} & \xrightarrow{-d_{\nu_* \eta_{2,K}}^0} & \nu_{2,*} \eta_{2,K}^0 \end{array}$$

commutes. Define

$$\varphi_{2,K,\nu}^0 : ((\eta_1^\bullet, \nu_1)_K)^0 \longrightarrow ((\eta_2^\bullet, \nu_2)_K)^0$$

to be the morphism induced by $\tilde{\varphi}_{2,K,\nu}^0$. We claim that $\varphi_{2,K,\nu}^0$ extends to a morphism $\varphi_{2,K,\nu}^\bullet$ of complexes

$$\begin{array}{ccccccc} (\eta_1^\bullet, \nu_1)_K & 0 & \longrightarrow & \eta_{1,K}^{-1} & \longrightarrow & ((\eta_1^\bullet, \nu_1)_K)^0 & \longrightarrow & \eta_{1,K}^1 & \longrightarrow & 0 \\ & & & \downarrow \varphi_{2,K}^{-1} & & \downarrow \varphi_{2,K}^0 & & \downarrow \varphi_2^1 & & \\ (\eta_2^\bullet, \nu_2)_K & 0 & \longrightarrow & \eta_{2,K}^{-1} & \longrightarrow & ((\eta_2^\bullet, \nu_2)_K)^0 & \longrightarrow & \eta_{2,K}^1 & \longrightarrow & 0 \end{array} \quad (2.4.39)$$

As we recorded in (2.4.12), for $i \in \{1, 2\}$, the morphism $\eta_{i,K}^{-1} \rightarrow ((\eta_i^\bullet, \nu_i)_K)^0$ is induced by (writing d_i^\bullet for the differential in the complex η_i^\bullet)

$$\begin{array}{ccc} \eta_{i,K}^{-1} & \longrightarrow & \eta_{i,K}^0 \times_{\eta_{i,K}^1} \nu_{i,*} \eta_{i,K}^0 \\ x & \longmapsto & (d_i^0(x), 0) \end{array}$$

Since $\tilde{\varphi}_{2,K,\nu}^0$ is given by $\varphi_{2,K}^0$ in the first factor, and since $\varphi_{2,K}^0$ is assumed to commute with the differentials (being that φ_2^\bullet was chosen to be a morphism of complexes), we see that the left square in (2.4.39) commutes.

As we recorded in (2.4.13), for $i \in \{1, 2\}$, the morphism $((\eta_i^\bullet, \nu_i)_K)^0 \rightarrow \eta_{i,K}^1$ is induced by

$$\begin{array}{ccc} \eta_{i,K}^0 \times_{\eta_{i,K}^1} \nu_{i,*} \eta_{i,K}^0 & \longrightarrow & \eta_{i,K}^1 \\ (a, b) & \longmapsto & (d_i^1(y), 0) \end{array}$$

Arguing as before, we see that the right square in (2.4.39) commutes. This has shown that $\varphi_{2,K,\nu}^0$ gives a morphism of complexes $\varphi_{2,K,\nu}^\bullet$. We use this to define the functor (2.4.38) by

$$(-)_K : \begin{cases} \mathrm{Hom}_{\mathrm{Ext}_{\mathrm{fin}^p}^\nu}((H_1^\bullet, \nu_1), (H_2^\bullet, \nu_2)) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ext}_{\mathrm{fin}^p_K}}(H_{1,K}^\bullet, H_{2,K}^\bullet) \\ \varphi^\bullet & \longmapsto & \varphi_{K,\nu}^\bullet \end{cases}.$$

We now show that $(-)_K$ is exact. Let

$$0 \longrightarrow (\eta_1^\bullet, \nu_1) \xrightarrow{\varphi_2^\bullet} (\eta_2^\bullet, \nu_2) \xrightarrow{\varphi_3^\bullet} (\eta_3^\bullet, \nu_3) \longrightarrow 0 \quad (2.4.40)$$

be a short exact sequence of objects of $\mathrm{Ext}_{\mathrm{fin}^p}^\nu$. Recall that this means that (2.4.40) is a short exact sequence of cochain complexes

$$0 \longrightarrow \eta_{1,K}^\bullet \xrightarrow{\varphi_2^\bullet} \eta_{2,K}^\bullet \xrightarrow{\varphi_3^\bullet} \eta_{3,K}^\bullet \longrightarrow 0 \quad (2.4.41)$$

that is compatible with the monodromy homomorphisms. We wish to show that

$$(\eta_1^\bullet, \nu_1)_K \xrightarrow{\varphi_{2,K,\nu}^\bullet} (\eta_2^\bullet, \nu_2)_K \xrightarrow{\varphi_{3,K,\nu}^\bullet} (\eta_3^\bullet, \nu_3)_K \quad (2.4.42)$$

is short exact in each degree. The sequence (2.4.42) is short exact in degrees -1 and 1 by the assumption that (2.4.40) is short exact. By Lemma 2.4.20, to show (2.4.42) is short exact, it suffices to show that $\ker \varphi_{3,K,\nu}^0 \subset \mathrm{im} \varphi_{2,K,\nu}^0$. We show this now.

Let us reiterate notation expand notation. For $i \in \{1, 2, 3\}$, we have

$$((\eta_i^\bullet, \nu_i)_K)^0 = \frac{\eta_{i,K}^0 \times_{\eta_{i,K}^1} \nu_{i,*} \eta_{i,K}^0}{\mathrm{im}(\Delta_i^-)} \quad (2.4.43)$$

where Δ_i^- is the morphism

$$\Delta_i^- : \begin{cases} \eta_{i,K}^{-1} & \longrightarrow & \eta_{i,K}^0 \times_{\eta_{i,K}^1} \nu_{i,*} \eta_{i,K}^0 \\ x & \longmapsto & (d_{\eta_{i,K}}^0(x), -d_{\nu_{i,*} \eta_{i,K}}^0(x)) \end{cases}.$$

We will represent the class of a pair $(a, b) \in \eta_{i,K}^0 \times_{\eta_{i,K}^1} \nu_{i,*} \eta_{i,K}^0$ in the quotient $((\eta_i^\bullet, \nu_i)_K)^0$ by $[a, b]$. Recall that the morphisms $\varphi_{j,K,\nu}^0$ (for $j \in \{2, 3\}$) are defined

using morphisms

$$\tilde{\varphi}_{j,K,\nu}^0 := (\varphi_{j,K}^0 \times \nu_* \varphi_j^0) : \eta_{1,K}^0 \times_{\eta_{1,K}^1} \nu_* \eta_{1,K}^0 \longrightarrow \eta_{j,K}^0 \times_{\eta_{j,K}^1} \nu_* \eta_{j,K}^0$$

Let $[a_2, b_2] \in ((\eta_2^\bullet, \nu_2)_K)^0$. By the description of $((\eta_2^\bullet, \nu_2)_K)^0$ given in (2.4.43), we have $[a_2, b_2] \in \ker \varphi_{3,K,\nu}^0$ if and only if there exists $x_3 \in \eta_{3,K}^{-1}$ such that

$$\tilde{\varphi}_{3,K,\nu}^0(a_2, b_2) = (\varphi_{3,K}^0(a_2), \nu_* \varphi_3^0(b_2)) = \Delta_3^-(x_3).$$

By assumption, (2.4.41) is short exact in every degree, so there exists $x_2 \in \eta_{2,K}^{-1}$ such that $\varphi_{3,K}^{-1}(x_2) = x_3$. As we noted when we proved that the $\tilde{\varphi}_{i,K,\nu}^0$ descend to morphisms between Baer sums, the diagram

$$\begin{array}{ccccc} \eta_{1,K}^0 \times_{\eta_{1,K}^1} \nu_{1,*} \eta_{1,K}^0 & \xrightarrow{\tilde{\varphi}_{2,K,\nu}^0} & \eta_{2,K}^0 \times_{\eta_{2,K}^1} \nu_{2,*} \eta_{2,K}^0 & \xrightarrow{\tilde{\varphi}_{3,K,\nu}^0} & \eta_{3,K}^0 \times_{\eta_{3,K}^1} \nu_{3,*} \eta_{3,K}^0 \\ \Delta_1^- \uparrow & & \Delta_2^- \uparrow & & \Delta_3^- \uparrow \\ \eta_{1,K}^{-1} & \xrightarrow{\varphi_{2,K}^{-1}} & \eta_{2,K}^{-1} & \xrightarrow{\varphi_{3,K}^{-1}} & \eta_{3,K}^{-1} \end{array}$$

commutes. Therefore,

$$\begin{aligned} \tilde{\varphi}_{3,K,\nu}^0((a_2, b_2) - \Delta_2^-(x_2)) &= \tilde{\varphi}_{3,K,\nu}^0((a_2, b_2)) - \tilde{\varphi}_{3,K,\nu}^0(\Delta_2^-(x_2)) \\ &= \Delta_3^-(x_3) - \Delta_3^-(\varphi_{3,K}^{-1}(x_2)) \\ &= \Delta_3^-(x_3) - \Delta_3^-(x_3) = 0. \end{aligned}$$

This has shown that $(a_2 - d_{\eta_{2,K}}^0(x_2), b_2 + d_{\nu_{2,*}\eta_{2,K}}^0(x_2)) \in \ker \tilde{\varphi}_{3,K,\nu}^0$, which is to say that we have

$$\begin{aligned} a_2 - d_{\eta_{2,K}}^0(x_2) &\in \ker \varphi_{3,K}^0 \\ b_2 + d_{\nu_{2,*}\eta_{2,K}}^0(x_2) &\in \ker \nu_* \varphi_3^0. \end{aligned}$$

By exactness of (2.4.41), there exists $a_1 \in \eta_{1,K}^0$ such that

$$\varphi_{2,K}^0(a_1) = a_2 - d_{\eta_{2,K}}^0(x_2).$$

By Proposition 2.4.26, there exists $b_1 \in \nu_{1,*}\eta_{1,K}^0$ such that

$$\nu_* \varphi_{2,K}^0(b_1) = b_2 + d_{\nu_{2,*}\eta_{2,K}}^0(x_2).$$

We claim that (a_1, b_1) gives a pair in the fiber product $\eta_{1,K}^0 \times_{\eta_{1,K}^1} \nu_{1,*} \eta_{1,K}^0$, which is to say that the images of a_1 and b_1 in $\eta_{1,K}^1$ are equal. The following diagram of Cartesian products commutes because $\varphi_{1,K}^\bullet$ and $\nu_{1,*} \varphi_{1,K}^\bullet$ are morphisms of complexes:

$$\begin{array}{ccc} \eta_{1,K}^0 \times \nu_{1,*} \eta_{1,K}^0 & \xrightarrow{\varphi_{2,K}^0 \times \nu_* \varphi_{2,K}^0} & \eta_{2,K}^0 \times \nu_{2,*} \eta_{2,K}^0 \\ d_{\eta_{1,K}^1}^1 \times d_{\nu_{1,*} \eta_{1,K}^1}^1 \downarrow & & \downarrow d_{\eta_{2,K}^1}^1 \times d_{\nu_{2,*} \eta_{2,K}^1}^1 \\ \eta_{1,K}^1 \times \eta_{1,K}^1 & \xrightarrow{\varphi_{2,K}^1 \times \varphi_{2,K}^1} & \eta_{2,K}^1 \times \eta_{2,K}^1 \end{array}$$

The image (a_2, b_2) of (a_1, b_1) in the bottom right corner has the form $(c, c) \in \eta_{2,K}^1 \times \eta_{2,K}^1$ by the assumption that (a_2, b_2) defines a pair in the relevant fiber product. By the assumption that (2.4.40) is short exact, $\varphi_{2,K}^1$ is injective. Since the diagram commutes, we see that images of a_1 and b_1 in $\eta_{1,K}^1$ are equal. Thus we have a pair $(a_1, b_1) \in \eta_{1,K}^0 \times_{\eta_{1,K}^1} \nu_{1,*} \eta_{1,K}^0$ that satisfies

$$\tilde{\varphi}_{2,K,\nu}((a_1, b_1)) = (a_2, b_2) + \Delta_2^-(x_2).$$

This implies that

$$\varphi_{2,K,\nu}([a_1, b_1]) = [a_2, b_2].$$

Since $[a_2, b_2]$ was an arbitrary element of $\ker \varphi_{3,K,\nu}^0 \subset ((\eta_2^\bullet, \nu_2)_K)^0$, we conclude that $\ker \varphi_{3,K,\nu}^0 \subset \text{im } \varphi_{2,K,\nu}^0$. By Lemma 2.4.20, which we noted before applies in the current situation (i.e., to the complex (2.4.42)), we conclude that the functor $(-)_K$ of (2.4.38) is exact. \square

Definition 2.4.44 (generic fiber functor on $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu$ and $\text{fin}_{\mathcal{O}_K}^{p,N}$).

We call the functor $(-)_K$ of Proposition 2.4.37, the **generic fiber functor** on $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu$. This defines an exact functor on $\text{fin}_{\mathcal{O}_K}^{p,N}$ by sending $(H, N) \in \text{fin}_{\mathcal{O}_K}^{p,N}$ to

$$(H, N)_K := (\text{cé}(H), N)_K.$$

We show that the generic fiber functor just defined is compatible with the usual generic fiber functor on $\text{fin}_{\mathcal{O}_K}^p$. Recall that there is a fully faithful exact functor $\text{fin}_{\mathcal{O}_K}^p \rightarrow \text{fin}_{\mathcal{O}_K}^{p,N}$ sending an object H to the pair $(\text{cé}(H), 0)$.

Proposition 2.4.45.

Let (η^\bullet, ν) be an object of $\text{Ext}_{\text{fin}^p \mathcal{O}_K}^\nu$. If $\nu = 0$, then $(\eta^\bullet, \nu)_K = \eta_K^\bullet$.

Proof. By definition,

$$(\eta^\bullet, \nu)_K := \eta_K^\bullet +_{\text{B}} \nu_* \eta_K^\bullet,$$

so to prove our contention it is enough to show that $\nu_* \eta_K^\bullet$ is 0 in the group of extensions, i.e., that $\nu_* \eta_K^0$ a split extension. Recall that to define $\nu_* \eta_K^0$ we chose n of p large enough to kill η^1 , set $E_n = E_{\varpi, K}[n]$, and formed the pushout

$$\begin{array}{ccccccc} \theta_{n, K}^\varpi \otimes \eta_K^1 & 0 & \longrightarrow & \eta_K^1(1) & \longrightarrow & \eta_K^1 \otimes E_n & \longrightarrow & \eta_K^1 & \longrightarrow & 0 \\ & & & \downarrow \nu & & \downarrow & & \parallel & & \\ \nu_*(\theta_{n, K}^\varpi \otimes \eta_K^1) & 0 & \longrightarrow & \eta_K^{-1} & \longrightarrow & \nu_* \eta_K^0 & \longrightarrow & \eta_K^1 & \longrightarrow & 0 \end{array}.$$

That is, we defined $\nu_* \eta_K^0$ by

$$\nu_* \eta_K^0 := \frac{(\eta_K^1 \otimes E_n) \oplus \eta_K^{-1}}{(y, 0) - (0, \nu(y)) : y \in \eta_K^1(1)}.$$

When $\nu = 0$, this is simply

$$\frac{(\eta_K^1 \otimes E_n) \oplus \eta_K^{-1}}{(\eta_K^1 \otimes \mathbb{Z}/n\mathbb{Z}(1)) \oplus 0} \simeq \eta_K^1 \oplus \eta_K^{-1}.$$

We have shown that $\nu_* \eta_K^0$ is split. This completes the proof. \square

2.5 Log finite flat \mathcal{O}_K -groups and semi-stable abelian K -varieties

Throughout this section, let $n \in \mathbb{Z}_{\geq 2}$ be a power of p , and assume the notation of Situation 1.6.11 in the semi-stable case, so that A_K is a semi-stable abelian K -variety with Néron model A and $\mathcal{Q}_K = F(\mathcal{O}_K, \mathfrak{m}_K)(A^\circ)$ its associated log 1-motive.

We saw in Proposition 1.6.2 that the finite K -group $A_K[n]$ is isomorphic to $\mathcal{Q}_K[n]$. We also saw in Proposition 1.6.4 that the log 1-motive \mathcal{Q}_K can be decomposed (using the notation of oc. cit.) as $\mathcal{Q}_{\varpi, K} = \mathcal{Q}_{\varpi, K}^1 + \mathcal{Q}_{\varpi, K}^2$, where $\mathcal{Q}_{\varpi, K}^1 = (\mathcal{Q}_{\varpi}^1)_K$ for some \mathcal{O}_K -1-motive \mathcal{Q}_{ϖ}^1 , and $\mathcal{Q}_{\varpi, K}^2$ is given by a homomorphism $\iota_{K, \varpi}^2 : Y_K \rightarrow T_K \hookrightarrow \tilde{A}_K$. The morphism $\iota_{K, \varpi}^2$ is determined by the

monodromy pairing μ , which gives rise to a level- n monodromy homomorphism (2.3.4)

$$\nu_n : Y/nY(1) \longrightarrow T[n] \hookrightarrow \tilde{A}[n]. \quad (2.5.1)$$

From the extension $\eta(\mathcal{Q}_\varpi^1, n)$ defined as in Example 2.2.12, we obtain a class

$$(\eta(\mathcal{Q}_\varpi^1, n), \nu_n) \in \text{Ext}_{\text{fin}_{\mathcal{O}_K}^\nu}^\nu(Y/nY, \tilde{A}[n]). \quad (2.5.2)$$

We have the following result.

Proposition 2.5.3 ([3], Theorem 19).

There exists an isomorphism

$$(\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K \simeq \eta(\mathcal{Q}_K, n)$$

of objects of $\text{Ext}_{\text{fin}_K^\nu}^\nu(Y_K/nY_K, \tilde{A}_K[n])$. In particular, the degree-zero term in the generic fiber $(\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K$ is isomorphic to $A_K[n]$.

Though it is not strictly necessary to do so, we prefer to work in the category $\text{fin}_{\mathcal{O}_K}^{p,N}$. For this we must construct a morphism $N_n : \mathcal{Q}_\varpi^1[n]^{\text{ét}}(1) \rightarrow \mathcal{Q}_\varpi^1[n]^\circ$. We do this as in Remark 2.4.8. Specifically, since $T[n]$ is connected, ν_n factors through $\tilde{A}[n]^\circ$, and since Y/nY is étale, the surjection in the short exact sequence

$$0 \longrightarrow \tilde{A}[n] \longrightarrow \mathcal{Q}_\varpi^1[n] \longrightarrow Y/nY \longrightarrow 0$$

factors through $\mathcal{Q}_\varpi^1[n]^{\text{ét}}$, so, using these morphisms, we can form

$$N_n : \mathcal{Q}_\varpi^1[n]^{\text{ét}}(1) \longrightarrow (Y/nY)(1) \xrightarrow{\nu_n} T[n] \longrightarrow \mathcal{Q}_\varpi^1[n]^\circ. \quad (2.5.4)$$

This gives an object of $\text{fin}_{\mathcal{O}_K}^{p,N}$.

Definition 2.5.5 ($(\mathcal{Q}_\varpi^1[n], N_n)$).

As in Situation 1.6.11, write \mathcal{Q}_K for the log 1-motive attached to A_K , and write $\mathcal{Q}_K = \mathcal{Q}_{\varpi,K}^1 + \mathcal{Q}_{\varpi,K}^2$ for its Raynaud decomposition with respect to the uniformizer ϖ of \mathcal{O}_K . Letting N_n be as in (2.5.4), we obtain an object of $\text{fin}_{\mathcal{O}_K}^{p,N}$ by forming the pair $(\mathcal{Q}_\varpi^1[n], N_n)$.

Proposition 2.5.6.

There is a canonical isomorphism of finite K -groups

$$\Psi : (\mathcal{Q}_{\varpi}^1[n], N_n)_K^0 \longrightarrow (\eta(\mathcal{Q}_{\varpi}^1, n), \nu_n)_K^0.$$

Proof. Introduce the abbreviated notation

$$Q_n^{\circ} = \mathcal{Q}_{\varpi}^1[n]_K^{\circ}, \quad Q_n = \mathcal{Q}_{\varpi}^1[n]_K, \quad Q_n^{\acute{e}t} = \mathcal{Q}_{\varpi}^1[n]_K^{\acute{e}t}, \quad Y_n = Y_K/nY_K$$

$$\tilde{A}_n = \tilde{A}_K[n], \quad \tilde{A}_n^{\circ} = \tilde{A}[n]_K^{\circ}, \quad \tilde{A}_n^{\acute{e}t} = \tilde{A}[n]_K^{\acute{e}t},$$

$$\nu = \nu_{n,K}, \quad N = N_{n,K}, \quad E_n = E_{\varpi,K}[n]$$

Taking generic fibers in Example 2.2.16 in the case $\mathcal{M} = \mathcal{Q}_{\varpi}^1$, we have a diagram of objects of fin_K^p

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_n^{\circ} = \tilde{A}_n^{\circ} & \longrightarrow & Q_n & \longrightarrow & Q_n^{\acute{e}t} & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow \pi & & \\ 0 & \longrightarrow & \tilde{A}_n & \longrightarrow & Q_n & \longrightarrow & Y_n & \longrightarrow & 0 \end{array}$$

Using the notation of (3.2.15), define N_*Q_n and ν_*Q_n to be the terms in degree zero in the short exact sequences $N_*\text{c}\acute{e}(\mathcal{Q}_{\varpi}^1[n])_K \in \text{Ext}(Q_n^{\acute{e}t}, Q_n^{\circ})$ and $\nu_*\eta(\mathcal{Q}_{\varpi}^1, n)_K \in \text{Ext}(Y_n, \tilde{A}_n)$. The diagrams obtained by pushout fit into a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_n^{\acute{e}t}(1) & \longrightarrow & Q_n^{\acute{e}t} \otimes E_n & \longrightarrow & Q_n^{\acute{e}t} & \longrightarrow & 0 \\ & & \downarrow N & \searrow & \downarrow N' & \searrow \pi \otimes \text{id} & \downarrow \parallel & \searrow \pi & \\ 0 & \longrightarrow & Y_n(1) & \longrightarrow & Y_n \otimes E_n & \longrightarrow & Y_n & \longrightarrow & 0 \\ & & \downarrow \nu & \swarrow & \downarrow \nu' & \swarrow & \downarrow \parallel & \swarrow & \\ 0 & \longrightarrow & Q_n^{\circ} & \longrightarrow & N_*Q_n & \longrightarrow & Q_n^{\acute{e}t} & \longrightarrow & 0 \\ & & \downarrow & \searrow & \downarrow \psi & \swarrow & \downarrow \parallel & \swarrow & \\ 0 & \longrightarrow & \tilde{A}_n & \longrightarrow & \nu_*Q_n & \longrightarrow & Y_n & \longrightarrow & 0 \end{array} \quad (2.5.7)$$

Here, the morphism

$$\psi : N_*Q_n \longrightarrow \nu_*Q_n$$

arises from the universal property of the cofiber product. That is, ψ is the morphism of coproducts

$$(Q_n^{\text{ét}} \otimes E_n) \coprod_{Q_n^{\text{ét}}(1)} Q_n^\circ \longrightarrow (Y_n \otimes E_n) \coprod_{Y_n(1)} \tilde{A}_n$$

which is determined by the morphism $\pi \otimes \text{id} : Q_n^{\text{ét}} \otimes E_n \rightarrow Y_n \otimes E_n$ and the inclusion $Q_n^\circ \rightarrow \tilde{A}_n$. We have, by definition of the generic fiber functor on $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^\nu}^\nu$, equalities

$$(Q_n, N_n)_K^0 := \frac{\{(e, q) \in N_*Q_n \oplus Q_n : \bar{e} = \bar{q} \in Q_n^{\text{ét}}\}}{\{(c, -c) : c \in Q_n^\circ\}}$$

and

$$(\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K^0 := \frac{\{(\epsilon, q) \in \nu_*Q_n \oplus Q_n : \bar{\epsilon} = \bar{q} \in Y_n\}}{\{(a, -a) : a \in \tilde{A}_n\}}.$$

Claim 2.5.8.

The morphism $\psi : N_*Q_n \rightarrow \nu_*Q_n$ induces a morphism

$$\Psi : \begin{cases} (Q_n, N_n)_K^0 & \longrightarrow & (\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K^0 \\ [e, q] & \longmapsto & [\psi(e), q] \end{cases},$$

where, for each finite K -group, we write $[-, -]$ for the class of the pair $(-, -)$ in the relevant fiber product.

Proof. We must prove that the formula for Ψ is well-defined. The groups $(Q_n, N_n)_K^0$ and $(\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K^0$ are subquotients of $N_*Q_n \oplus Q_n$ and $\nu_*Q_n \oplus Q_n$, respectively. We first show that $\psi \oplus \text{id}$ restricts to the relevant subsets. To see this, it is enough consider the pullbacks of ψ to $Q_n^{\text{ét}} \otimes E_n$ and Q_n° . If $e \in Q_n^\circ$, the image \bar{e} of e in $Q_n^{\text{ét}}$ is 0, so $\bar{e} = 0$ in Y_n . Therefore, if $(e, q) \in N_*Q_n \oplus Q_n$ has the property that $\bar{e} = \bar{q}$ in $Q_n^{\text{ét}}$, then $\bar{e} = \bar{q}$ in Y_n , so the pair $(\psi(e), q) = (e, q) \in \tilde{A}_n \oplus Q_n \subset \nu_*Q_n \oplus Q_n$ gives an element of $(\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K^0$. Any other element of N_*Q_n is of the form $e = N'(e')$ for some $e' \in Q_n^{\text{ét}} \otimes E_n$. Suppose $e = q$ in $Q_n^{\text{ét}}$. By commutativity of the back-right square and the top-right square of (2.5.7), we see that $e' = q$ in $Q_n^{\text{ét}}$ and $(\pi \otimes \text{id})(e') = \pi(q)$ in Y_n , so $\psi(e) = \nu'((\pi \otimes \text{id})(e'))$ has the same image in Y_n as q . To conclude, we need to show that $\Psi([c, -c]) = 0$ if $c \in Q_n^\circ$. But ψ restricts to the inclusion of Q_n° into $\tilde{A}_n \subset N_*Q_n$. Since $c \in \tilde{A}_n$, we have $\Psi([c, -c]) = [c, -c] = 0 \in (\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K^0$, as desired. \square

We now show that Ψ is an isomorphism of finite K -groups.

First, note that it is sufficient to show Ψ is injective because the finite groups $(Q_n, N_n)_K^0$ and $(\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K^0$ have the same cardinality. To see this, recall that Q_n gives a class in both $\text{Ext}(Y_n, \tilde{A}_n)$ and $\text{Ext}(Q_n^{\text{ét}}, Q_n^\circ)$. Since $(Q_n, N_n)_K$ gives a class in $\text{Ext}(Q_n^{\text{ét}}, Q_n^\circ)$ and $(\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K$ gives a class in $\text{Ext}(Y_n, \tilde{A}_n)$, we have

$$\#((Q_n, N_n)_K^0) = \#Q_n = \#(\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K^0.$$

We prove Ψ is injective. Let $[e, q] \in (Q_n, N_n)_K^0$. By definition, $[\psi(e), q] = 0$ if there is some $a \in \tilde{A}_n$ such that

$$(\psi(e), q) = (a, -a) \in \tilde{A}_n \oplus \tilde{A}_n \subset \nu_* Q_n \oplus Q_n. \quad (2.5.9)$$

That is, $[e, q]$ is in the kernel of Ψ if and only if $q \in \tilde{A}_n$ and $\psi(e) = -q$. Hence, if $[e, q] \in \ker \Psi$, then $q = -a$. By the universal property characterizing the cofiber product

$$N_* Q_n = (Q_n^{\text{ét}} \otimes E_n) \coprod_{Q_n^{\text{ét}}(1)} Q_n^\circ,$$

to prove injectivity it is sufficient to consider the cases where $e \in Q_n^\circ$ and where $e \in \text{im } N'$, with N' the morphism $Q_n^{\text{ét}} \otimes E_n \rightarrow N_* Q_n$ in (2.5.7).

Suppose that $e \in Q_n^\circ$ and $[e, q] \in \ker \Psi$. Then $\bar{e} = 0$, so $\bar{q} = 0$, so $q \in Q_n^\circ$. The map $\psi|_{Q_n^\circ}$ is the composition of inclusions $Q_n^\circ = \tilde{A}_n^\circ \hookrightarrow \tilde{A}_n \hookrightarrow \nu_* Q_n$. Hence

$$(e, q) = (\psi(e), q) = (a, -a) = (-q, q) \in Q_n^\circ \oplus Q_n^\circ,$$

so $[e, q] = 0$ in $(Q_n, N_n)_K^0$.

Suppose now that $[e, q] \in \ker \Psi$ with $e \in \text{im } N'$, say, $e = N'(e')$. Then $\psi(e) = \nu'((\pi \otimes \text{id})(e'))$, and this is in \tilde{A}_n by (2.5.9). Hence the image of $\psi(e)$ in Y_n is 0, so by the commuting of the front-right square in the diagram (2.5.7) we have $(\pi \otimes \text{id})(e') \in Y_n(1)$. This shows that

$$\psi(e) = \nu'((\pi \otimes \text{id})(e')) = \nu((\pi \otimes \text{id})(e')) \in Q_n^\circ,$$

so also q is in Q_n° by (2.5.9). But then $q = 0$ in $Q_n^{\text{ét}}$, so since $\bar{e} = \bar{q}$ in $Q_n^{\text{ét}}$ we also have $e \in Q_n^\circ$. As we have seen, this gives $\psi(e) = e$, so that (2.5.9) is the equation $e = -q$ in Q_n° . We see that $[e, q] = 0$. This completes the proof. \square

Proposition 2.5.10 (Corollary of Proposition 2.5.3 and Proposition 2.5.6).

Fix ϖ and fix an isomorphism $(\eta(\mathcal{Q}_\varpi^1, n), \nu_n)_K \simeq \eta(\mathcal{Q}_K, n)$ as in Proposition 2.5.3. Then the term in degree zero of the complex $(\mathcal{Q}_\varpi^1[n], N_n)_K$ admits a canonical isomorphism of finite K -groups

$$(\mathcal{Q}_\varpi^1[n], N_n)_K^0 \simeq A_K[n].$$

Therefore, $A_K[n]$ lies in a short exact sequence of finite K -groups

$$0 \longrightarrow \mathcal{Q}_\varpi^1[n]_K^\circ \longrightarrow A_K[n] \longrightarrow \mathcal{Q}_\varpi^1[n]_K^{\text{ét}} \longrightarrow 0. \quad (2.5.11)$$

In Chapter 3, we will define various classes of $\mathbb{Z}_p[\Gamma_K]$ -module. The assumption that A_K is semi-stable, which is in place in Proposition 2.5.10, will tell us that $A_K[n]$ is a “torsion 1-semi-stable” $\mathbb{Z}_p[\Gamma_K]$ -module. Inside of $A_K[n]$ is a largest $\mathbb{Z}_p[\Gamma_K]$ -submodule $\text{Crys}_1(A_K[n])$ belonging to the smaller class of “torsion 1-crystalline $\mathbb{Z}_p[\Gamma_K]$ -modules”. Proposition 2.5.10 will be our starting point for constructing a finite flat \mathcal{O}_K -group with generic fiber $\text{Crys}_1(A_K[n])$.

Chapter 3

Crys_r and its continuous derived functors

Let K be a p -adic field as in Situation 0.1.2. In this chapter, we introduce p -adic Hodge theoretic classes of representations of Γ_K . For an abelian K -variety A_K having semi-stable reduction, the rational p -adic Tate module $T_p A_K \otimes \mathbb{Q}_p$ belongs to a particular class, the class of semi-stable representation, and its quotients $A_K[p^m]$ will all belong to the class of torsion 1-semi-stable representations. For finitely many integers $m \geq 1$ (depending on the ramification of K and the abelian variety A_K), it may be that $A_K[p^m]$ belongs to a smaller class, the class of 1-crystalline representations. In Chapter 4, we will study this phenomenon and relate it to nature of the Néron component group attached to A_K . For this, we will need to develop, following [31] and [39], an appropriate framework for doing homological algebra.

For $r \in \mathbb{Z}_{\geq 0}$, following [39], we define categories of r -crystalline representations. The notions of r -crystallinity for finite-dimensional \mathbb{Q}_p -representations, and for finite-rank \mathbb{Z}_p -modules that are free or torsion, are established in the literature (e.g., in [36]). For these representations it is clear how one should define the functor Crys_r that assigns maximal r -crystalline subrepresentations. However, in order to construct the derived functors of Crys_r , we need to work on a category with enough injectives, and so we need to define r -crystallinity for a much broader class of $\mathbb{Z}_p[\Gamma_K]$ -modules that includes modules not of finite type over \mathbb{Z}_p . We carefully develop certain categories of such modules.

The method for defining derived functors of Crys_r , developed in this chapter was introduced in the paper [31] of Kim and Marshall. They attribute the core idea, namely, of defining $R^i \text{Crys}_r$ for $i > 0$ using inverse systems of torsion modules, to Jannsen, who used the technique in [25]. The dissertation [39] of Susan Marshall gave an expanded account of essential results on the functors $R^i \text{Crys}_r$, which was very helpful to us. Our development of categories of general r -crystalline $\mathbb{Z}_p[\Gamma_K]$ -modules follows Marshall's in spirit, but departs in a few ways; most notably, we use a more modern result, Theorem 3.2.4, to demonstrate compatibility of two notions of crystallinity (see Lemma 3.2.6) of $\mathbb{Z}_p[\Gamma_K]$ -modules free of finite rank over \mathbb{Z}_p . We note that this compatibility receives a different treatment in [31].

Conventions

Throughout this chapter, we will use the notation of Situation 0.1.2.

3.1 Admissible representations

We find it clarifying to describe certain properties of crystalline and semi-stable representations in the broader context of admissible representations.

Definition 3.1.1 ([9] 5.1: $\text{Rep}_F(\Gamma)$; (F, Γ) -regularity; D_B ; α_V ; B -admissibility; $\text{Rep}_F^B(\Gamma)$).

Let F be a topological field, let Γ be a topological group, and let B be an integral domain which is an $F[\Gamma]$ -algebra. Let $\text{Rep}_F(\Gamma)$ denote the category of continuous representations of Γ on finite-dimensional F -vector spaces.

Suppose $E := B^\Gamma$ is a field. Equipping $\text{Frac}(B)$ with its natural action of Γ , we say that B is (F, Γ) -**regular** if $\text{Frac}(B)^\Gamma = B^\Gamma$ and if every nonzero $b \in B$ such that $Fb \subset B$ is Γ -stable satisfies $b \in B^\times$.

In the above setup, for each finite-dimensional F -representation V of Γ we define an E -vector space with trivial Γ action by setting (with reference to the diagonal action of Γ on $V \otimes_F B$)

$$D_B(V) := (B \otimes_F V)^\Gamma.$$

There is a natural morphism of $B[\Gamma]$ -modules

$$\alpha_V : B \otimes_E D_B(V) \longrightarrow B \otimes_F V.$$

We say that V is **B -admissible** if $\dim_E(D_B(V)) = \dim_F V$. Define $\text{Rep}_F^B(\Gamma)$ to be full subcategory of $\text{Rep}_F(\Gamma)$ consisting of the B -admissible representations.

Proposition 3.1.2 ([9] 5.2.1).

Assume the setting of Definition 3.1.1.

- (i) The morphism α_V is injective. The inequality $\dim_E(D_B(V)) \leq \dim_F V$ always holds. The representation V is B -admissible if and only if α_V is an isomorphism of B -algebras.
- (ii) Let Vec_E denote the category of finite-dimensional E -vector spaces. Then D_B defines an exact and faithful functor

$$D_B : \text{Rep}_F^B(\Gamma) \longrightarrow \text{Vec}_E.$$

When formed in the category $\text{Rep}_F(\Gamma)$, subquotients and finite direct sums of objects of $\text{Rep}_F^B(\Gamma)$, are again objects of $\text{Rep}_F^B(\Gamma)$.

- (iii) When formed in the category $\text{Rep}_F(\Gamma)$, finite tensor products, F -linear duals, exterior powers and symmetric powers of objects of $\text{Rep}_F^B(\Gamma)$ are again objects of $\text{Rep}_F^B(\Gamma)$. The functor D_B on $\text{Rep}_F^B(\Gamma)$ commutes with formation of duals and tensor products.

Proof. Everything is proved in [9] 5.2.1 except the statement about closure under formation of finite direct sums. Since D_B is exact, it preserves colimits. From this the claim is easy to see, since for any two objects V_1, V_2 of $\text{Rep}_F^B(\Gamma)$

$$\begin{aligned} \dim_E(D_B(V_1 \oplus V_2)) &= \dim_E(D_B(V_1)) + \dim_E(D_B(V_2)) \\ &= \dim_F V_1 + \dim_F V_2 \\ &= \dim_F(V_1 \oplus V_2). \end{aligned}$$

□

We are primarily interested in B -admissible \mathbb{Q}_p -representations in the case where B is one of the period rings B_{crys} , B_{st} and B_{dR} of Fontaine. We recall the definition of these rings now for completeness, ignoring many important and subtle details relating to topology, dependence of the constructions on choices, and so on. Details can be found in [9] and [16].

Let C be any complete algebraically closed nonarchimedean extension of \mathbb{Q}_p . Let \mathcal{O}_C denote the ring of elements of C of norm at most 1. The tilt of \mathcal{O}_C is

defined to be the inverse limit

$$\mathcal{O}_C^b := \lim_{x \rightarrow x^p} \mathcal{O}_C. \quad (3.1.3)$$

This is the ring of integers of a complete and algebraically closed nonarchimedean field of characteristic p . Write

$$A_{\text{inf}} := W(\mathcal{O}_C^b) \quad (3.1.4)$$

for the ring of Witt vectors of \mathcal{O}_C^b . The ring A_{inf} carries a bijective Frobenius morphism $\varphi_{A_{\text{inf}}}$ arising from the p -power map on $\mathcal{O}_C/p\mathcal{O}_C$. Define an element

$$\tilde{\epsilon} \in \mathcal{O}_C^b \quad (3.1.5)$$

by setting $\tilde{\epsilon}^{(0)} = 1$ choosing, for $m \geq 1$, elements $\tilde{\epsilon}^{(m)} \in \mathcal{O}_C$ such that $(\tilde{\epsilon}^{(m)})^p = \tilde{\epsilon}^{(m-1)}$. Define

$$\mu := [\tilde{\epsilon}] - 1 \in A_{\text{inf}}, \quad \xi := \frac{\mu}{\varphi_{A_{\text{inf}}}^{-1}(\mu)}.$$

The element ξ generates the kernel of the surjective homomorphism

$$\theta : \begin{cases} A_{\text{inf}} & \longrightarrow \mathcal{O}_C \\ \sum_{m \geq 0} [a_m]p^m & \longmapsto \sum_{m \geq 0} a_m^0 p^m \end{cases}$$

(here a_m^0 is the 0-th component of $a_m = (a_m^0, a_m^1, \dots) \in \mathcal{O}_C^b$). Writing $(-)^{\hat{\xi}}$ for the ξ -adic completion, set

$$B_{\text{dR}}^+ := A_{\text{inf}}[1/p]^{\hat{\xi}}, \quad B_{\text{dR}} := B_{\text{dR}}^+[1/\xi].$$

Writing $(-)^{\hat{p}}$ for p -adic, let

$$A_{\text{crys}} := \left(A_{\text{inf}} \left[\frac{\xi^n}{n!} : n \in \mathbb{Z}_{\geq 1} \right] \right)^{\hat{p}}.$$

This ring is naturally a subring of B_{dR}^+ . The ring of crystalline periods is

$$B_{\text{crys}} := A_{\text{crys}}[1/\mu]. \quad (3.1.6)$$

We define the ring of semi-stable periods B_{st} by

$$B_{\text{st}} := B_{\text{crys}}[\log([\tilde{p}]]) \subset B_{\text{dR}}$$

for an appropriately defined logarithm $\log([\tilde{p}]) : \text{Frac}(\mathcal{O}_C^\flat)^\times \rightarrow B_{\text{dR}}^+$, where \tilde{p} is an element of \mathcal{O}_C^\flat with $\tilde{p}^{(0)} = p \in \mathbb{Z} \subset \mathcal{O}_C$. We summarize some well-known results about these rings.

Proposition 3.1.7.

Assume the notation of Situation 0.1.2

- (i) *The rings B_{dR} , B_{st} and B_{crys} are (\mathbb{Q}_p, Γ_K) -regular.*
- (ii) *Let V be an object of $\text{Rep}_{\mathbb{Q}_p}(\Gamma_K)$. If V is B_{st} -admissible then it is B_{dR} -admissible. If V is B_{crys} -admissible then it is B_{st} -admissible.*

Definition 3.1.8 (de Rham; semi-stable; crystalline).

*We call an object of $\text{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ **de Rham** (resp., **semi-stable**, resp., **crystalline**) if it is B_{dR} - (resp., B_{st} -, resp., B_{crys} -) admissible.*

Any de Rham representation has an associated multi-set of Hodge-Tate weights, a collection of integers with multiplicities. To define these, recall that C is a complete algebraically closed extension of \mathbb{Q}_p , and for $i \in \mathbb{Z}$, let $C(i)$ to be the 1-dimensional C -module Ce with $\text{Gal}(\overline{K}/\mathbb{Q}_p)$ -action given by $\sigma(ce) = \chi(\sigma)^i \sigma(c)e$, where χ is the cyclotomic character.

Definition 3.1.9 (Hodge-Tate weights).

*Let V be in $\text{Rep}_{\mathbb{Q}_p}^{B_{\text{dR}}}(\Gamma_K)$. An integer i is a **Hodge-Tate weight** of V if*

$$(V \otimes_{\mathbb{Q}_p} C(-i))^{\Gamma_K} \neq 0,$$

*where $V \otimes_{\mathbb{Q}_p} C(-i)$ is given the diagonal action of Γ_K . The **multiplicity** of i as a Hodge-Tate weight of V is defined to be $\dim_C(V \otimes_{\mathbb{Q}_p} C(-i))^{\Gamma_K}$.*

We record some properties of Hodge-Tate weights are immediately checked from the definitions.

Proposition 3.1.10.

Let V_1 and V_2 be objects of $\text{Rep}_{\mathbb{Q}_p}^{B_{\text{dR}}}(\Gamma_K)$ with Hodge-Tate weights in $[0, r_1]$ and $[0, r_2]$, respectively.

- (i) *If $V_1 \subset V_2$ then $r_1 \leq r_2$.*

(ii) If V_1 is a quotient of V_2 , then $r_1 \leq r_2$.

(iii) $V_1 \oplus V_2$ is an object of $\text{Rep}_{\mathbb{Q}_p}^{\text{BdR}}(\Gamma_K)$ with Hodge-Tate weights in $[0, \max\{r_1, r_2\}]$.

(iv) $V_1 \otimes V_2$, is an object of $\text{Rep}_{\mathbb{Q}_p}^{\text{BdR}}(\Gamma_K)$ with Hodge-Tate weights in $[0, r_1 + r_2]$.

Remark 3.1.11. By our convention, the p -adic cyclotomic character, which we review below, has unique Hodge-Tate weight 1 with multiplicity 1.

Example 3.1.12. Any unramified representation is crystalline with 0 as its only Hodge-Tate weight. In particular, the trivial character $\Gamma_K \rightarrow \{\text{id}\} \subset \text{GL}_1(\mathbb{Q}_p)$ is crystalline. Conversely, any de Rham representation with 0 as its only Hodge-Tate weight is unramified (see [9] 8.3.5).

By Proposition 3.1.10, if V_1 is unramified and V_2 is de Rham (resp. semi-stable, resp. crystalline), then $V_1 \otimes_{\mathbb{Q}_p} V_2$ is de Rham (resp. semi-stable, resp. crystalline) with the same Hodge-Tate weights as V_2 , with known, possibly greater multiplicities.

Example 3.1.13. The p -adic cyclotomic character

$$\chi : \text{Gal}(\overline{K}/\mathbb{Q}_p) \longrightarrow \text{GL}_1(\mathbb{Z}_p) \subset \text{GL}_1(\mathbb{Q}_p)$$

is crystalline. This homomorphism can be defined as follows. Let $m \in \mathbb{Z}_{\geq 1}$, let $n = p^m$, and let $\zeta \in \overline{K}^\times[n]$. For any $\sigma \in \text{Gal}(\overline{K}/\mathbb{Q}_p)$, the element $\sigma(\zeta)$ has order n , so it can be written as $\zeta^{\chi_n(\sigma)}$ for some unique integer $\chi_n(\sigma)$ between 0 and $n - 1$. The integer $\chi_n(\sigma)$ is independent of the choice of ζ above. Reducing modulo n defines the mod n -cyclotomic character

$$\chi_n : \text{Gal}(\overline{K}/\mathbb{Q}_p) \longrightarrow \text{GL}_1(\mathbb{Z}/n\mathbb{Z}). \quad (3.1.14)$$

For $i \in \mathbb{Z}$ we write $\mathbb{Z}/n\mathbb{Z}(i)$ for the $\mathbb{Z}/n\mathbb{Z}$ -module $\mathbb{Z}/n\mathbb{Z}$ equipped with the action of Γ_K given by (choosing $\{1\} \subset \mathbb{Z}/n\mathbb{Z}$ as a basis over $\mathbb{Z}/n\mathbb{Z}$)

$$\sigma(c \cdot 1) = \chi(\sigma)^i c \cdot 1 \quad (c \in \mathbb{Z}/n\mathbb{Z}).$$

One usually write $\mathbb{Z}/n\mathbb{Z}$ for $\mathbb{Z}/n\mathbb{Z}(0)$. A choice of generator of $\overline{K}^\times[n]$ gives rise to an isomorphism $\mathbb{Z}/n\mathbb{Z}(1) \simeq \overline{K}^\times[n]$. For example, using the element $\tilde{e} \in \mathcal{O}_C^b$ of (3.1.5), we have an isomorphism

$$\begin{aligned} \mathbb{Z}/n\mathbb{Z}(1) &\longrightarrow \overline{K}^\times[n] \\ c \cdot 1 &\longmapsto (\tilde{e}^{(m)})^c \end{aligned} \quad (3.1.15)$$

The mod n characters χ_n give rise to the p -adic cyclotomic character via

$$\chi(\sigma) := \lim_{m \geq 1} \chi_{p^m}(\sigma) \in \mathrm{GL}_1(\mathbb{Z}_p).$$

In general, any crystalline \mathbb{Q}_p -character of Γ_K is the tensor product an unramified \mathbb{Q}_p -character with an integer power of χ . In particular, a character of Γ_K is crystalline if and only if it is de Rham.

Example 3.1.16. Let A_K be an abelian variety over K . Then

$$V_p A_K := T_p E_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a $2 \dim(A_K)$ -dimensional de Rham representation with Hodge-Tate weights in $[0, 1]$. This follows from a much more general statement about étale cohomology groups which we will not need. When A_K has semi-stable reduction, $V_p A_K$ is semi-stable. When A_K has good reduction, $V_p A_K$ is crystalline. More than this is true; see Corollary 4.3.3.

Example 3.1.17. Let $q \in K^\times$ have $|q| < 1$. The Tate curve $E_{q,K}$ has bad semi-stable reduction. Let $m \in \mathbb{Z}_{\geq 1}$ and let $n = p^m$. Corresponding to the log 1-motive $\mathcal{T}_{q,K}$ defined in (1.5.6), we have, via Proposition 1.6.2 and (2.2.13), a short exact sequence

$$\theta_{n,K}^q \quad 0 \longrightarrow \mathbb{Z}/n\mathbb{Z}(1) \longrightarrow E_{q,K}[n] \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \quad (3.1.18)$$

To see this, first note that Raynaud extension \tilde{E}_q is $\mathbb{G}_{m,\mathcal{O}_K}$. The morphism ψ_q of (1.5.9) induces a Γ_K -equivariant isomorphism of abelian groups

$$\bar{\psi}_q : \bar{K}^\times / q^\mathbb{Z} \xrightarrow{\sim} E_q(\bar{K}). \quad (3.1.19)$$

The restriction of the quotient map $\bar{K}^\times \rightarrow \bar{K}^\times / q^\mathbb{Z}$ to $\bar{K}^\times[n]$ is an injection, so $\bar{\psi}_q$ gives an injective homomorphism of $\bar{K}^\times[n]$ into $E_q[n](\bar{K})$. Taking the isomorphism $\mathbb{Z}/n\mathbb{Z}(1) \xrightarrow{\sim} \bar{K}^\times[n]$ of (3.1.15), we obtain an injection $\mathbb{Z}/n\mathbb{Z}(1) \rightarrow E_q[n](\bar{K})$ determined by (recalling that $n = p^m$)

$$1 \mapsto \bar{\psi}_q(\tilde{\epsilon}^{(m)}) = (X(\tilde{\epsilon}^{(m)}, q), Y(\tilde{\epsilon}^{(m)}, q)) \in \mathbb{A}^2(\bar{K})$$

with X and Y as in (1.5.8). Set $x_1 := \bar{\psi}_q(\tilde{\epsilon}^{(m)})$, and let x_2 be any element of

$E_{q,K}[n]$ not in the $\mathbb{Z}/n\mathbb{Z}$ -span of x_1 . Then

$$E_{q,K}[n] \simeq (\mathbb{Z}/n\mathbb{Z})x_1 \oplus (\mathbb{Z}/n\mathbb{Z})x_2$$

as $\mathbb{Z}/n\mathbb{Z}$ -modules but not as $\mathbb{Z}/n\mathbb{Z}[\Gamma_K]$ -modules. To describe the action of Γ_K , let $z_2 \in \overline{K}^\times$ satisfy $\bar{\psi}_q(z_2 q^{\mathbb{Z}}) = x_2$. Then $z_2 q^{\mathbb{Z}}$ has order n , i.e., $z_2^n = q^s$ for some $s \in \mathbb{Z}$. Let $\sigma \in \Gamma_K$. Writing $\sigma(z_2)^n = \sigma(q^s) = q^s = z_2^n$, we see that $\sigma(z_2)/z_2 \in \overline{K}^\times[n]$. We may, therefore, write

$$\sigma(z_2) = (\bar{\epsilon}^{(m)})^{c_{q,n}(\sigma)} z_2$$

for some unique integer $c_{q,n}(\sigma)$ between 0 and $n-1$. By Γ_K -equivariance of $\bar{\psi}_q$, this equation becomes $\sigma(x_2) = c_{q,n}(\sigma)x_1 + x_2$. The elements $c_{q,n}(\sigma)$ determine a 1-cocycle

$$c_{q,n} : \Gamma_K \longrightarrow \mathbb{Z}/n\mathbb{Z}. \quad (3.1.20)$$

Altogether, we see that, in the basis $\{x_1, x_2\}$, the action of Γ_K on $E_{q,K}[n]$ has the matrix form $\begin{pmatrix} \chi_n & c_{q,n} \\ & 1 \end{pmatrix}$. A direct calculation with the ring B_{st} shows that $V_p E_{q,K}$ is semi-stable with Hodge-Tate weights 0 and 1—in fact, B_{st} is defined essentially to make this statement hold.

Remark 3.1.21. The \mathbb{Q}_p -representation $V_p E_{q,K}$ of Example 3.1.17 is an extension of an unramified representation by the cyclotomic character, which is crystalline with Hodge-Tate weights in $[0, 1]$. The representation $V_p E_{q,K}$ is semi-stable but not crystalline. In particular, a Yoneda extension of a crystalline representation by another crystalline representation need not be crystalline.

3.2 Some categories of $\mathbb{Z}_p[\Gamma_K]$ -modules

Let A_K be as in the semi-stable case of Situation 1.6.11. In order to define $R^1 \text{Crys}_1(T_p A_K)$ as in [31] and [39], we need to define various categories of $\mathbb{Z}_p[\Gamma_K]$ -modules relating to $\text{Rep}_{\mathbb{Q}_p}^{B_{\text{crys}}}(\Gamma_K)$, but which are allowed to have torsion, and which need not be finitely generated. We do this and give some essential properties of these modules in this section.

For clarity, we include the definition of an inverse systems.

Definition 3.2.1.

Given any category \mathcal{C} , define the category $\text{Inv } \mathcal{C}$, whose objects are called “inverse systems of objects of \mathcal{C} ” as follows. The objects are sequences $(C_m)_{m \geq 1}$ of objects of \mathcal{C} indexed by $\mathbb{Z}_{\geq 1}$ equipped with morphisms $t_{m',m} : C_m \rightarrow C_{m'}$ for all $m' \leq m$ satisfying the transitivity relation

$$t_{m'',m'} \circ t_{m',m} = t_{m'',m} : C_m \rightarrow C_{m'} \rightarrow C_{m''}.$$

A morphism $f : (C'_m)_{m \geq 1} \rightarrow (C_m)_{m \geq 1}$ is a collection of morphisms $f_m : C'_m \rightarrow C_m$ such that $f_{m'} \circ t'_{m',m} = t_{m',m} \circ f_m$ for all $m' \leq m$.

We now define our categories of $\mathbb{Z}_p[\Gamma_K]$ -modules with integral and torsion coefficients. We would prefer to do this without introducing so much notation, but we do so here because we need to be very precise in order to define Crys_r and its derived functors, and this allows us to avoid repeating hypotheses relating to finiteness and topology.

Definition 3.2.2.

Let $r \in \mathbb{Z}_{\geq 0}$. Let $*$ \in $\{\text{st, crys}\}$.

(i) Let $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ denote the category of topological $\mathbb{Z}_p[\Gamma_K]$ -modules, i.e., the category of topological spaces M equipped with a continuous morphism $\mathbb{Z}_p[\Gamma_K] \times M \rightarrow M$ giving M the structure of a $\mathbb{Z}_p[\Gamma_K]$ -module.

(ii) Let $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$ denote the full subcategory of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ such that

$$\text{Ob}(\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)) = \left\{ L \text{ such that } L \text{ is finite type and free} \right. \\ \left. \text{over } \mathbb{Z}_p \text{ and } L \text{ has the } p\text{-adic topology} \right\}.$$

(iii) Let $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$ denote the full subcategory of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ such that

$$\text{Ob}(\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}) = \left\{ M \text{ such that } M \text{ is a torsion } \mathbb{Z}_p\text{-} \right. \\ \left. \text{module with the discrete topology.} \right\}.$$

(iv) Let $\text{Rep}_{\mathbb{Q}_p}^{*,r}(\Gamma_K)$ denote the full subcategory of $\text{Rep}_{\mathbb{Q}_p}^{B_*}(\Gamma_K)$ such that

$$\text{Ob}(\text{Rep}_{\mathbb{Q}_p}^{*,r}(\Gamma_K)) = \left\{ V \text{ such that } V \text{ all Hodge-Tate weights} \right. \\ \left. \text{in the interval } [0, r] \right\}.$$

ever, it is not clear that an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$ that is free over \mathbb{Z}_p is actually an object of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$. That this holds is due to Tong Liu.

Theorem 3.2.4 ([36], 7.2 and 7.3).

Let $r \in \mathbb{Z}_{\geq 1}$. Let $$ \in $\{\text{st}, \text{crys}\}$. Let L be an object of $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$. Then L is an object of $\text{Rep}_{\text{free}}^{\text{crys},r}(\Gamma_K)$ (resp., $\text{Rep}_{\text{free}}^{\text{st},r}(\Gamma_K)$) if and only if, for each $m \geq 1$, $L/p^m L$ is an object of $\text{Rep}_{\text{tors}}^{\text{crys},r}(\Gamma_K)$ (resp. $\text{Rep}_{\text{tors}}^{\text{st},r}(\Gamma_K)$).*

By Proposition 3.1.2, the de Rham, semi-stable and crystalline conditions on representations with \mathbb{Q}_p -coefficients are all preserved under certain constructions. We need the same to be true semi-stable and crystalline representations with integral and torsion coefficients.

Proposition 3.2.5.

Let \mathcal{C} be any of the categories of Definition 3.2.2 defined in (i)-(vii). Note that, by definition, \mathcal{C} is a full subcategory of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$.

- (a) *A $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ -subobject of an object of \mathcal{C} is also an object of \mathcal{C} .*
- (b) *Suppose \mathcal{C} is one of the categories containing objects with nontrivial torsion. If M is a $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ -quotient of an object of \mathcal{C} , then M is an object of \mathcal{C} .*
- (c) *If M is a $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ -direct sum of finitely-many objects of \mathcal{C} , then M is an object of \mathcal{C} .*

Proof. We prove the statements (a)-(c) one category at a time.

- (i) For $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$, the statements (a)-(c) are tautologically true.
- (ii) Let L be an object of $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$. A \mathbb{Z}_p -submodule L' of L is free of finite rank. By assumption, L' is a $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$, so it has the induced topology, which is necessarily the p -adic topology. We see that (a) holds. We do not claim (b) holds for this category. The direct sum of two objects L_1, L_2 of $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$ has the product topology. By assumption L_1 and L_2 have the product topology coming from isomorphisms of topological \mathbb{Z}_p -modules $L_i \simeq \mathbb{Z}_p^{d_i}$ where $d_i \in \mathbb{Z}_{\geq 0}$ ($i = 1, 2$). The direct sum $L_1 \oplus L_2$, then, admits an isomorphism of topological \mathbb{Z}_p -modules $L_1 \oplus L_2 \simeq \mathbb{Z}_p^{d_1+d_2}$. Claim (c) for $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$ follows.
- (iii) Let V be an object of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$. The discrete topology induces the discrete topology and gives the discrete topology to any quotient. Also, a finite product of two spaces with the discrete topology has the discrete topology. Claims (a)-(c) follow.

(iv) The category $\text{Rep}_{\mathbb{Q}_p}^{*,r}(\Gamma_K)$ is a full subcategory of the form $\text{Rep}_F^B(\Gamma)$ defined in Definition 3.1.1. Therefore, by Proposition 3.1.2, subquotients and finite direct sums of objects of $\text{Rep}_{\mathbb{Q}_p}^{*,r}(\Gamma_K)$ are again objects of $\text{Rep}_{\mathbb{Q}_p}^{B,*}(\Gamma_K)$. That the constructions preserve $\text{Rep}_{\mathbb{Q}_p}^{*,r}(\Gamma_K)$ follows from Proposition 3.1.10.

(v) Let L be an object of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$. Let L' be a $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ -subobject of L . We have seen that L' is in $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$. Since $L'[1/p] \subset L[1/p]$ is a $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ -subobject. We have just shown that this implies that $L'[1/p]$ is an object of $\text{Rep}_{\mathbb{Q}_p}^{*,r}(\Gamma_K)$. This shows that L' is an object of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$.

We don't claim (b) holds for $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$. Let L_1 and L_2 be objects of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$. Then $(L_1 \oplus L_2)[1/p] \simeq L_1[1/p] \oplus L_2[1/p]$, so claim (c) from claim (a) for $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$.

(vi) We now consider the category $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$.

We prove (a). Let V be an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, and choose some surjection $\pi : L \rightarrow V$ with L in $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$. Suppose $V' \subset V$ is a $\mathbb{Z}_p[\Gamma_K]$ -submodule and let $L' = \pi^{-1}(V') \subset L$. Then L' is a $\mathbb{Z}_p[\Gamma_K]$ -submodule of L . Equipping L' with the induced topology, we have that L' is an object of $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$. Since $L'[1/p] \subset L[1/p]$, we see that L' is an object of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$. This shows that V' , being a quotient of L' , is an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$.

We prove (b). Adopt the notation of the previous paragraph. Let U be an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$ and suppose there is an surjection $V \rightarrow U$ of objects of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$. Composing this with π gives a map $\bar{\pi} : L \rightarrow U$. Arguing as before, we see that $L/\ker(\bar{\pi}) = U$ realizes U as an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$. We prove (c). For $i = 1, 2$ let $L_i^{(2)} \subset L_i^{(1)}$ be an inclusion of objects of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$. Then $L_1^{(1)} \oplus L_2^{(1)}$ realizes $L_1^{(1)}/L_1^{(2)} \oplus L_2^{(1)}/L_2^{(2)}$ as an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$.

(vii) Let M be an object of the category $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$. Note that the proofs of Lemma 3.2.6 and Lemma 3.2.7 rely on the result we are presently proving, but only in the case of the category $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, whose proof is already complete. Hence, to prove (a)-(b) for $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$, we may use these lemmas.

Being finitely generated over \mathbb{Z}_p , we can write $M = M_{\text{free}} \oplus M_{\text{tors}}$ where M_{tors} is the maximal torsion submodule and M_{free} is torsion-free. By Lemma 3.2.6, M_{free} and M_{tors} are, respectively, objects of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$ and $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$.

We prove (a). Let M' be a $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ -subobject of M . Then $M' = M'_{\text{free}} \oplus M'_{\text{tors}}$. By (a) for the categories $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$ and $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, M'_{free} is an object of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$ and M'_{tors} is an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$. By Lemma 3.2.7, M' is an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$.

We prove (c). Suppose M_1 and M_2 are objects of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$, say with $M_i = \lim_{j \geq 1} M_{i,j}$ where $M_{i,j}$ is an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$ for all i, j . Then $M_1 \oplus M_2$ is the inverse limit of the natural inverse system formed by the modules $M_{1,j} \oplus M_{2,j}$. By (c) for the category $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, $M_{1,j} \oplus M_{2,j}$ is an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, so we see that $M_1 \oplus M_2$ is an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$.

We prove (b). First, let L and L' be objects of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$ that are free. Then, by Lemma 3.2.6, L and L' are objects of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$, in particular, they are each p -adically complete. By definition of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, $L/p^m L$ and $L'/p^m L'$ are objects of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$. For each $m \in \mathbb{Z}_{\geq 1}$ we have a short exact sequence

$$0 \longrightarrow L'/p^m L' \longrightarrow L/p^m L \longrightarrow L''_m \longrightarrow 0$$

with L''_m an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$ by the claim (c) for that category. These short exact sequences are compatible with the quotient maps defining the inverse systems $(L'/p^m L')_{m \geq 1}$ and $(L/p^m L)_{m \geq 1}$. By [51, Tag 03CA], the functor $\lim_{m \geq 1}$, using p -adic completeness of L' and L , gives a short exact sequence

$$0 \longrightarrow L' \longrightarrow L \longrightarrow \lim_{m \geq 1} L''_m \longrightarrow 0.$$

This has shown that L/L' is an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$.

For the general case of (b), let M be an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$ and let M' be a subobject of M in that category. Then

$$M/M' = (M_{\text{free}}/M'_{\text{free}}) \oplus (M_{\text{tors}}/M'_{\text{tors}}).$$

We have just shown that $M_{\text{free}}/M'_{\text{free}}$ is an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$, and we saw above that $M_{\text{tors}}/M'_{\text{tors}}$ is an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, hence an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$. By (c), M/M' is an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$.

Granting Lemma 3.2.6 and Lemma 3.2.7, which we prove next, this completes the proof. \square

Lemma 3.2.6.

Let $* \in \{\text{st, crys}\}$. Let M be an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$ and let M' be a $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ -subobject of M .

(i) If M' is free then M' is an object of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$.

(ii) If M' is torsion then M' is an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$.

(iii) Writing $M = M_{\text{free}} \oplus M_{\text{tors}}$ for the decomposition of M as a direct sum of a free \mathbb{Z}_p -module and its maximal torsion \mathbb{Z}_p -submodule. Then the decomposition $M = M_{\text{free}} \oplus M_{\text{tors}}$ is preserved by the action of Γ_K , and M_{free} and M_{tors} are objects of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$ and $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, respectively.

Proof. We may assume $M = \lim_{i \in \mathbb{Z}_{\geq 1}} M_i$ with M_i in $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$.

Consider claim (i). Assume the subobject $M' \subset M$ is free over \mathbb{Z}_p . Since M_i is finite type and torsion, there exists some $k(i) \in \mathbb{Z}_{\geq 1}$ such that $p^{k(i)} M_i = 0$. Therefore $p^{k(i)} M' \subset \ker(M \rightarrow M_i)$, so $M'/p^{k(i)} M' \subset M_i$. Since M_i is in $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, Proposition 3.2.5 for $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$ implies that $M'/p^{k(i)} M'$ is in $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$. Since the $k(i)$ tend to infinity as i tends to infinity (else M is torsion), we see that in fact each $M'/p^m M'$ ($m \in \mathbb{Z}_{\geq 1}$) is an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$. Since M' is assumed p -adically complete and free over \mathbb{Z}_p , Theorem 3.2.4 implies that M' is an object of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$.

We now prove (ii). We will build a finite direct sum of the M_i 's into which M' injects. Using Proposition 3.2.5 for $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, we will conclude that M is in $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$. Since M is torsion and of finite type over \mathbb{Z}_p , its underlying set is finite, so M' is a finite set, say

$$M' = \{m_1, \dots, m_d\}.$$

For $j \in \{1, \dots, d\}$, write $m_j = (m_{j,i})_{i \geq 1} \in M \subset \prod_{i \in \mathbb{Z}_{\geq 1}} M_i$. For each $j' \in \{1, \dots, d\}$ with $j' \neq j$, choose an index $i(j; j')$ such that the image of $m_j - m_{j'}$ in $M_{i(j; j')}$ is nonzero. Define

$$I(j) := \{i(j, j') : j' \in \{1, \dots, d\}, j' \neq j\}, \quad I := \prod_{j=1}^d I(j).$$

By construction, the natural morphism

$$M \longrightarrow \bigoplus_{\alpha \in I} M_{\alpha}$$

is injective; if m_j and $m_{j'}$ map to the same element of M , then they agree in the $i(j; j')$ -th component, which is not possible unless $j = j'$. As we noted above, this proves (ii).

We prove (iii). The claim that the decomposition $M = M_{\text{free}} \oplus M_{\text{tors}}$ is preserved by the action of Γ_K is immediate from the fact that, for $\sigma \in \Gamma_K$ and $m \in M$, σm is torsion if and only if m is torsion. Now, the modules M_{free} and M_{tors} are $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ -submodules of M that are, respectively free and torsion, so parts (i) and (ii) prove the claim. \square

Lemma 3.2.7.

Let L be an object of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$ and let V be an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$. Then $L \oplus V$ is an object of $\text{Rep}_{\text{ftype}}^{*,r}(\Gamma_K)$.

Proof. For $m \in \mathbb{Z}_{\geq 1}$, let $M_m := (L/p^m L) \oplus V$. Define an inverse system as follows: for $m' \leq m$, let $M_m \rightarrow M_{m'}$ be the product of the mod- $p^{m'}$ reduction map with the identity morphism on V . Since $L/p^m L$ is a quotient of the object L of $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$, it is an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$ by definition. By Proposition 3.2.5 (c) for the category $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, M_m is an object of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$. It is immediately verified that $L \oplus V \simeq \lim_{m \geq 1} M_m$. This completes the proof. \square

Example 3.2.8. Every object M of $\text{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ contains an object of $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$ of maximal rank, so that $L[1/p] = M$. Hence any of the \mathbb{Q}_p -representations described in Examples 3.1.12, 3.1.13, 3.1.16, 3.1.17 all give rise to examples of objects of the categories $\text{Rep}_{\text{free}}^{*,r}(\Gamma_K)$.

Example 3.2.9. We are particularly interested in representations coming from abelian varieties. If A_K is an abelian K -variety of dimension g , then $T_p A_K$ is an object is a Γ_K -stable \mathbb{Z}_p -lattice in a $2g$ -dimensional de Rham representation with Hodge-Tate weights in $[0, 1]$. Each of the quotients $T_p A_K / p^m T_p A_K = A_K[p^m]$ is free over $\mathbb{Z}/p^m \mathbb{Z}$ of rank $2g$. If A_K has semi-stable (resp., crystalline) reduction each of $T_p A_K$ and $A_K[p^m]$ ($m \geq 1$) is 1-semi-stable (resp., 1-crystalline).

In Chapter 4, we will use the following result about Baer sums via our use of Proposition 3.2.13.

Proposition 3.2.10.

Let $* \in \{\text{st, crys}\}$. Suppose we have two short exact sequences

$$\eta(V_i) \quad 0 \longrightarrow V' \xrightarrow{\alpha_i} V_i \xrightarrow{\beta_i} V'' \longrightarrow 0 \quad i = 1, 2$$

in $\text{Rep}_{\text{tors}}(\Gamma_K)$. Let V_+ be the middle term in the Baer sum $\eta(V_1) +_{\text{B}} \eta(V_2)$

(i) If V_1 and V_2 are objects of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, then so is V_+ .

(ii) If any two of V_1 , V_2 and V_+ is in $\text{Rep}_{\text{tors}}^{\text{crys},r}(\Gamma_K)$, then all three are in $\text{Rep}_{\text{tors}}^{\text{crys},r}(\Gamma_K)$.

Proof. Recall that $\text{Rep}_{\text{tors}}(\Gamma_K)$ is an abelian category which is equivalent to fin_K^p . Therefore, as we saw in Proposition 2.4.10, the set

$$\text{Ext}(V'', V') = \text{Ext}_{\text{Rep}_{\text{tors}}(\Gamma_K)}(V'', V')$$

of Yoneda extensions of V'' by V' has the structure of an abelian group under the Baer sum. We recall this construction. Let

$$U_1 := V_1 \times_{V''} V_2 := \{(v_1, v_2) \in V_1 \oplus V_2 : \beta_1(v_1) = \beta_2(v_2)\} \subset V_1 \oplus V_2$$

and

$$U_2 := \{(i_1(v'), -\alpha_2(v')) : v' \in V'\} \subset V_1 \oplus V_2.$$

Recall from Proposition 2.4.10 that the Baer sum is defined to be an extension

$$\eta(V_1) +_{\text{B}} \eta(V_2) \quad 0 \longrightarrow V' \longrightarrow V_+ \longrightarrow V'' \longrightarrow 0$$

where $V_+ := U_1/U_2$.

By Proposition 3.2.5, $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$ is stable under finite direct sums and passage to subquotients. Therefore, since U_1 and U_2 are visibly $\mathbb{Z}_p[\Gamma_K]$ -stable inside of $V_1 \oplus V_2$, they are both objects of $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$, so V_+ is in $\text{Rep}_{\text{tors}}^{*,r}(\Gamma_K)$.

Taking a Baer difference, (ii) follows from (i). \square

In Definition 3.2.2 we defined many categories of $\mathbb{Z}_p[\Gamma_K]$ -modules that are not \mathbb{Q}_p -vector spaces. For our later applications, we will mostly be interested in objects of $\text{Rep}_{\text{free}}^{\text{st},1}(\Gamma_K)$, $\text{Rep}_{\text{free}}^{\text{crys},1}(\Gamma_K)$, $\text{Rep}_{\text{tors}}^{\text{st},1}(\Gamma_K)$ and $\text{Rep}_{\text{tors}}^{\text{crys},1}(\Gamma_K)$. These can be understood in terms of the following theorem, which is a summary work done by many people over the course of fifty years.

Theorem 3.2.11 (Tate; Raynaud; Kisin; Kim; see Remark 3.2.12).

One has the following relationships between 1-crystalline $\mathbb{Z}_p[\Gamma_K]$ -modules and finite flat \mathcal{O}_K -groups. Let $\text{pdiv}_{\mathcal{O}_K}$ denote the category of p -divisible groups over \mathcal{O}_K .

(i) *The Tate module of any object of $\text{pdiv}_{\mathcal{O}_K}$ is an object of $\text{Rep}_{\text{free}}^{\text{crys},1}(\Gamma_K)$.*

(ii) The generic fiber of any object of $\text{fin}_{\mathcal{O}_K}^p$ is an object of $\text{Rep}_{\text{tors}}^{\text{crys},1}(\Gamma_K)$.

(iii) Every object of $\text{Rep}_{\text{free}}^{\text{crys},1}(\Gamma_K)$ is the Tate module of an object of $\text{pdiv}_{\mathcal{O}_K}$.

(iv) Every object of $\text{Rep}_{\text{tors}}^{\text{crys},1}(\Gamma_K)$ is the generic fiber of an object of $\text{fin}_{\mathcal{O}_K}^p$.

Remark 3.2.12. Part (i) was proved by Tate in different terms in [52], Section 4, Corollary 2. Part (ii) is originally due to Oort (as noted in [9]), but it also follows Raynaud’s stronger result that every object of $\text{fin}_{\mathcal{O}_K}^p$ is the kernel of an isogeny of abelian \mathcal{O}_K -schemes (see [4], 3.1.1). Parts (iii) and (iv) were proven by Kisin in [33] (see also [34]) for $p > 2$ (and, in some cases, for $p = 2$), and by Wansu Kim in the case for $p = 2$. See [32] 2.3 for an explanation of Kisin’s work, and 4.1 for the extension to the case $p = 2$.

Proposition 3.2.13.

Let (η^\bullet, ν) be an object of $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu$ and let $\nu_*\eta_K^0$ be defined as in Definition 2.4.14. Then $(\eta^\bullet, \nu)_K^0$ is 1-semi-stable. Moreover, $(\eta^\bullet, \nu)_K^0$ is 1-crystalline if and only if $\nu_*\eta_K^0$ is 1-crystalline.

Proof. An object of $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^\nu$ is a pair (η^\bullet, ν) where η^\bullet is a three-term complex in degrees $-1, 0, 1$

$$\eta^\bullet = [\eta^{-1} \longrightarrow \eta^0 \longrightarrow \eta^1]$$

of objects of $\text{fin}_{\mathcal{O}_K}^p$ with η^1 étale, and where ν is a homomorphism

$$\nu : \eta^1(1) \longrightarrow \eta^{-1}.$$

Let n be a power of p large enough to kill η^1 . Recall that we construct a torsion $\mathbb{Z}_p[\Gamma_K]$ -module from this data, which we call the “generic fiber” of the pair (η^\bullet, ν) , as follows. Tensor the short exact sequence

$$\theta_{n,K}^\varpi \quad 0 \longrightarrow \mu_{n,K} \longrightarrow E_{\varpi,K}[n] \longrightarrow \mathbb{Z}/n\mathbb{Z}_K \longrightarrow 0 \quad (3.2.14)$$

over $\mathbb{Z}/n\mathbb{Z}$ with η_K^1 , and push out by ν as in

$$\begin{array}{ccccccc} \theta_{n,K}^\varpi \otimes \eta_K^1 & 0 & \longrightarrow & \eta_K^1(1) & \longrightarrow & \eta_K^1 \otimes E_{q,K}[n] & \longrightarrow & \eta_K^1 & \longrightarrow & 0 \\ & & & \downarrow \nu & & \downarrow & & \parallel & & \\ \nu_*(\theta_{n,K}^\varpi \otimes \eta_K^1) & 0 & \longrightarrow & \eta_K^{-1} & \longrightarrow & \nu_*\eta_K^0 & \longrightarrow & \eta_K^1 & \longrightarrow & 0 \end{array}.$$

The generic fiber $(\eta^\bullet, \nu)_K$ is the Baer sum in $\text{Ext}_{\text{fin}_K^p}(\eta_K^1, \eta_K^{-1})$

$$(\eta^\bullet, \nu)_K := \eta_K^\bullet +_{\text{B}} \nu_*(\theta_{n,K}^\varpi \otimes \eta_K^1). \quad (3.2.15)$$

By Theorem 3.2.11, all of the terms of η_K^\bullet are 1-crystalline as $\mathbb{Z}_p[\Gamma_K]$ -modules. By Example 2.1.8, η_K^1 is unramified, which by Example 3.1.12 means that η_K^1 is 0-crystalline. Since $E_{\varpi,K}[n]$ is 1-semi-stable, it follows from Proposition 3.1.10 that $\eta_K^1 \otimes E_{q,K}[n]$ is 1-semi-stable. Therefore $(\eta_K^1 \otimes E_{q,K}[n]) \oplus \eta_K^{-1}$ is 1-semi-stable. Since $\nu_*\eta_K^0$, being the pushout of $\eta_K^1(1) \rightarrow \eta_K^1 \otimes E_{q,K}[n]$ by ν , is a quotient of $(\eta_K^1 \otimes E_{q,K}[n]) \oplus \eta_K^{-1}$, we see that $\nu_*\eta_K^0$ is 1-semi-stable. By Proposition 3.2.10, the degree-zero term $(\eta^\bullet, \nu)_K^0$ in $(\eta^\bullet, \nu)_K$ is 1-semi-stable. Proposition 3.2.10 also shows that $(\eta^\bullet, \nu)_K^0$ is 1-crystalline if and only if $\nu_*\eta_K^0$ is 1-crystalline. \square

3.3 Fully faithfulness theorems

Let $L \in \text{Rep}_{\text{free}}^{\text{st},r}(\Gamma_K)$ and assume L is not an object of $\text{Rep}_{\text{free}}^{\text{crys},r}(\Gamma_K)$. By Theorem 3.2.4, we know that $L/p^m L$ is not an object of $\text{Rep}_{\text{tors}}^{\text{crys},r}(\Gamma_K)$ for all m sufficiently large. However, it can happen that, for finitely many values of m , the $\mathbb{Z}_p[\Gamma_K]$ -module $L/p^m L$ is r -crystalline. This is a phenomenon of central importance to our study of Néron component groups in Chapter 4. Example 3.3.5 illustrates this phenomenon.

The crystalline condition has a special relationship with the subgroup $\Gamma_{\tilde{\varpi}}$ defined in Situation 0.1.2. Let $\text{Rep}_{\text{free}}(\Gamma_{\tilde{\varpi}})$ denote the category of finite free \mathbb{Z}_p -modules equipped with a continuous action of $\Gamma_{\tilde{\varpi}}$.

Proposition 3.3.1 ([33] [2]; [8] 3.4.3 in the case $r = 1$).

For any $r \in \mathbb{Z}_{\geq 0}$, the restriction functor

$$\text{Res}_{\Gamma_{\tilde{\varpi}}}^{\Gamma_K} : \text{Rep}_{\text{free}}^{\text{crys},r}(\Gamma_K) \longrightarrow \text{Rep}_{\text{free}}(\Gamma_{\tilde{\varpi}})$$

is fully faithful.

Remark 3.3.2. Note that the construction of the category $\text{Rep}_{\text{free}}^{\text{crys},r}(\Gamma_K)$ (like the construction of B_{crys}) does not depend on a choice of uniformizer; our choice of pseudo-uniformizer $\tilde{\varpi}$ of \mathcal{O}_C^b with 0-th component ϖ plays no special role.

In practice, Proposition 3.3.1 means that one can detect whether a \mathbb{Q}_p -linear map of r -crystalline representations is Γ_K -equivariant by testing whether it is

equivariant for the actions of the subgroup $\Gamma_{\tilde{\omega}} \subset \Gamma_K$. In general, the analogous statement for torsion $\mathbb{Z}_p[\Gamma_K]$ -modules fails for subtle reasons which are related to the bound in Raynaud's theorem (Proposition 2.2.1). Recall that e_K is the ramification index of K/\mathbb{Q}_p .

Proposition 3.3.3 ([43] 1.2; others in special cases, e.g. $r = 1$ $p > 2$ case by [8] 3.4.3. See the introduction to [43] for a detailed history of this result.).

Let $r \in \mathbb{Z}_{\geq 0}$. Suppose $e_K(r - 1) < p - 1$. Then the restriction functor

$$\mathrm{Res}_{\Gamma_{\tilde{\omega}}}^{\Gamma_K} : \mathrm{Rep}_{\mathrm{tors}}^{\mathrm{crys}, r}(\Gamma_K) \longrightarrow \mathrm{Rep}_{\mathrm{tors}}(\Gamma_{\tilde{\omega}})$$

is fully faithful.

Example 3.3.4. The Galois closure of $K(\tilde{\omega}^{(m)})$ in \overline{K} is $K(\tilde{\omega}^{(m)}, \tilde{\epsilon}^{(m)})$. The group $\Gamma_{\tilde{\omega}}$ fixes $\tilde{\omega}^{(m)}$ but, since K/\mathbb{Q}_p is finitely ramified, the group $\Gamma_{\tilde{\omega}}$ does not fix $\tilde{\epsilon}^{(m)}$ for $m \gg 0$. This shows that $\Gamma_{\tilde{\omega}}$ remembers ‘most’ of the action of the p -adic cyclotomic character on $\Gamma_{\mathbb{Q}_p}$. Since K may contain finitely many of the roots of unity $\tilde{\epsilon}^{(m)}$, the mod p^m -cyclotomic character may be trivial. However, if the mod p^m -cyclotomic is non-trivial, then its restriction to $\Gamma_{\tilde{\omega}}$ will be nontrivial.

Example 3.3.5. Let $m \in \mathbb{Z}_{\geq 1}$. Consider a Tate curve $E_{q,K}$. Since $E_{q,K}$ is not 1-crystalline, Theorem 3.2.4 shows that, if m is sufficiently large, $E_{q,K}[p^m]$ is not 1-crystalline. It can happen, however, that $E_{q,K}[p^m]$ is 1-crystalline for finitely many values of m . To see this, recall that for any nonzero $z \in \mathcal{O}_K$ we may define a 1-cocycle $c_{z,p^m} : \Gamma_K \rightarrow \mathbb{Z}/p^m\mathbb{Z}$ by choosing a sequence $\tilde{z} \in A_{\mathrm{inf}}$ with $\tilde{z}^0 = z$ and sending $\sigma \in \Gamma_K$ to the integer mod p^m satisfying

$$\sigma(\tilde{z}^{(m)}) = (\tilde{\epsilon}^{(m)})^{c_{z,p^m}(\sigma)} \tilde{z}^{(m)},$$

where $\tilde{\epsilon}$ is the element defined in Situation 0.1.2. Kummer theory asserts that $z \mapsto c_{z,p^m}$ gives an isomorphism of groups

$$K^\times / K^{\times p^m} \xrightarrow{\sim} H^1(\Gamma_K, \mathbb{Z}/p^m\mathbb{Z}(1)). \quad (3.3.6)$$

Here, the group structure on $K^\times / K^{\times p^m}$ is the obvious one, and the group structure on $H^1(\Gamma_K, \mathbb{Z}/p^m\mathbb{Z}(1))$ comes from addition of cocycles. As we saw in Example 3.1.17, $E_{q,K}$ is attached to the cocycle c_{q,p^m} . Write $q = u\varpi^v$ with $u \in \mathcal{O}_K^\times$ and $v = v_K(q) \in \mathbb{Z}_{\geq 1}$. Since (3.3.6) is a homomorphism we have an

equality

$$c_{q,p^m} = c_{u,p^m} + v c_{\varpi,p^m} : \Gamma_K \longrightarrow \mathbb{Z}/p^m\mathbb{Z}.$$

It is well known (see [9] 9.2.8, and take the reduction) that the cocycle c_{u,p^m} give a 1-crystalline class in $\text{Ext}(\mathbb{Z}/p^m\mathbb{Z}, \mathbb{Z}/p^m\mathbb{Z}(1))$. Therefore, if $p^m|v$, then c_{q,p^m} gives a 1-crystalline extension. We will now relate this fact to the size of the Néron component group of $E_{q,K}$. We find it helpful to rephrase the above argument along the way.

Choose $\tilde{q}, \tilde{\varpi}^v \in A_{\text{inf}}$, and define $\tilde{u} \in A_{\text{inf}}$ by $\tilde{u}^{(m)} = \tilde{q}^{(m)}/\tilde{\varpi}^{v(m)}$. Since ϖ is a uniformizer of \mathcal{O}_K , there are no roots of ϖ in K , so Γ_K acts non-trivially on every p^m -th root of ϖ . However, it can happen that K contains a p^m -th root of ϖ^v ; this happens if and only if $p^m|v$. Recalling that the Néron component group of $E_{q,K}$ is cyclic of order $v_K(q)$, we have

$$c_{\varpi^v,p^m} = 0 \quad \text{if and only if} \quad m \leq v_p(v_K(q)) = \#\Phi_{E_{q,K}}[p^\infty]. \quad (3.3.7)$$

For $\sigma \in \Gamma_K$, we have

$$\begin{aligned} (\tilde{\epsilon}^{(m)})^{c_{q,p^m}(\sigma)} \tilde{q}^{(m)} &= \sigma(\tilde{q}^{(m)}) \\ &= \sigma(\tilde{u}^{(m)}) \sigma(\tilde{\varpi}^{v(m)}) \\ &= (\tilde{\epsilon}^{(m)})^{c_{u,p^m}(\sigma)} (\tilde{\epsilon}^{(m)})^{c_{\varpi^v,p^m}(\sigma)} \tilde{q}^{(m)}. \end{aligned}$$

By (3.3.7),

$$c_{q,p^m} = c_{u,p^m} \quad \text{if and only if} \quad m \leq \#\Phi_{E_{q,K}}[p^\infty]. \quad (3.3.8)$$

As we described above, the rank-2 $\mathbb{Z}/p^m\mathbb{Z}$ -module $E(c_{u,p^m})$ is 1-crystalline. Hence we obtain that

$$E_{q,K}[p^m] \quad \text{is 1-crystalline if} \quad m \leq \#\Phi_{E_{q,K}}[p^\infty].$$

In fact, by Proposition 3.3.3, this is an equivalence. To see this, note that when $r = 1$ then the condition $e_K(r-1) < p-1$ is always satisfied, even for $p = 2$. By definition, the elements of $\Gamma_{\tilde{\varpi}}$ all fix $\tilde{\varpi}^v$, so $c_{\tilde{\varpi}^v,p^m}(\sigma) = 0$ for all $\sigma \in \Gamma_{\tilde{\varpi}}$, so

$$c_{q,p^m}(\sigma) = c_{u,p^m}(\sigma) \quad (\sigma \in \Gamma_{\tilde{\varpi}^v}).$$

This shows that there is an isomorphism of $\mathbb{Z}_p[\Gamma_{\bar{\omega}}]$ -modules

$$E_{q,K}[p^m]|_{\Gamma_{\bar{\omega}}} \simeq E(c_{u,p^m})|_{\Gamma_{\bar{\omega}}}.$$

If $E_{q,K}[p^m]$ is 1-crystalline then Proposition 3.3.3 implies that in fact there is an isomorphism of $\mathbb{Z}_p[\Gamma_K]$ -modules

$$E_{q,K}[p^m] \simeq E(c_{u,p^m}).$$

In particular, we have the following: if q has a v -th root in K , which is to say that u has a v -th root in K , then 1-crystallinity of $E_{q,K}[p^m]$ is equivalent to splitness of $E_{q,K}[p^m]$ as an extension of $\mathbb{Z}/p^m\mathbb{Z}$ by $\mathbb{Z}/p^m\mathbb{Z}(1)$.

3.4 Crys_r and its continuous derived functors

Following Marshall's thesis [39], we define functors Crys_r and construct their derived functors. As we shall see, the derived functors are not constructed in the usual way, but rather are constructed using inverse systems of torsion $\mathbb{Z}_p[\Gamma_K]$ -modules. As we noted in [39], this technique for defining derived functors has its origin in Jannsen's work [25].

We first define the maximal r -crystalline subobjects, then show that there is a functor picking out such things.

Definition 3.4.1 (Crys_r).

For each object M of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ and for $r \in \mathbb{Z}_{\geq 0}$, define $\text{Crys}_r(M)$ as the sum in M of all $C \subset M$ such that C is an object $\text{Rep}_{\text{fitype}}^{\text{crys}, r}(\Gamma_K)$.

Proposition 3.4.2.

Let M be an object of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$. Let $r \in \mathbb{Z}_{\geq 0}$.

- (i) The object $\text{Crys}_r(M)$ of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ is an object of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{crys}, r}$.
- (ii) If M is finite type and torsion with the discrete topology, then $\text{Crys}_r(M)$ is an object of $\text{Rep}_{\text{tors}}^{\text{crys}, r}(\Gamma_K)$.
- (iii) If M is an object of $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$, then $\text{Crys}_r(M)$ is an object of $\text{Rep}_{\text{free}}^{\text{crys}, r}(\Gamma_K)$.
- (iv) If M is an object of $\text{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, then $\text{Crys}_r(M)$ is an object of $\text{Rep}_{\mathbb{Q}_p}^{\text{crys}, r}(\Gamma_K)$.

Proof. We prove (i). We need only verify that the sum defining $\text{Crys}_r(M)$ is a colimit. Let $S \subset 2^M$ be the set consisting of those submodules of M that

are objects of $\text{Rep}_{\text{ftype}}^{\text{crys},r}(\Gamma_K)$ so that $\text{Crys}_r(M) = \sum_{C \in S} C \subset M$. Let \mathcal{S} be the category such that $\text{Ob } \mathcal{S} = S$ and for $C_1, C_2 \in S$,

$$\text{Hom}_{\mathcal{S}}(C_1, C_2) := \begin{cases} \text{the inclusion } C_1 \hookrightarrow C_2 & \text{if } C_1 \subset C_2 \\ \emptyset & \text{otherwise.} \end{cases}$$

Define a functor $F : \mathcal{S} \rightarrow \text{Rep}_{\text{ftype}}^{\text{crys},r}(\Gamma_K)$ by sending $C \in S$ to the same object C of $\text{Rep}_{\text{ftype}}^{\text{crys},r}(\Gamma_K)$ and sending any morphism of objects of \mathcal{S} to the corresponding inclusion of objects of $\text{Rep}_{\text{ftype}}^{\text{crys},r}(\Gamma_K)$. We claim that $\text{Crys}_r(M)$ is isomorphic to the filtered colimit $\text{colim}_{\mathcal{S}} F$. We must show that $\text{Crys}_r(M)$ satisfies the universal property characterizing $\text{colim}_{\mathcal{S}} F$. Let Z be an object of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ and let $(\varphi_C : C \rightarrow Z)_{C \in S}$ be a compatible collection of morphisms. Then the φ_C determine a unique morphism $\varphi : \text{Crys}_r(M) \rightarrow Z$ compatible with the inclusions $C \subset \text{Crys}_r(M)$. To see this, let $x \in \text{Crys}_r(M)$. Then x lies in some finite sum (in M) of objects of $\text{Rep}_{\text{ftype}}^{\text{crys},r}(\Gamma_K)$, which is again an object $C \in S$. Define $\varphi(x) = \varphi_C(x)$. We claim this is well-defined. Indeed, suppose that $x \in C' \subset M$ for some $C' \in S$ which is possibly distinct from C . Then $C' \cap C$ is in S . By compatibility of the φ_C 's, $\varphi_C(x) = \varphi_{C \cap C'}(x) = \varphi_{C'}(x)$. This has shown that φ is well-defined. Now let $x, y \in M$. Then there is some $C \in S$ such that $x + y \in C$, so $\varphi(x + y) = \varphi_C(x + y) = \varphi_C(x) + \varphi_C(y) = \varphi(x) + \varphi(y)$. We see that φ is a homomorphism. It follows that $\text{Crys}_r(M) \approx \text{colim}_{\mathcal{S}} F$, as claimed. This has shown that $\text{Crys}_r(M)$ is an object of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{crys},r}$.

We prove (ii) and (iii). When M is finite type, the \mathbb{Z}_p -submodule $\text{Crys}_r(M) \subset M$ is finite type and torsion with the discrete topology with \mathbb{Z}_p -rank bounded by that of M . It is therefore equal to one of the summands defining $\text{Crys}_r(M)$ as a sum of finite type submodules of M . If M is torsion (resp., free), then $\text{Crys}_r(M)$ is torsion (resp., free). We are then done by Lemma 3.2.6. \square

Example 3.4.3. We look at the example of a Tate curve. Recall that $L = T_p E_{q,K}$ has a basis x_1, x_2 where the action of Γ_K has the matrix form $\begin{pmatrix} \chi & c_q \\ & 1 \end{pmatrix}$, where c_q is the limit of the cocycles c_{q,p^m} described in Example 3.1.17. It is obvious from the explicit matrix form of the action that the only proper Γ_K -stable subspace of $L[1/p]$ is the copy of $\mathbb{Q}_p(1)$ spanned by x_1 , which is crystalline. Therefore, any 1-crystalline Γ_K -stable subspace of L is a copy of $\mathbb{Z}_p(1)$ of the form $p^m \mathbb{Z}_p x_1$ for some $m \geq 0$. Since L is not 1-crystalline, we see that

$$\text{Crys}_1(L) = \mathbb{Z}_p x_1 \subset \mathbb{Q}_p x_1 = \text{Crys}_1(L[1/p]).$$

In Example 3.3.5 we saw that the $L/p^m L$ can be 1-crystalline. In fact, letting $c = \#\Phi_{E_{q,K}}[p^\infty]$, it follows from Example 3.3.5 that

$$\text{Crys}_1(L/p^m L) = L/p^m L \quad \text{if } m \leq c$$

and

$$\text{Crys}_1(L/p^m L) \supset \mathbb{Z}/p^m \mathbb{Z}x_1 + p^{m-c}(L/p^m L) \quad \text{if } m > c.$$

It will follow from Proposition 4.2.22 that this containment is an equality.

Proposition 3.4.4.

Let $r \in \mathbb{Z}_{\geq 0}$. There exists a functor $\text{Crys}_r : \text{Mod}_{\mathbb{Z}_p, \Gamma_K} \rightarrow \text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{crys}, r}$ that sends an object M of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ to $\text{Crys}_r(M)$ as defined in Definition 3.4.1. This functor is right adjoint to the forgetful functor $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{crys}, r} \rightarrow \text{Mod}_{\mathbb{Z}_p, \Gamma_K}$, and hence it is left exact and commutes with limits.

Proof. We have shown in Proposition 3.4.2 that Crys_r is well-defined on objects. Now, for any morphism $f : M \rightarrow M'$ of objects of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$, define $\text{Crys}_r(f) := f|_{\text{Crys}_r(M)}$. For this to be well-defined, we need that $f|_{\text{Crys}_r(M)}$ factors through $\text{Crys}_r(M')$. Writing $\text{Crys}_r(M) = \text{colim } M_i$ with $M_i \in \text{Rep}_{\text{ftype}}^{\text{crys}, r}(\Gamma_K)$, each f_{M_i} factors through its image, which is an object of $\text{Rep}_{\text{ftype}}^{\text{crys}, r}(\Gamma_K)$ contained in M' . By the universal property of the colimit, we conclude that $f|_{\text{Crys}_r(M)}$ factors through $\text{Crys}_r(M')$.

We check the claimed adjointness property. Let F be the relevant forgetful functor. We must show that there are natural bijections

$$\text{Hom}_{\text{Mod}_{\mathbb{Z}_p, \Gamma_K}}(F(C), M) = \text{Hom}_{\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{crys}, r}}(C, \text{Crys}_r(M))$$

for $C \in \text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{crys}, r}$ and $M \in \text{Mod}_{\mathbb{Z}_p, \Gamma_K}$, which is just to say that an equivariant homomorphism $f : C \rightarrow M$ factors through $\text{Crys}_r(M)$. Writing C as a colimit of finite type objects and arguing as above, we see that f factors through $\text{Crys}_r(M)$. The claim that a right-adjoint functor is left exact and commutes with limits is standard. \square

For $L \in \text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$, the relationship between $\text{Crys}_r(L)$ and $\text{Crys}_r(L[1/p])$ can be understood via the following basic lemma.

Lemma 3.4.5.

Let $L \in \text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$, and let $W \subset L[1/p]$ be a stable \mathbb{Q}_p -subspace. Then $W \cap L$ is a lattice in W .

Proof. Let w_1, \dots, w_d be a basis for W and let e_1, \dots, e_n be a basis for L . Write each w_i as $w_i = \sum_{j=0}^n c_{ij} e_j$ with $c_{ij} \in \mathbb{Q}_p$. Choose m such that $p^m c_{ij} \in \mathbb{Z}_p$ for all i, j . Then the $p^m w_1, \dots, p^m w_d$, which still form a basis for W , all lie in L . This proves the claim. \square

Remark 3.4.6. Applying Lemma 3.4.5 to $W = \text{Crys}_r(L[1/p])$ we find that

$$\text{Crys}_r(L)[1/p] = \text{Crys}_r(L[1/p]). \quad (3.4.7)$$

One could take this as a definition.

We wish to study the derived functors of Crys_r , but to define these using the usual formalism of Grothendieck we must restrict Crys_r to some category with enough injectives. Let us illustrate one potential issue. We might wish to restrict to $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$, but the category $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$ does not have enough injectives. To see this, suppose that $I \in \text{Rep}_{\text{free}}^{\text{crys}, r}(\Gamma_K)$. Consider the diagram

$$\begin{array}{ccc} I & \xrightarrow{p} & I \\ \parallel & \swarrow \psi? & \\ I & & \end{array}$$

If I is injective, then there is a morphism $\psi : I \rightarrow I$ such that $\psi(px) = x$ for all $x \in I$. But then $x = p\psi(x)$, so I is p -divisible. This is not possible for a finite free \mathbb{Z}_p -module of positive rank. A natural divisible avatar for L that preserves the information of the quotients of L is the Pontryagin dual $L \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

Proposition 3.4.8.

Let $L \in \text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$. Then $L \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is naturally an object of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$. The natural morphism

$$\text{colim}_{m \geq 1} \text{Crys}_r(L/p^m L) \longrightarrow \text{Crys}_r(L \otimes \mathbb{Q}_p/\mathbb{Z}_p),$$

where the colimit is with respect to maps $L/p^m L \rightarrow L/p^{m+1} L$ given by multiplication by p , is an isomorphism.

Proof. By definition of Crys_r , $\text{Crys}_r(L \otimes \mathbb{Q}_p/\mathbb{Z}_p) = \text{colim}_{\mathcal{I}} F$ with $F(i)$ an object of $\text{Rep}_{\text{fitype}}^{\text{crys}, r}(\Gamma_K)$ for each $i \in \mathcal{I}$. Since $L \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is torsion and each $F(i)$ is finite type and r -crystalline, we see that for each $i \in \mathcal{I}$ there is some $m(i) \in \mathbb{Z}_{\geq 1}$ such that $F(i) \rightarrow \text{Crys}_r(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ factors through $\text{Crys}_r(L/p^{m(i)} L)$. This induces an inverse to the map $\text{colim}_{m \geq 1} \text{Crys}_r(L/p^m L) \longrightarrow \text{Crys}_r(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)$. \square

Proposition 3.4.9.

The categories $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$, $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$ and $\text{Inv Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$ (defined via Definition 3.2.1) each have enough injectives.

Proof. We essentially follow the proof given by Marshall. The claim for $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ follows from [51, Tag 04JE], which is the stronger result that this category admits injective hulls. Let M be an object of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$. Let $M \rightarrow I$ be an injection of M into an injective object of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$. Since M is torsion, this injection factors through I_{tors} , so we need only show that I_{tors} is an injective object in the category $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$. Suppose we have morphisms $f : T \rightarrow I_{\text{tors}}$ and $g : T \rightarrow T'$ in $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$. Then f and g are also morphisms in the category $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$, so we get a morphism $f' : T' \rightarrow I$ such that $f = f' \circ g : T \rightarrow I$. But T' is torsion, so f' factors through I_{tors} . This shows that I_{tors} is injective, finishing the proof of the claim for $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$. The claim for $\text{Inv Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$ is immediate from what we have. \square

The following definitions will let us define, for each $L \in \text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$, groups that can reasonably (because Theorem 3.4.12 holds) be called $R^i \text{Crys}_r(L)$. We note again that this method of defining the derived functors $R^i \text{Crys}_r(L)$ is due to Marshall and Kim-Marshall in [39] and [31], who credit Jannsen for introducing the style of definition in [25].

Definition 3.4.10.

Let M be an object of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$. Define the following objects of $\text{Inv Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$.

- (i) $T(M) := (T_m)_{m \geq 1}$ with $T_m = M[p^m]$ and transition maps $T_m \rightarrow T_{m'}$, for $m' \leq m$, given by multiplication by $p^{m-m'}$.
- (ii) $V(M) := (V_m)_{m \geq 1}$ with $V_m = M$ and transition maps $V_m \rightarrow V_{m'}$, for $m' \leq m$, given by multiplication by $p^{m-m'}$.
- (iii) $\underline{M} := (M_m)_{m \geq 1}$ the constant system with $M_m = M$ and each transition morphism is the identity.

Proposition 3.4.11.

Let $L \in \text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$, $j \geq 0$, $r \geq 0$. Then $R^i \text{Crys}_r(L[1/p])$ has no p -torsion.

Proof. Every element of $V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is sequence of the form $(a_j/p^{m+j})_{j \geq 0}$, where $m \in \mathbb{Z}_{\geq 1}$, $a_j \in L$ for all $j \in \mathbb{Z}_{\geq 0}$, and $a_{j+1} \equiv a_j$ modulo p^{m+j} . We claim that the morphism $p : V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is an isomorphism. It is surjective because $(a_j/p^{m+j})_{j \geq 0} = p(a_j/p^{m+j+1})_{j \geq 0}$. Suppose that

$p(a_j/p^{m+j})_{j \geq 0} = (0)_{\geq 0}$. Then $pa_j/p^{m+j} = 0 \in L \otimes \mathbb{Q}_p/\mathbb{Z}_p$, i.e., $p^{m+j-1} \mid a_j$. But $a_j \equiv a_{j-1}$ modulo p^{m+j-1} , so $(a_j/p^{m+j})_{j \geq 0} = (0)_{\geq 0}$. This has shown that multiplication by p on $V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is an isomorphism. Taking the the long exact sequence associated to this isomorphism using the usual derived functor cohomology, we find that p is an isomorphism on $R^i \text{Crys}_r(V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p))$ for each $j \geq 0$. In particular, $R^i \text{Crys}_r(L[1/p])$ has no p -torsion. \square

Theorem 3.4.12 ([39], Proposition 3.2).

Let L be an object of $\text{Rep}_{\mathbb{Z}_p}(\Gamma_K)$ and let $r \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned} \lim \text{Crys}_r(T(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)) &= \text{Crys}_r(L) \\ \lim \text{Crys}_r(V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)) &= \text{Crys}_r(L[1/p]) \\ \lim \text{Crys}_r(\underline{L \otimes \mathbb{Q}_p/\mathbb{Z}_p}) &= \text{Crys}_r(L \otimes \mathbb{Q}_p/\mathbb{Z}_p). \end{aligned}$$

For $i \geq 1$, define

$$\begin{aligned} R^i \text{Crys}_r(L) &:= R^i(\lim \text{Crys}_r)(T(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)) \\ R^i \text{Crys}_r(L[1/p]) &:= R^i(\lim \text{Crys}_r)(V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)) \\ R^i \text{Crys}_r(L^D) &:= R^i(\lim \text{Crys}_r)(\underline{L \otimes \mathbb{Q}_p/\mathbb{Z}_p}). \end{aligned}$$

The morphisms

$$\pi_m = \text{id} \otimes p^m : L \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow L \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

defined for $m \in \mathbb{Z}_{\geq 1}$ give rise to a short exact sequence of objects of $\text{Inv Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$

$$0 \longrightarrow T(L \otimes \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \underline{L \otimes \mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow 0, \quad (3.4.13)$$

hence a long exact sequence in $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{crys}, r}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Crys}_r(L) & \longrightarrow & \text{Crys}_r(L[1/p]) & \longrightarrow & \text{Crys}_r(L \otimes \mathbb{Q}_p/\mathbb{Z}_p) \\ & & & & \swarrow & & \\ & & R^1 \text{Crys}_r(L) & \longrightarrow & R^1 \text{Crys}_r(L[1/p]) & \longrightarrow & R^1 \text{Crys}_r(L \otimes \mathbb{Q}_p/\mathbb{Z}_p) \\ & & & & \swarrow & & \\ & & R^2 \text{Crys}_r(L) & \longrightarrow & \dots & & \end{array} \quad (3.4.14)$$

Proof. Since the transition morphism in the system $\underline{L \otimes \mathbb{Q}_p/\mathbb{Z}_p}$ are all identity maps, it we clearly have

$$\lim \text{Crys}_r \left(\underline{L \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right) = \text{Crys}_r(L \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

By Proposition 3.4.4 the functor Crys_r preserves limits, so

$$\lim \text{Crys}_r(T(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)) = \lim_{m \geq 1} \text{Crys}_r(L/p^m L) = \text{Crys}_r(L).$$

Consider the modules

$$\lim \text{Crys}_r(V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)) = \lim_{x \mapsto px} \text{Crys}_r(L \otimes \mathbb{Q}_p/\mathbb{Z}_p) = \text{Crys}_r(\lim_{x \mapsto px} (L \otimes \mathbb{Q}_p/\mathbb{Z}_p)).$$

An element of $\lim_{x \mapsto px} (L \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is a sequence $(a_j/p^m)_{j \geq 1}$ with each $a_j \in L$ such that

$$p \frac{a_{j+1}}{p^{m_{j+1}}} = \frac{a_j}{p^{m_j}} \in L \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

Then $m_{j+1} = m_j + 1 = \dots = m_1 + j$ and the a_j satisfy $a_{j+1} - a_j \in p^{m_j} L$. Then $\lim a_j$ exists in L , and so we have a map

$$\begin{aligned} \lim_{x \mapsto px} (L \otimes \mathbb{Q}_p/\mathbb{Z}_p) &\longrightarrow L[1/p] \\ \left(\frac{a_j}{p^{m+j}} \right)_{j \geq 1} &\longmapsto \frac{\lim a_j}{p^m} \end{aligned}$$

which is clearly an isomorphism. Hence we have

$$\lim \text{Crys}_r(V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)) = \text{Crys}_r(L[1/p]).$$

We now construct the short exact sequence (3.4.13). Let $i = (i_m)_{m \geq 1} : T(L \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ be the morphism of inverse systems obtained by taking $i_m : p^{-m}L/L = (L \otimes \mathbb{Q}_p/\mathbb{Z}_p)[p^m] \rightarrow L \otimes \mathbb{Q}_p/\mathbb{Z}_p$ to be the inclusion. Represent any nonzero element x of $L \otimes \mathbb{Q}_p/\mathbb{Z}_p$ as a/p^{m+j} with $a \in L$ and $j \geq 1$.

Writing $a = \sum_{m \geq 0} a_m p^m$ with $a_m \in L$, we have

$$\begin{aligned} x = \frac{a}{p^{m+j}} &= \frac{a_0 + \cdots + a_{j-1} p^{j-1} + a_j p^j + \cdots + a_{n+j} p^{m+j}}{p^{m+j}} \\ &= \frac{a_0 + \cdots + a_{j-1} p^{j-1}}{p^{m+j}} + \frac{a_j + \cdots + a_{n+j} p^m}{p^m} \\ &\equiv \frac{a_0 + \cdots + a_{j-1} p^{j-1}}{p^{m+j}} \pmod{\frac{1}{p^m} L/L}. \end{aligned}$$

We see that the cokernel of i_m is isomorphic to $p^{-m}(L \otimes \mathbb{Q}_p/\mathbb{Z}_p) \subset L \otimes \mathbb{Q}_p/\mathbb{Z}_p$. But this subgroup is isomorphic to $L \otimes \mathbb{Q}_p/\mathbb{Z}_p$, so we have a short exact sequence

$$0 \longrightarrow \frac{1}{p^m} L/L \xrightarrow{[p^m] \circ i_m} L \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\pi_m} L \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

Explicitly, writing $x \in L \otimes \mathbb{Q}_p/\mathbb{Z}_p$ as before, the map $\pi_m = \text{id} \otimes p^m$ can be described on representatives of equivalence classes by

$$\pi_m(x) = \pi_m\left(\frac{a}{p^{m+j}}\right) = \frac{a_0 + \cdots + a_{j-1} p^{j-1}}{p^j}.$$

We check that the short exact sequences involving the π_m give a short exact sequence of inverse systems. The transition maps in the systems $T(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ and $V(L \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ are both multiplication by p , so they commute with the maps $[p^m] \circ i_m$. We must also check that the squares

$$\begin{array}{ccc} L \otimes \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\pi_{m+1}} & L \otimes \mathbb{Q}_p/\mathbb{Z}_p \\ p \downarrow & & \parallel \\ L \otimes \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\pi_m} & L \otimes \mathbb{Q}_p/\mathbb{Z}_p \end{array}$$

commute. This follows from the congruence

$$\pi_m\left(p \frac{a}{p^{m+1+j}}\right) = \pi_m\left(\frac{a}{p^{m+j}}\right) \equiv \frac{a_0 + \cdots + a_{j-1} p^{j-1}}{p^{m+j}} \equiv \pi_{m+1}\left(\frac{a}{p^{m+1+j}}\right) \pmod{\frac{1}{p^m} L/L}.$$

We have constructed the short exact sequence (3.4.13).

As we saw in Proposition 3.4.9, the category $\text{Inv Mod}_{\mathbb{Z}_p, \Gamma_K}^{\text{tors}}$ has enough injectives. With this, the existence of the long exact sequence is standard. \square

Chapter 4

Maximal 1-crystalline submodules and Néron component groups in the semi-stable case

Let K be a p -adic field, let A_K be a semi-stable abelian K -variety, and let $m \in \mathbb{Z}_{\geq 1}$. We have seen that the finite $\mathbb{Z}_p[\Gamma_K]$ -modules $A_K[p^m]$ are torsion 1-semi-stable in the sense that they are objects of the category $\text{Rep}_{\text{free}}^{\text{st},1}(\Gamma_K)$ defined in Definition 3.2.2. It is sometimes the case that $A_K[p^m]$ belongs to a smaller category, the category $\text{Rep}_{\text{free}}^{\text{crys},1}(\Gamma_K)$ of torsion 1-crystalline $\mathbb{Z}_p[\Gamma_K]$ -modules. In general, we have a functor Crys_1 (defined in Proposition 3.4.4) that produces the maximal 1-crystalline $\mathbb{Z}_p[\Gamma_K]$ -submodule $\text{Crys}_1(A_K[p^m]) \subset A_K[p^m]$ whose homological properties were developed in Section 3.4. In this section, we undertake a careful study of $\text{Crys}_1(A_K[p^m])$. This allows us prove a formula for the p -primary part of the Néron component group Φ_{A_K} of A_K ; in Theorem 4.3.7 we construct an isomorphism of unramified $\mathbb{Z}_p[\Gamma_K]$ -modules

$$\Phi_{A_K,\kappa}[p^\infty] \simeq R^1 \text{Crys}_1(T_p A_K)_{\text{tors}}. \quad (4.0.1)$$

In Chapter 0 write more on the history of this formula, its proof under restrictions on K in [31], and its older l -adic analogue ($l \neq p$) proved in [19].

We outline the strategy of proof. Our approach stems from Kisin's result (see Theorem 3.2.11) that a torsion $\mathbb{Z}_p[\Gamma_K]$ -module is 1-crystalline if and only if it is the generic fiber of a finite flat \mathcal{O}_K -group. By work of Fontaine, one knows that crystalline \mathbb{Q}_p -representations of Γ_K are classified by filtered φ -modules, and that semi-stable \mathbb{Q}_p -representations are classified by filtered φ -modules with the additional structure of a monodromy operator. One might hope, then, that torsion 1-semi-stable representations can be classified by finite flat \mathcal{O}_K -groups with additional monodromy structure. This idea was pursued in some depth by Kato in his theory of log finite flat group schemes. We refer the reader to the introduction of Section 2.4 for more on the history of log finite flat group schemes.

In Proposition 2.5.10, we showed that the object $(\mathcal{Q}_\varphi^1[p^m], N_{p^m})$ of $\text{fin}_{\mathcal{O}_K}^{p,N}$ defined in Definition 2.5.5 has generic fiber isomorphic to $A_K[p^m]$. As we noted above, $\text{Crys}_1(A_K[p^m])$ is the generic fiber of a finite flat \mathcal{O}_K -group. Such a finite flat \mathcal{O}_K -group need not be unique or have good functorial properties. Nonetheless, we construct such an object in a fairly natural way from $(\mathcal{Q}_\varphi^1[p^m], N_{p^m})$ by killing monodromy as in Remark 2.4.5. In this way we obtain a finite flat \mathcal{O}_K group that we call ${}^*\mathcal{Q}_\varphi^1[p^m]$. We stress that, while it is clear ${}^*\mathcal{Q}_\varphi^1[p^m]$ gives a large 1-crystalline subobject of $A_K[p^m]$, it is not clear that ${}^*\mathcal{Q}_\varphi^1[p^m]$ is equal to all of $\text{Crys}_1(A_K[p^m])$. To show this, we appeal to the torsion fully faithfulness result of Proposition 3.3.3.

Our application of Proposition 3.3.3 to prove that

$${}^*\mathcal{Q}_\varphi^1[p^m]_K \simeq \text{Crys}_1(A_K[p^m]) \tag{4.0.2}$$

involves a decomposition of $V = A_K[p^m]$ into a 1-semi-stable piece N_*W built from the torsion in a Tate curve and a 1-crystalline piece W ; that this decomposition exists is (essentially) the content of the statement that $A_K[p^m]$ is the generic fiber of an object of $\text{fin}_{\mathcal{O}_K}^{p,N}$, which is defined to be a certain Baer sum. Our strategy is to understand $\text{Crys}_1(N_*W)$. Since N_*W has an explicit construction involving the monodromy homomorphism $N = N_{p^m}$, the connected-étale sequence of $\mathcal{Q}_\varphi^1[p^m]$, and the torsion subgroup $E_{\varphi,K}[p^m]$ in the Tate curve with parameter φ , we are able to understand $\text{Crys}_1(N_*W)$ with sufficient precision to describe $\text{Crys}_1(V)$. Ultimately, this rests on a simple lemma, Lemma 4.2.12, pertaining to the the Kummer cocycle attached to $E_{\varphi,K}[p^m]$.

With (4.0.2) established, it is straightforward to prove that the Raynaud extension \tilde{A} , defined as in Proposition 1.4.1, satisfies

$$T_p \tilde{A}_K = \text{Crys}_1(T_p A_K). \quad (4.0.3)$$

If A_K has bad reduction, $\text{Crys}_1(T_p A_K)/p^m \text{Crys}_1(T_p A_K)$ may be strictly smaller than $\text{Crys}_1(A_K[p^m])$. In fact, the difference in size between these two groups is directly related to size of the p -primary part of the Néron component group Φ_{A_K} attached to A_K ; by a direct analysis of ${}^* \mathcal{Q}_{\varpi}^1[p^m]_K$ using results of [19] relating to the monodromy pairing, we show that there are isomorphisms of unramified $\mathbb{Z}_p[\Gamma_K]$ -modules

$$\frac{{}^* \mathcal{Q}_{\varpi}^1[p^m]_K}{\tilde{A}_K[p^m]} \simeq \frac{\text{Crys}_1(T_p A_K \otimes \mathbb{Z}/p^m \mathbb{Z})}{\text{Crys}_1(T_p A_K) \otimes \mathbb{Z}/p^m \mathbb{Z}} \simeq \Phi_{A_K, \kappa}[p^m]. \quad (4.0.4)$$

The isomorphism on the right was constructed in [39] and [31] under the assumption that $p > 2$ and K/\mathbb{Q}_p is unramified. This was done by constructing a finite flat \mathcal{O}_K -group with generic fiber $\text{Crys}_1(A_K[p^m])$. This construction makes use of the strong result Proposition 2.2.1 of Raynaud that requires that the ramification index e_K of K/\mathbb{Q}_p satisfies $e_K < p - 1$. Our construction, which removes the restriction $e_K < p - 1$, gives a new description of $\text{Crys}_1(A_K[p^m])$ as a submodule of $A_K[p^m]$. In a sense, we replace Raynaud's result on torsion 1-crystalline $\mathbb{Z}_p[\Gamma_K]$ -modules with the torsion fully faithfulness result described earlier.

With (4.0.4), we are able to follow the strategy of [39] and [31] to construct an isomorphism as in (4.0.1) using the long exact sequence constructed in Theorem 3.4.12.

Conventions

In this chapter, assume the notation of Situation 1.6.11 in the case where A_K has semi-stable reduction. We sometimes reiterate the choices made in Situation 1.6.11 to emphasize the dependence of our constructions on these choices. We will be interested in understanding the modules $\text{Crys}_1(A_K[n])$ as defined in Definition 3.4.1. It should be noted here that this module is torsion 1-crystalline as a Γ_K -module, but the dependence on K is suppressed. For instance, we will sometimes write “is 1-crystalline” to mean “is an object of $\text{Rep}_{\text{free}}^{\text{crys}, 1}(\Gamma_K)$ ”.

4.1 Preliminaries

Let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . We have seen in Proposition 2.5.10 that $A_K[n]$ is isomorphic to the degree-zero term in the complex

$$(\text{c}\acute{e}(\mathcal{Q}_{\varpi}^1[n]), N_n)_K := \text{c}\acute{e}(\mathcal{Q}_{\varpi}^1[n])_K +_{\text{B}} N_{n,*}(\theta_{n,K}^{\varpi} \otimes \mathcal{Q}_{\varpi}^1[n]_K^{\acute{e}\text{t}}). \quad (4.1.1)$$

Let us recall what this means. Since A_K is semi-stable, it has, by Proposition 1.5.12, an associated log 1-motive \mathcal{Q}_K . With respect to our fixed uniformizer ϖ , we can attach to \mathcal{Q}_K an \mathcal{O}_K -1-motive \mathcal{Q}_{ϖ}^1 . The monodromy pairing gives a homomorphism $N_n : \mathcal{Q}_{\varpi}^1[n]^{\acute{e}\text{t}}(1) \rightarrow \mathcal{Q}_{\varpi}^1[n]^{\circ}$ of finite flat \mathcal{O}_K -groups. The pair $(\mathcal{Q}_{\varpi}^1[n], N_n)$ is an object of the category $\text{fin}_{\mathcal{O}_K}^{p,N}$, which is to say that the connected-étale sequence $\text{c}\acute{e}(\mathcal{Q}_{\varpi}^1[n])$ gives an object $(\text{c}\acute{e}(\mathcal{Q}_{\varpi}^1[n]), N_n)$ of $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^{\vee}$. The generic fiber in the sense of Definition 2.4.44 is the Baer sum in $\text{Ext}_{\text{fin}_K^p}(\mathcal{Q}_{\varpi}^1[n]_K^{\acute{e}\text{t}}, \mathcal{Q}_{\varpi}^1[n]_K^{\circ})$ of two terms. The first term is the generic fiber of the connected-étale sequence $\text{c}\acute{e}(\mathcal{Q}_{\varpi}^1[n])$. The second term is constructed from N_n as in Definition 2.4.14. By Proposition 2.5.10, there is a short exact sequence of $\mathbb{Z}_p[\Gamma_K]$ -modules

$$0 \longrightarrow \mathcal{Q}_{\varpi}^1[n]_K^{\circ} \longrightarrow A_K[n] \longrightarrow \mathcal{Q}_{\varpi}^1[n]_K^{\acute{e}\text{t}} \longrightarrow 0. \quad (4.1.2)$$

To lighten notation, define

$$\begin{aligned} V &:= A_K[n], & W &:= \mathcal{Q}_{\varpi}^1[n]_K, \\ V^{\circ} &:= \mathcal{Q}_{\varpi}^1[n]_K^{\circ}, & V^{\acute{e}\text{t}} &:= \mathcal{Q}_{\varpi}^1[n]_K^{\acute{e}\text{t}}. \end{aligned} \quad (4.1.3)$$

Then (4.1.2) becomes

$$0 \longrightarrow V^{\circ} \longrightarrow V \longrightarrow V^{\acute{e}\text{t}} \longrightarrow 0 \quad (4.1.4)$$

Using, as we did before, $(-)^0$ to denote the term in degree zero of an object of $\text{Ext}_{\text{fin}_{\mathcal{O}_K}^p}^{\vee}$, define

$$N := N_{n,K}, \quad N_*W := (N_*(\theta_{n,K}^{\varpi} \otimes \mathcal{Q}_{\varpi}^1[n]_K^{\acute{e}\text{t}}))^0. \quad (4.1.5)$$

Since W and N_*W come from objects of $\text{Ext}_{\text{fin}_K^p}(\mathcal{Q}_\varpi^1[n]_K^{\text{ét}}, \mathcal{Q}_\varpi^1[n]_K^\circ)$, they fit into short exact sequences

$$0 \longrightarrow V^\circ \longrightarrow W \longrightarrow V^{\text{ét}} \longrightarrow 0 \quad (4.1.6)$$

and

$$0 \longrightarrow V^\circ \longrightarrow N_*W \longrightarrow V^{\text{ét}} \longrightarrow 0. \quad (4.1.7)$$

Definition 4.1.8 ($\text{SubMod}(M, V^\circ)$; $(-)^{\text{ét}}$ and cé_{V° on $\text{SubMod}(V, V^\circ)$).

Let M be one of V , W , or N_*W . Define $\text{SubMod}(M, V^\circ)$ to be the subcategory of $\text{Mod}_{\mathbb{Z}_p, \Gamma_K}$ consisting of those $\mathbb{Z}_p[\Gamma_K]$ -submodules of M containing the image of the map $(V^\circ \rightarrow M)$ of (4.1.4), or (4.1.6) or (4.1.7), as the case may be, where the morphisms of $\text{SubMod}(M, V^\circ)$ the inclusions of submodules. For U in $\text{SubMod}(M, V^\circ)$ we define

$$U^{\text{ét}} := U/V^\circ. \quad (4.1.9)$$

Writing $i : U \hookrightarrow M$ for the inclusion of the submodule $U \subset M$, we regard $U^{\text{ét}}$ as a submodule of $V^{\text{ét}}$ via the injection $\bar{i} : U^{\text{ét}} \hookrightarrow V^{\text{ét}}$ induced by i . In this setup, write $\text{cé}_{V^\circ}(U)$ for the natural short exact sequence

$$\text{cé}_{V^\circ}(U) \quad 0 \longrightarrow V^\circ \longrightarrow U \longrightarrow U^{\text{ét}} \longrightarrow 0.$$

Thus, the sequences (4.1.4), (4.1.6) and (4.1.7) are $\text{cé}_{V^\circ}(V)$, $\text{cé}_{V^\circ}(W)$ and $\text{cé}_{V^\circ}(N_*W)$, respectively.

Lemma 4.1.10.

The category $\text{SubMod}(V, V^\circ)$ contains a unique object C maximal with respect to the property of being 1-crystalline, and this is equal to $\text{Crys}_1(V)$.

Proof. First, note that V° , being the generic fiber of the finite flat \mathcal{O}_K -group $\mathcal{Q}_\varpi^1[n]^\circ$, is 1-crystalline. This shows that $\text{SubMod}(V, V^\circ)$ contains a unique object C maximal with respect to the property of being 1-crystalline. Recall that $\text{Crys}_1(V)$ is defined in Definition 3.4.1 as the internal sum of all 1-crystalline submodules of V . From this, it is clear that $V^\circ \subset \text{Crys}_1(V)$, so that $\text{Crys}_1(V)$ is an object of $\text{SubMod}(V, V^\circ)$. By the maximality property of C , $\text{Crys}_1(V) \subset C$. By the maximality property of $\text{Crys}_1(V)$, $\text{Crys}_1(V) \subset C$. \square

Using the monodromy morphisms ν_n and N_n , we will characterize $\text{Crys}_1(V)$ among the objects of the category $\text{SubMod}(V, V^\circ)$. This amounts to characterizing a particular subgroup of $V^{\text{ét}}$.

Remark 4.1.11. This notational convention of adorning $\mathbb{Z}_p[\Gamma_K]$ -modules with the symbols ét and \circ must be used with care. Given a finite flat $\mathbb{Z}_p[\Gamma_K]$ -module V , the notions V° and $V^{\text{ét}}$ are not well-defined without a preferred finite flat \mathcal{O}_K -group scheme giving rise to V . For instance, If V is the abelian group $\mathbb{Z}/p\mathbb{Z}$ with the trivial action of Γ_K , and if K contains a p -th root of unity, V can be realized as both $(\mu_{p, \mathcal{O}_K})_K$ and $(\mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K})_K$. But μ_{p, \mathcal{O}_K} is connected and $\mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K}$ is étale. If we thought of V as coming from μ_{p, \mathcal{O}_K} , then we would have $V^\circ = V$ and $V^{\text{ét}} = 0$, while if we thought of V as coming from $\mathbb{Z}/p\mathbb{Z}_{\mathcal{O}_K}$ then we would have $V^\circ = 0$ and $V^{\text{ét}} = V$. Thus, the notation is dangerous for the uninitiated. We only apply the notation $(-)^{\circ}$ in one case V° , where there is a preferred finite flat \mathcal{O}_K -group scheme with generic fiber V° , namely $\mathcal{Q}_{\varpi}^1[n]_K^\circ$. Our use of the notation $(-)^{\text{ét}}$ is always via the definition in (4.1.9).

4.2 Description of $\text{Crys}_1(A_K[n])$

Until Proposition 4.2.22, fix U in $\text{SubMod}(V, V^\circ)$ with $i : U \hookrightarrow V$ the inclusion. Let $\bar{i} : U^{\text{ét}} \hookrightarrow V^{\text{ét}}$ be the induced injection. We are concerned with $\bar{i}^* \text{cé}_{V^\circ}(N_*W) \in \text{Ext}(U^{\text{ét}}, V^\circ)$, the pullback of $\text{cé}_{V^\circ}(N_*W) \in \text{Ext}(V^{\text{ét}}, V^\circ)$ under \bar{i} . This is defined so that the rightmost square in

$$\begin{array}{ccccccccc} \bar{i}^* \text{cé}_{V^\circ}(N_*W) : & 0 & \longrightarrow & V^\circ & \longrightarrow & \bar{i}^* N_*W & \longrightarrow & U^{\text{ét}} & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow \bar{i} & & \\ \text{cé}_{V^\circ}(N_*W) : & 0 & \longrightarrow & V^\circ & \longrightarrow & N_*W & \longrightarrow & V^{\text{ét}} & \longrightarrow & 0 \end{array} \quad (4.2.1)$$

is Cartesian. Write U' for the degree-zero term in

$$\text{cé}_{V^\circ}(U) \text{ } {}_{-B} \bar{i}^* \text{cé}_{V^\circ}(N_*W) \quad 0 \longrightarrow V^\circ \longrightarrow U' \longrightarrow U^{\text{ét}} \longrightarrow 0. \quad (4.2.2)$$

Proposition 4.2.3.

The morphism i induces an injection $i' : U' \hookrightarrow W$ of $\mathbb{Z}_p[\Gamma_K]$ -modules.

Proof. Define

$${}_{-B} \bar{i}^* N_*W := ({}_{-B} \bar{i}^* \text{cé}_{V^\circ}(N_*W))^0.$$

To be sure, this is isomorphic to the term in degree zero $(\bar{i}^* \text{cé}_{V^\circ}(N_*W))^0$ as a $\mathbb{Z}_p[\Gamma_K]$ -module, but it has a different extension structure; we find the notation psychologically useful. Similarly, define

$$-_{\text{B}}N_*W := (-_{\text{B}} \text{cé}_{V^\circ}(N_*W))^0.$$

By definition (see Proposition 2.4.10), the Baer difference has the form

$$U' = \frac{U \times_{U^{\text{ét}}} (-_{\text{B}}\bar{i}^*N_*W)}{\Delta_{U'}(V^\circ)}.$$

We explain the notation. The $\mathbb{Z}_p[\Gamma_K]$ -module $U \times_{U^{\text{ét}}} (-_{\text{B}}\bar{i}^*N_*W)$ is the fiber product of U and \bar{i}^*N_*W over $U^{\text{ét}}$ taken with respect to the natural surjection $U \rightarrow U^{\text{ét}}$ and the surjection $\bar{i}^*N_*W \rightarrow U^{\text{ét}}$ in $-_{\text{B}}\bar{i}^* \text{cé}_{V^\circ}(N_*W)$. By definition of Baer negation, the surjection $-\bar{i}^*N_*W \rightarrow U^{\text{ét}}$ in $-_{\text{B}}\bar{i}^* \text{cé}_{V^\circ}(N_*W)$ is obtained by negating the surjection $\bar{i}^*N_*W \rightarrow U^{\text{ét}}$ in $i^* \text{cé}_{V^\circ}(N_*W)$. The map $\Delta_{U'}$ is the morphism $V^\circ \rightarrow U \times_{U^{\text{ét}}} (-_{\text{B}}\bar{i}^*N_*W)$ given by taking obvious diagonal map and negating the second coordinate. By (4.1.1), we have

$$\text{cé}_{V^\circ}(V) = \text{cé}_{V^\circ}(W) +_{\text{B}} \text{cé}_{V^\circ}(N_*W).$$

Using this, and the same notational conventions we just described, we have

$$W = \frac{V \times_{V^{\text{ét}}} (-_{\text{B}}N_*W)}{\Delta_W(V^\circ)}.$$

There is a commuting diagram of $\mathbb{Z}_p[\Gamma_K]$ -modules

$$\begin{array}{ccccc}
U \times_{U^{\text{ét}}} (-_{\text{B}}\bar{i}^*N_*W) & \xrightarrow{\quad\quad\quad} & U & \xrightarrow{\quad i \quad} & V \\
\downarrow & \searrow \tilde{i}' & \downarrow & \downarrow & \downarrow \\
& & V \times_{V^{\text{ét}}} (-_{\text{B}}N_*W) & \xrightarrow{\quad\quad\quad} & V \\
& & \downarrow & \downarrow & \downarrow \\
-_{\text{B}}\bar{i}^*N_*W & \xrightarrow{\quad\quad\quad} & U^{\text{ét}} & \xrightarrow{\quad \bar{i} \quad} & V^{\text{ét}} \\
& \searrow & \downarrow & \downarrow & \downarrow \\
& & -_{\text{B}}N_*W & \xrightarrow{\quad\quad\quad} & V^{\text{ét}}
\end{array} \tag{4.2.4}$$

constructed as follows. Start with the bottom square, which is Cartesian. Add the rightmost face, then form the front and back faces as Cartesian squares.

The morphism \tilde{i}' is induced by the Cartesian property of its target.

To show that \tilde{i}' induces a morphism $U' \rightarrow W$, we must show that the diagram

$$\begin{array}{ccc} V^\circ & \xrightarrow{\Delta_{U'}^-} & U \times_{U\text{ét}} (-_B \bar{i}^* N_* W) \\ \parallel & & \downarrow \tilde{i}' \\ V^\circ & \xrightarrow{\Delta_W^-} & V \times_{U\text{ét}} (-_B N_* W) \end{array} \quad (4.2.5)$$

commutes. To see this, note that the injection $\Delta_{U'}^-$ is induced by the morphisms

$$V^\circ \longrightarrow U \quad (4.2.6)$$

$$V^\circ \longrightarrow -_B \bar{i}^* N_* W \quad (4.2.7)$$

(the second one one being negated) appearing in the complexes $\text{cé}_{V^\circ}(U)$ and $-_B \bar{i}^* \text{cé}_{V^\circ}(N_* W)$, respectively. Since \tilde{i}' induces the inclusion $U \subset V$ on U we see that \tilde{i}' sends $V^\circ \subset U$ to $V^\circ \subset V$. Since \tilde{i}' agrees with the morphism $\bar{i}^* N_* W \rightarrow N_* W$ of (4.2.1), it sends commutes with the morphisms $V^\circ \rightarrow \bar{i}^* N_* W$ and $V^\circ \rightarrow N_* W$. This has shown that (4.2.5) commutes. We conclude that \tilde{i}' induces a morphism

$$i' : U' \longrightarrow W.$$

We show that i' is injective. Let $(x, y) \in U \times_{U\text{ét}} (-_B \bar{i}^* N_* W)$ and write $[x, y]$ for the class of (x, y) in $U -_B \bar{i}^* N_* W$. To say that $i'([x, y]) = 0$ is to say that $\tilde{i}'((x, y))$ lands in $\Delta_W^-(V^\circ)$. Suppose $\tilde{i}'((x, y)) = \Delta_W^-(z)$ for $z \in V^\circ$. By commutativity of (4.2.5), we see that $(x, y) = \Delta_{U'}^-(z)$. This shows that $[x, y] = 0$. We conclude that i' is injective. \square

Proposition 4.2.8.

U is 1-crystalline if and only if $\bar{i}^ N_* W$ is 1-crystalline.*

Proof. Recall that U' was defined in (4.2.2) to satisfy

$$\text{cé}_{V^\circ}(U') = \text{cé}_{V^\circ}(U) -_B \bar{i}^* \text{cé}_{V^\circ}(N_* W). \quad (4.2.9)$$

Since W is defined to be the generic fiber of the finite flat \mathcal{O}_K -group $\mathcal{Q}_\omega^1[n]$, W is 1-crystalline by Theorem 3.2.11. We just showed in Proposition 4.2.3 that U' is isomorphic to a $\mathbb{Z}_p[\Gamma_K]$ -submodule of W , so by Proposition 3.2.5, U' is 1-crystalline. Applying Proposition 3.2.10 to (4.2.9) completes the proof. \square

We now develop a precise criterion for 1-crystallinity of \bar{i}^*N_*W . Combining the diagram defining N_*W as a pushout and the diagram defining \bar{i}^*N_*W as a pullback, we have a commuting diagram of $\mathbb{Z}/n\mathbb{Z}[\Gamma_K]$ -modules with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V^{\text{ét}}(1) & \longrightarrow & E_{\varpi,K}[n] \otimes V^{\text{ét}} & \longrightarrow & V^{\text{ét}} & \longrightarrow & 0 \\
& & \downarrow N & & \downarrow N & & \parallel & & \\
0 & \longrightarrow & V^\circ & \longrightarrow & N_*W & \longrightarrow & V^{\text{ét}} & \longrightarrow & 0 \\
& & \parallel & & \uparrow & & \uparrow \bar{i} & & \\
0 & \longrightarrow & V^\circ & \longrightarrow & \bar{i}^*N_*W & \longrightarrow & U^{\text{ét}} & \longrightarrow & 0
\end{array} \tag{4.2.10}$$

Note that we abuse notation by using N for both the map from $V^{\text{ét}}$ and the map from $E_{\varpi,K}[n] \otimes V^{\text{ét}}$. Recall that the top row arises by tensoring

$$\theta_{n,K}^{\varpi} : \quad 0 \longrightarrow \mu_{n,K} \longrightarrow E_{\varpi,K}[n] \longrightarrow \mathbb{Z}/n\mathbb{Z}_K \longrightarrow 0$$

with $V^{\text{ét}}$. Choose the ordered basis $\{x_1, x_2\}$ of $E_{\varpi,K}[n]$ constructed in Example 3.1.17. In this basis, the action of Γ_K is via

$$\begin{pmatrix} \chi & c_{\varpi,n} \\ & 1 \end{pmatrix}.$$

where χ is the mod- n cyclotomic character and $c_{\varpi,n}$ is the $\mathbb{Z}/n\mathbb{Z}$ -valued cocycle (3.1.20) in the case $q = \varpi$. Suppose now that $n = p^m$. Recall that the cocycle $c_{\varpi,n}$ satisfies the relation

$$\sigma(\tilde{\varpi}^{(m)}) = (\tilde{\epsilon}^{(m)})^{c_{\varpi,n}(\sigma)} \tilde{\varpi}^{(m)}, \tag{4.2.11}$$

for all $\sigma \in \Gamma_K$, where $\tilde{\varpi}$ and $\tilde{\epsilon}$ are as in Situation 0.1.2. We will need the following result.

Lemma 4.2.12.

There exists $\sigma \in \Gamma_K$ such that $c_{\varpi,n}(\sigma) \in (\mathbb{Z}/n\mathbb{Z})^\times$.

Proof. Suppose $p \mid c_{\varpi,n}(\sigma)$ for all σ . Recall that $n = p^m$ with $m \geq 1$. Raising the relation defining $c_{\varpi,n}$ to the p^{m-1} -th power, we see that $(\tilde{\varpi}^{(m)})^{p^{m-1}}$ is fixed by all of Γ_K , which is to say that K contains a p -th root of ϖ . Since ϖ is a uniformizer of \mathcal{O}_K , this is impossible, so we conclude that $c_{\varpi,n}(\sigma) \in (\mathbb{Z}/n\mathbb{Z})^\times$ for some σ . \square

With our choice of basis of $E_{\varpi,K}[n]$, we may write elements of the tensor product $E_{\varpi,K}[n] \otimes V^{\acute{e}t}$ uniquely in the form

$$x_1 \otimes y_1 + x_2 \otimes y_2 \quad (y_1, y_2 \in V^{\acute{e}t}).$$

Using this basis for $E_{\varpi,K}[n]$, the morphisms in the top row of (4.2.10) are simply the inclusion

$$\begin{aligned} V^{\acute{e}t}(1) = \mathbb{Z}/n\mathbb{Z}(1) \otimes V^{\acute{e}t} &\longrightarrow E_{\varpi,K}[n] \otimes V^{\acute{e}t} \\ x_1 \otimes y_1 &\longmapsto x_1 \otimes y_1 \end{aligned}$$

and

$$\begin{aligned} E_{\varpi,K}[n] \otimes V^{\acute{e}t} &\longrightarrow V^{\acute{e}t} \\ x_1 \otimes y_1 + x_2 \otimes y_2 &\longmapsto y_2. \end{aligned}$$

Recall that N_*W is defined so that the upper-left square in (4.2.10) is co-Cartesian. This is to say that

$$N_*W = \frac{(E_{\varpi,K}[n] \otimes V^{\acute{e}t}) \oplus V^\circ}{\{(x_1 \otimes y_1, 0) - (0, N(x_1 \otimes y_1)) : y_1 \in V^{\acute{e}t}\}}. \quad (4.2.13)$$

Given $(x_1 \otimes y_1 + x_2 \otimes y_2, z) \in (E_{\varpi,K}[n] \otimes V^{\acute{e}t}) \oplus V^\circ$, we denote its class in N_*W by $[x_1 \otimes y_1 + x_2 \otimes y_2, z]$. Hence (using the monodromy relation in the denominator of (4.2.13)) we have

$$N_*W = \{[x_2 \otimes y_2, z] : y_2 \in V^{\acute{e}t}, z \in V^\circ\}.$$

By definition of the fiber product, the vertical injection $\bar{i}^*N_*W \hookrightarrow N_*W$ in (4.2.10) is the inclusion of the submodule

$$\bar{i}^*N_*W = \{[x_2 \otimes u, z] : u \in U^{\acute{e}t}, z \in V^\circ\} \subset N_*W.$$

There is a homomorphism of finite abelian groups

$$s : \begin{cases} U^{\acute{e}t} &\longrightarrow \bar{i}^*N_*W \\ u &\longmapsto [x_2 \otimes u, 0] \end{cases} \quad (4.2.14)$$

which is a section of the quotient map $\bar{i}^*N_*W \rightarrow U^{\acute{e}t}$. The morphism s need not be Γ_K -equivariant, but we will now show it is $\Gamma_{\bar{\omega}}$ -equivariant.

Lemma 4.2.15.

The homomorphism of finite abelian groups s is $\Gamma_{\tilde{\omega}}$ -equivariant.

Proof. Restrict (4.2.10) to $\Gamma_{\tilde{\omega}}$. It is immediate from (4.2.11) that $c_{\tilde{\omega},n}|_{\Gamma_{\tilde{\omega}}} = 0$, which is to say that

$$E_{\tilde{\omega},K}[n]|_{\Gamma_{\tilde{\omega}}} = (\mathbb{Z}/n\mathbb{Z}(1) \oplus \mathbb{Z}/n\mathbb{Z})|_{\Gamma_{\tilde{\omega}}} \quad (4.2.16)$$

with x_1 spanning $\mathbb{Z}/n\mathbb{Z}(1)|_{\Gamma_{\tilde{\omega}}}$ and x_2 spanning $\mathbb{Z}/n\mathbb{Z}|_{\Gamma_{\tilde{\omega}}}$. The homomorphism $y_2 \mapsto x_2 \otimes y_2$ is a $\Gamma_{\tilde{\omega}}$ -equivariant section of the quotient map

$$(E_{\tilde{\omega},K}[n] \otimes V^{\text{ét}})|_{\Gamma_{\tilde{\omega}}} \rightarrow V^{\text{ét}}|_{\Gamma_{\tilde{\omega}}}.$$

We see that the extension

$$0 \longrightarrow V^\circ|_{\Gamma_{\tilde{\omega}}} \longrightarrow \bar{i}^*N_*W|_{\Gamma_{\tilde{\omega}}} \longrightarrow U^{\text{ét}}|_{\Gamma_{\tilde{\omega}}} \longrightarrow 0 \quad (4.2.17)$$

is split by the section $U^{\text{ét}}|_{\Gamma_{\tilde{\omega}}} \rightarrow \bar{i}^*N_*W|_{\Gamma_{\tilde{\omega}}}$ induced by s . \square

The following key proposition uses the difficult “torsion fully faithfulness theorem” described in Proposition 3.3.3.

Proposition 4.2.18.

The $\mathbb{Z}_p[\Gamma_K]$ -module \bar{i}^*N_*W is 1-crystalline if and only if the $\Gamma_{\tilde{\omega}}$ -equivariant section s defined in (4.2.14) is Γ_K -equivariant.

Proof. Recall that a torsion $\mathbb{Z}_p[\Gamma_K]$ -module is 1-crystalline if and only if it is the generic fiber of a finite flat \mathcal{O}_K -group. Therefore $V^\circ = \mathcal{Q}_{\tilde{\omega}}^1[n]_K^\circ$ is 1-crystalline. Since $V^{\text{ét}}$ is the generic fiber of the étale \mathcal{O}_K -group $\mathcal{Q}_{\tilde{\omega}}^1[n]_K^{\text{ét}}$, $V^{\text{ét}}$ is unramified. Therefore, the $\mathbb{Z}_p[\Gamma_K]$ -submodule $U^{\text{ét}} \subset V^{\text{ét}}$ is also unramified. By Example 2.1.8, $U^{\text{ét}}$ and $V^{\text{ét}}$ are 1-crystalline. Suppose first that s is Γ_K -equivariant. Then $\bar{i}^*N_*W \simeq V^\circ \oplus U^{\text{ét}}$ as $\mathbb{Z}_p[\Gamma_K]$ -modules. Since $U^{\text{ét}}$ and V° are 1-crystalline, \bar{i}^*N_*W is 1-crystalline.

Now suppose that \bar{i}^*N_*W is 1-crystalline. Since $U^{\text{ét}}$ is 1-crystalline, Proposition 3.3.3 implies that the $\Gamma_{\tilde{\omega}}$ -equivariant morphism s is Γ_K -equivariant. \square

Remark 4.2.19. In Proposition 4.2.18 we have made use of non-canonical choices using our chosen element $\tilde{\omega} \in \mathcal{O}_C^\flat$. On the one hand, $\tilde{\omega}$ gives a subgroup $\Gamma_{\tilde{\omega}} \subset \Gamma_K$ that we use when applying Proposition 3.3.3. On the other hand, we used $\varpi = \tilde{\omega}^{(0)}$ when we formed $E_{\tilde{\omega},K}[n]$ and performed Raynaud decomposition.

We could have used a different uniformizer, say $u\varpi$ where $u \in \mathcal{O}_K^\times$, to perform Raynaud decomposition. In that case, $E_{u\varpi, K}[n]$ is not split as a $\mathbb{Z}_p[\Gamma_{\varpi}]$ -module; see Example 3.3.5.

Returning to the situation of the above Proposition 4.2.18, we now relate the splitness of \bar{i}^*N_*W as an extension of $U^{\text{ét}}$ by V° to the monodromy morphism. Recall that $N(-1) : V^{\text{ét}} \rightarrow V^\circ(-1)$ is a morphism of $\mathbb{Z}_p[\Gamma_K]$ -modules.

Proposition 4.2.20.

The section s is Γ_K -equivariant if and only if $U^{\text{ét}} \subset \ker N(-1)$.

Proof. Suppose $U^{\text{ét}} \subset \ker N(-1)$. With reference to our basis $\{x_1, x_2\}$ for $E_{\varpi, K}[n] \otimes V^{\text{ét}}$, this means that $N(x_1 \otimes u) = 0$ for all $u \in U^{\text{ét}}$. Let $\sigma \in \Gamma_K$. We have

$$\begin{aligned} \sigma s(u) - s(\sigma u) &= \sigma[x_2 \otimes u, 0] - [x_2 \otimes \sigma u, 0] \\ &= [c_{\varpi, n}(\sigma)x_1 \otimes \sigma u + x_2 \otimes \sigma u, 0] - [x_2 \otimes \sigma u, 0] \\ &= [c_{\varpi, n}(\sigma)x_1 \otimes \sigma u, 0] \\ &= [0, N(x_1 \otimes c_{\varpi, n}(\sigma)\sigma u)] \\ &= [0, 0]. \end{aligned}$$

This shows that s is Γ_K -equivariant.

Conversely, suppose s is Γ_K -equivariant and let $u \in U^{\text{ét}}$. For any $\sigma \in \Gamma_K$, we see from the above calculation that we have an equality of elements of \bar{i}^*N_*W

$$[0, 0] = \sigma s(u) - s(\sigma u) = [0, N(x_1 \otimes c_{\varpi, n}(\sigma)\sigma u)],$$

which is to say that, for some $y_1 \in V^{\text{ét}}$ depending on u , we have the equality of elements of $(E_{\varpi, K}[n] \otimes V^{\text{ét}}) \oplus V^\circ$

$$(0, N(x_1 \otimes c_{\varpi, n}(\sigma)\sigma u)) = (x_1 \otimes y_1, -N(x_1 \otimes y_1)).$$

We see that we must have $y_1 = 0$, hence $N(x_1 \otimes y_1) = 0$, hence

$$N(x_1 \otimes c_{\varpi, n}(\sigma)\sigma u) = 0. \tag{4.2.21}$$

By Lemma 4.2.12, we may choose σ such that $c_{\varpi, n}(\sigma) \in (\mathbb{Z}/n\mathbb{Z})^\times$. It follows

from (4.2.21) that

$$0 = N(x_1 \otimes \sigma u) = \chi(\sigma^{-1})\sigma N(x_1 \otimes u),$$

whence $N(x_1 \otimes u) = 0$. As u was an arbitrary element of $U^{\text{ét}}$, we conclude that $U^{\text{ét}} \subset \ker N(-1)$. \square

Proposition 4.2.22.

Let U be an object of $\text{SubMod}(V, V^\circ)$. Then U is 1-crystalline if and only if the induced injection $U^{\text{ét}} = U/V^\circ \hookrightarrow V^{\text{ét}} = V/V^\circ$ factors through

$$\ker N(-1) \subset V^{\text{ét}}.$$

Proof. This results from combining Proposition 4.2.8, Proposition 4.2.18, and Proposition 4.2.20. More specifically, by Proposition 4.2.8, U is 1-crystalline if and only if \bar{i}^*N_*W is 1-crystalline. By Proposition 4.2.18, \bar{i}^*N_*W is 1-crystalline if and only if the morphism s defined in (4.2.14) is Γ_K -equivariant. By Proposition 4.2.20, the morphism s is Γ_K equivariant if and only if the inclusion $U^{\text{ét}} \hookrightarrow V^{\text{ét}}$ factors through $\ker N(-1) \subset V^{\text{ét}}$. Altogether, we have the claim. \square

Theorem 4.2.23.

Let p be any rational prime. Let K be a p -adic field in the sense of Section 0.1, and fix a uniformizer ϖ of \mathcal{O}_K . Let A_K be a semi-stable abelian K -variety, and let $n \in \mathbb{Z}_{\geq 2}$, be a power of p .

Let $(\mathcal{Q}_\varpi^1[n], N_n)$ be the log finite flat \mathcal{O}_K -group of Proposition 2.5.10. Then there is an isomorphism of $\mathbb{Z}_p[\Gamma_K]$ -modules

$$A_K[n] \times_{\mathcal{Q}_\varpi^1[n]_K^{\text{ét}}} \ker N_{n,K}(-1) \simeq \text{Crys}_1(A_K[n]) \quad (4.2.24)$$

given by projection onto the first factor of $A_K[n] \times_{\mathcal{Q}_\varpi^1[n]_K^{\text{ét}}} \ker N_{n,K}(-1)$. After fixing an isomorphism $(\mathcal{Q}_\varpi^1[n], N_n)_K^0 \simeq A_K[n]$ as in Proposition 2.5.10, there is a canonical isomorphism of $\text{Crys}_1(A_K[n])$ with the generic fiber of the finite flat \mathcal{O}_K -group

$${}^* \mathcal{Q}_\varpi^1[n] := \mathcal{Q}_\varpi^1[n] \times_{\mathcal{Q}_\varpi^1[n]_K^{\text{ét}}} (\ker N_n)(-1), \quad (4.2.25)$$

where the Tate twist is as in Proposition 2.2.23.

Proof. Recall that the generic fiber (in the sense of Definition 2.4.44) of the log finite flat \mathcal{O}_K -group $(\mathcal{Q}_\varpi^1[n], N_n)$ was shown in Proposition 2.5.10 (which uses the semi-stability assumption) to be isomorphic to $A_K[n]$. By definition of the generic fiber functor, we know that in fact $A_K[n]$ lies in a short exact sequence

$$0 \longrightarrow \mathcal{Q}_\varpi^1[n]_K^\circ \longrightarrow A_K[n] \longrightarrow \mathcal{Q}_\varpi^1[n]_K^{\acute{e}t} \longrightarrow 0. \quad (4.2.26)$$

Since $\ker N_{n,K} \subset \mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}(1)$, exactness of twisting (Proposition 2.2.23) gives an inclusion $\ker N_{n,K}(-1) \subset \mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}$. The fiber product in the statement of the theorem is taken with respect to the surjection in (4.2.26) and the inclusion $\ker N_{n,K}(-1) \subset \mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}$. The fiber product consists of those pairs $(x, y) \in A_K[n] \times \ker N_{n,K}(-1)$ such that x and y map to the same element of $\mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}$ under the two maps to $\mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}$. Writing a bar for the surjection in (4.2.26), we therefore have

$$A_K[n] \times_{\mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}} \ker N_{n,K}(-1) = \{x \in A_K[n] : \bar{x} \in \ker N_{n,K}(-1)\}. \quad (4.2.27)$$

Said differently, $A_K[n] \times_{\mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}} \ker N_{n,K}(-1)$ is the maximal $\mathbb{Z}_p[\Gamma_K]$ -submodule $M \subset A_K[n]$ containing $\mathcal{Q}_\varpi^1[n]_K^\circ$ with the property that the natural injection of $M/\mathcal{Q}_\varpi^1[n]_K^\circ$ into $\mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}$ factors through $\ker N_{n,K}(-1)$.

We saw in Lemma 4.1.10 that $\text{Crys}_1(A_K[n])$ is an object of the category $\text{SubMod}(A_K[n], \mathcal{Q}_\varpi^1[n]_K^\circ)$, and that it is the object of $\text{SubMod}(A_K[n], \mathcal{Q}_\varpi^1[n]_K^\circ)$ maximal with respect to the property of being 1-crystalline. We showed in Proposition 4.2.22 that an object of $\text{SubMod}(A_K[n], \mathcal{Q}_\varpi^1[n]_K^\circ)$ is 1-crystalline if and only if its quotient by $\mathcal{Q}_\varpi^1[n]_K^\circ$ is contained in $\ker N_{n,K}(-1)$. As we explained above, the maximal such module is the fiber product (4.2.24). We conclude that

$$A_K[n] \times_{\mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}} \ker N_{n,K}(-1) \simeq \text{Crys}_1(A_K[n])$$

via the projection onto the first factor.

For the claim that ${}^*\mathcal{Q}_\varpi^1[n]$ has generic fiber $\text{Crys}_1(A_K[n])$, recall from Remark 2.4.5 that the object $({}^*\mathcal{Q}_\varpi^1[n], 0)$ of $\text{fin}_{\mathcal{O}_K}^{p,N}$ is a subobject of $(\mathcal{Q}_\varpi^1[n], N_n)$. By Proposition 2.4.37, and Proposition 2.4.45,

$${}^*\mathcal{Q}_\varpi^1[n]_K = ({}^*\mathcal{Q}_\varpi^1[n], 0)_K \subset (\mathcal{Q}_\varpi^1[n], N_n)_K = A_K[n].$$

By its definition in (4.2.25) as a pullback, ${}^*\mathcal{Q}_\varpi^1[n]$ gives a class in

$$\mathrm{Ext}_{\mathrm{fin}_{\mathcal{O}_K}^p}((\ker N_n)(-1), \mathcal{Q}_\varpi^1[n]^\circ),$$

so its generic fiber ${}^*\mathcal{Q}_\varpi^1[n]_K$ gives a class in $\mathrm{Ext}_{\mathrm{fin}_K^p}(\ker N_{n,K}(-1), \mathcal{Q}_\varpi^1[n]_K^\circ)$. The description in (4.2.24), $\mathrm{Crys}_1(A_K[n])$ shows that it is the pullback of the class of (4.2.26) under the inclusion $\ker N_{n,K}(-1) \subset \mathcal{Q}_\varpi^1[n]_K^{\acute{e}t}$. This shows that $\mathrm{Crys}_1(A_K[n])$ gives a class in $\mathrm{Ext}_{\mathrm{fin}_K^p}(\ker N_{n,K}(-1), \mathcal{Q}_\varpi^1[n]_K^\circ)$. It follows that ${}^*\mathcal{Q}_\varpi^1[n]_K$ and $\mathrm{Crys}_1(A_K[n])$ are 1-crystalline $\mathbb{Z}_p[\Gamma_K]$ -submodules of $A_K[n]$ having the same order. By maximality of $\mathrm{Crys}_1(A_K[n])$, we have

$${}^*\mathcal{Q}_\varpi^1[n]_K = \mathrm{Crys}_1(A_K[n]) \subset A_K[n]$$

This completes the proof. \square

Theorem 4.2.23 is our main technical result. In the next section, we provide some applications to the study of $T_p A_K$ and the p -primary part of Φ_{A_K} .

4.3 Néron component groups and $R^1 \mathrm{Crys}_1$ in the semi-stable case

Recall that the Tate module $T_p \tilde{A}_K$ is the Tate module of a p -divisible group (see Example 2.2.11). We therefore know that $T_p \tilde{A}_K$ is an object of $\mathrm{Rep}_{\mathrm{free}}^{\mathrm{crys},1}(\Gamma_K)$.

Theorem 4.3.1.

Fix notation as in Theorem 4.2.23. Then the isomorphisms of Theorem 4.2.23

$$A_K[p^m] \times_{\mathcal{Q}_\varpi^1[p^m]_K^{\acute{e}t}} \ker N_{p^m,K}(-1) \simeq \mathrm{Crys}_1(A_K[p^m])$$

for $m \in \mathbb{Z}_{\geq 1}$ are compatible in m , and give rise to an isomorphism

$$\mathrm{Crys}_1(T_p A_K) \simeq T_p \tilde{A}_K.$$

Proof. As noted in Remark 2.3.5, the morphisms $\nu_{p^m} : Y/p^m Y \rightarrow \tilde{A}[p^m]$ are compatible in m . Since N_{p^m} is obtained by pre-composing with canonical morphisms $\mathcal{Q}_\varpi^1[p^m]^{\acute{e}t}(1) \rightarrow (Y/p^m Y)(1)$ (see (2.5.4)), we see that the N_{p^m} are compatible in m . It follows that the finite flat \mathcal{O}_K -groups $\ker N_{p^m}$ are compatible in m . The compatibility of the isomorphisms of Theorem 4.2.23 follows from the

description of these isomorphisms as $\text{pr}_{1,K}$ on $A_K[p^m] \times_{\mathcal{Q}_{\varpi}^1[p^m]_{\text{ét}}} \ker N_{p^m,K}(-1)$.
By Proposition 3.4.4,

$$\text{Crys}_1(T_p A_K) = \lim_m \text{Crys}_1(T_p A_K / p^m T_p A_K) = \lim_m \text{Crys}_1(A_K[p^m]),$$

so by Theorem 4.2.23 we have isomorphisms

$$\text{Crys}_1(T_p A_K) \simeq \lim_m {}^* \mathcal{Q}_{\varpi}^1[p^m]_K. \quad (4.3.2)$$

Since $T_p \tilde{A}[p^\infty]$ is 1-crystalline, it is contained in $\text{Crys}_1(T_p A_K)$. The quotient $\text{Crys}_1(T_p A_K) / p^m \text{Crys}_1(T_p A_K)$ is torsion 1-crystalline, as is $\tilde{A}[p^m]_K$, so we find

$$\tilde{A}[p^m]_K \subset \text{Crys}_1(T_p A_K) / p^m \text{Crys}_1(T_p A_K) \subset \text{Crys}_1(A[p^m]_K) = {}^* \mathcal{Q}_{\varpi}^1[p^m]_K.$$

The finite flat \mathcal{O}_K -group ${}^* \mathcal{Q}_{\varpi}^1[p^m]$ is defined by pullback as in

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Q}_{\varpi}^1[p^m]^\circ & \longrightarrow & {}^* \mathcal{Q}_{\varpi}^1[p^m] & \longrightarrow & (\ker N_{p^m})(-1) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{Q}_{\varpi}^1[p^m]^\circ & \longrightarrow & \mathcal{Q}_{\varpi}^1[p^m] & \longrightarrow & \mathcal{Q}_{\varpi}^1[p^m]_{\text{ét}} \longrightarrow 0 \end{array}$$

It is immediate that the upper row is the connected-étale sequence of ${}^* \mathcal{Q}[p^m]^\circ$. It follows from (2.2.17) that ${}^* \mathcal{Q}[p^m]_{\text{ét}} = \ker N_{p^m}(-1)$ fits into the short exact sequence

$$0 \longrightarrow \tilde{A}[p^m]_{\text{ét}} \longrightarrow {}^* \mathcal{Q}_{\varpi}^1[p^m]_{\text{ét}} \xrightarrow{\pi_n} \ker \nu_{p^m}(-1) \longrightarrow 0.$$

By Proposition 2.3.7 (since Φ_{A_K} is finite), $T_p((\ker \nu_{p^m}(-1))_{m \geq 1}) = 0$. It follows that $\lim_m {}^* \mathcal{Q}_{\varpi}^1[p^m]_K = \lim_m \tilde{A}_K[p^m] = T_p \tilde{A}_K$. By (4.3.2), we have $\text{Crys}_1(T_p A_K) \simeq T_p \tilde{A}_K$. This completes the proof. \square

Corollary 4.3.3 ([10] 4.7; also [7] 5.3.4 when $p > 2$ (see also Breuil's Remarque 5.3.5 for earlier special cases)).

An abelian K -variety A_K has good reduction over \mathcal{O}_K if and only if its p -adic Tate module $T_p A_K$ is a 1-crystalline $\mathbb{Z}_p[\Gamma_K]$ -module.

Proof. Our proof will essentially follow the proof of Theorem 4.7 of [10], except we will invoke Theorem 4.3.1 where they invoke their equivalent statement Corollary 4.6, which they prove by very different means.

Suppose that A_K has good reduction. Then $T_p A_K$ is the generic fiber of the p -divisible group $A[p^\infty]$ over \mathcal{O}_K , which is 1-crystalline by Theorem 3.2.11.

Suppose $T_p A_K$ is 1-crystalline. We wish to prove that A_K has good reduction. In the proof of Theorem 4.7 in [10], it is shown using the results 7.3.3 and 7.5.1 of [14] that we may assume without loss of generality that A_K has semi-stable reduction. Make this assumption, and adopt the notation of Situation 1.6.11. By Proposition 2.5.3, there is a short exact sequence

$$0 \longrightarrow T_p \tilde{A}_K \longrightarrow T_p A_K \longrightarrow Y_K \otimes \mathbb{Q}_p / \mathbb{Z}_p \longrightarrow 0$$

where Y_K has \mathbb{Z}_p -rank equal to the dimension of the torus in \tilde{A}_K . By Theorem 4.3.1, the first morphism in this short exact sequence is an isomorphism. Therefore $Y = 0$, so $T = 0$ and $\tilde{A} = B$. Since $\tilde{A}_K = A_K$, we see that B , which is an abelian \mathcal{O}_K -scheme, has generic fiber A_K . By Proposition 1.3.2, B is the Néron model of A_K . Thus we see that A_K has good reduction if $T_p A_K$ is 1-crystalline. \square

Recall that A is the Néron model of A_K , which is assumed to be semi-stable, and that Φ_{A_K} (defined as in Definition 1.3.8) is the component group of A . It is shown in [19] that (taking generic fiber after taking finite parts)

$$\Phi_{A_K}[p^m] \simeq \frac{A[p^m]_K^f}{A^\circ[p^m]_K^f}.$$

This fact is used by Kim and Marshall (in the case where $p > 2$ and K is unramified) to prove (4.3.5) below. We give a different proof of (4.3.5) using Theorem 4.2.23 and Theorem 4.3.1

Lemma 4.3.4.

Let Φ_{A_K} be the component group of the Néron model of A . Let $n \in \mathbb{Z}_{\geq 2}$ be a power of p . Choose a uniformizer ϖ of \mathcal{O}_K and fix an isomorphism of finite K -groups $(\mathcal{Q}_\varpi^1[n], N_n)_K^0 \simeq A_K[n]$ as in Proposition 2.5.10. Associated to these choices is a canonical isomorphism of unramified $\mathbb{Z}_p[\Gamma_K]$ -modules

$$\Phi_{A_K}[n] \simeq \frac{\text{Crys}_1(T_p A_K \otimes \mathbb{Z}_p / n\mathbb{Z}_p)}{\text{Crys}_1(T_p A_K) \otimes \mathbb{Z}_p / n\mathbb{Z}_p}. \quad (4.3.5)$$

Proof. Choose a uniformizer ϖ of \mathcal{O}_K and form the finite flat \mathcal{O}_K -group ${}^*\mathcal{Q}_\varpi^1[n]$ of Theorem 4.2.23. Recall that ${}^*\mathcal{Q}_\varpi^1[n]$ is defined by pullback as in

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{A}[n]^\circ & \longrightarrow & {}^*\mathcal{Q}_\varpi^1[n] & \longrightarrow & (\ker N_n)(-1) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{A}[n]^\circ & \longrightarrow & \mathcal{Q}_\varpi^1[n] & \longrightarrow & \mathcal{Q}_\varpi^1[n]^{\text{ét}} \longrightarrow 0 \end{array} .$$

By the universal property of ${}^*\mathcal{Q}_\varpi^1[n]$ as a fiber product, we obtain a homomorphism $\tilde{A}[n] \rightarrow {}^*\mathcal{Q}_\varpi^1[n]$ arising from canonical morphism $\tilde{A}[n] \rightarrow \mathcal{Q}_\varpi^1[n]$ in the complex $\eta(\mathcal{Q}_\varpi^1, n)$ (defined as in (2.2.13)) and the zero map $\tilde{A}[n] \rightarrow \text{Ker}N_n(-1)$. We claim that the generic fiber of the morphism $\tilde{A}[n] \rightarrow {}^*\mathcal{Q}_\varpi^1[n]$ has cokernel isomorphic to $\Phi_{A_K}[n]$. Recall the short exact sequence

$$0 \longrightarrow \tilde{A}[n]^{\text{ét}} \longrightarrow \mathcal{Q}_\varpi^1[n]^{\text{ét}} \xrightarrow{\pi_n} Y/nY \longrightarrow 0$$

of (2.2.17), and recall that N_n is defined in (2.5.4) as

$$N_n : \mathcal{Q}_\varpi^1[n]^{\text{ét}}(1) \xrightarrow{\pi_n(1)} Y/nY(1) \xrightarrow{\nu_n} T[n] \hookrightarrow \mathcal{Q}_\varpi^1[n]^\circ .$$

By restricting $\pi_n(1)$ and untwisting, we obtain a short exact sequence

$$0 \longrightarrow \tilde{A}[n]^{\text{ét}} \longrightarrow (\ker N_n)(-1) \longrightarrow \ker \nu_n(-1) \longrightarrow 0,$$

where the morphism $\tilde{A}[n]^{\text{ét}} \rightarrow (\ker N_n)(-1)$ is the one obtained by descending the canonical homomorphism $\tilde{A}[n] \rightarrow {}^*\mathcal{Q}_\varpi^1[n]$ to étale parts. We see that

$${}^*\mathcal{Q}_\varpi^1[n] \longrightarrow (\ker N_n)(-1) \longrightarrow \frac{(\ker N_n)(-1)}{\tilde{A}[n]^{\text{ét}}}$$

induces an isomorphism

$$\frac{{}^*\mathcal{Q}_\varpi^1[n]}{\tilde{A}[n]} \simeq \frac{(\ker N_n)(-1)}{\tilde{A}[n]^{\text{ét}}} = \ker \nu_n(-1). \quad (4.3.6)$$

From our choice of isomorphism $(\mathcal{Q}_\varpi^1[n], N_n)_K^0 \simeq A_K[n]$, we obtain

$${}^*\mathcal{Q}_\varpi^1[n] \simeq \text{Crys}_1(A_K[n]) \quad \text{and} \quad T_p \tilde{A}_K \simeq \text{Crys}_1(T_p A_K)$$

as in Theorem 4.2.23 and Theorem 4.3.1, respectively. Using these, we obtain

from (4.3.6) and the canonical isomorphism $\ker \nu_n(-1) \simeq \Phi_{A_K}[n]$ of Proposition 2.3.7 an isomorphism

$$\Phi_{A_K}[n] \simeq \frac{\text{Crys}_1(T_p A_K \otimes \mathbb{Z}_p/n\mathbb{Z}_p)}{\text{Crys}_1(T_p A_K) \otimes \mathbb{Z}_p/n\mathbb{Z}_p}.$$

This completes the proof \square

Theorem 4.3.7.

Let $\Phi_{A_K}[p^\infty]$ denote the p -Sylow subgroup of Φ_{A_K} . Then the isomorphisms of Lemma 4.3.4 give rise to an isomorphism

$$\Phi_{A_K}[p^\infty] \simeq (R^1 \text{Crys}_1(T_p A_K))_{\text{tors}}. \quad (4.3.8)$$

Proof. By Proposition 3.4.8,

$$\text{colim}_{m \geq 1} \text{Crys}_1(T_p A_K \otimes \mathbb{Z}/p^m \mathbb{Z}) = \text{Crys}_1(T_p A_K \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

Hence, by exactness of directed limits, (4.3.5) gives

$$\Phi_{A_K}[p^\infty] = \frac{\text{Crys}_1(T_p A_K \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{\text{Crys}_1(T_p A_K) \otimes \mathbb{Q}_p/\mathbb{Z}_p}. \quad (4.3.9)$$

On the other hand, consider the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Crys}_1(T_p A_K) & \longrightarrow & \text{Crys}_1(T_p A_K) \otimes \mathbb{Q}_p & \longrightarrow & \text{Crys}_1(T_p A_K \otimes \mathbb{Q}_p/\mathbb{Z}_p) \\ & & & & & \swarrow & \\ & & R^1 \text{Crys}_1(T_p A_K) & \longrightarrow & R^1 \text{Crys}_1(V_p A_K) & \longrightarrow & \dots \end{array} \quad (4.3.10)$$

attached to $T_p A_K$ using Theorem 3.4.12. The kernel of the morphism $f : R^1 \text{Crys}_1(T_p A_K) \rightarrow R^1 \text{Crys}_1(V_p A_K)$ is exactly the right-hand side of (4.3.9), which is torsion, so we have

$$\Phi_{A_K}[p^\infty] \subset (R^1 \text{Crys}_1(T_p A_K))_{\text{tors}}.$$

On the other hand, $R^1 \text{Crys}_1(V_p A_K)$ is torsion-free (Proposition 3.4.11), so the kernel of f contains the full torsion subgroup of $R^1 \text{Crys}_1(T_p A_K)$. This proves the desired equality. \square

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