# Determinants and Non-Intersecting Paths

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# Aztec Basics

- Definition: The Aztec Diamond of order $n$ is the union of all unit squares in the plane whose coordinates lie on the integer lattice and whose *centers* satisfy the relationship $|x| + |y| \leq n$.
- Let $\mathrm{T}(A_n)$ denote the set of all domino tilings of the order $n$ Aztec diamond.
- $|\mathrm{T}(A_n)| = 2^{\frac{n(n+1)}{2}}$ (proof coming soon)

# Orientations

- Color the Aztec diamond in a checkerboard pattern so that the leftmost square of each row in the upper half of the diamond is *white*. This leads to four orientations of dominos.
- The horizontal tile with a white square on the left (located on the top of the diamond) is denoted North
- The Vertical tile with a white square on top (located on the left) is denoted West.
- The other orientations of horizontal (*black* on the left) and vertical (*black* on top) tiles are denoted South and East, respectively.

# Aztec Weightings

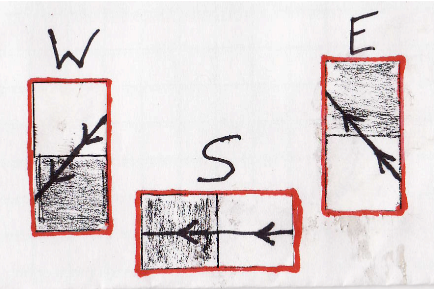
- Given a tiling $\tau \in \mathsf{T}(A_n)$, let $v(\tau)$ denote the number of vertical dominos in $\tau$.
- We can then define a probability measure on $\mathsf{T}(A_n)$ by giving horizontal dominos weight 1 and vertical dominos weight $\omega$. The weight of a tiling is then the normalized weight of the product of its tiles:

$$\mathbb{P}[\tau] = \frac{\omega^{v(\tau)}}{\sum_{\tau \in \mathsf{T}(A_n)} \omega^{v(\tau)}}.$$

- The uniform measure on tilings corresponds with $\omega = 1$.

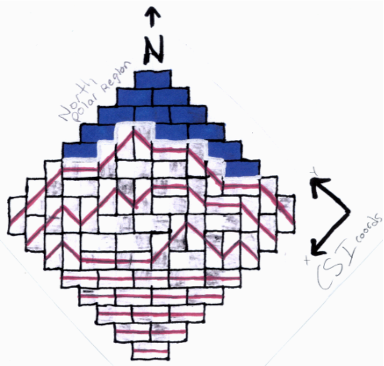
# Non-Intersecting Paths

- We can define a 1-1 mapping from T($A_n$) to families of $n$ non-intersecting lattice paths.
- Drawing the following line segments inside every South, East, and West domino in a tiling $\tau$ leads us to $n$ non-intersecting.



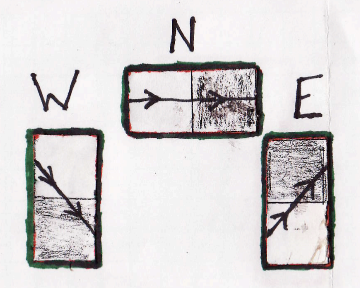
# Type 1 Paths

- These paths start at $A_j = (n+1-j, \frac{1}{2}-j)$ and end at $E_j = (-n-1+j, \frac{1}{2}-j)$ for $1 \leq j \leq n$. They are called type 1 DR-paths (the *DR* in honor of one of my amazing former professors from Georgia Tech, Dana Randall).



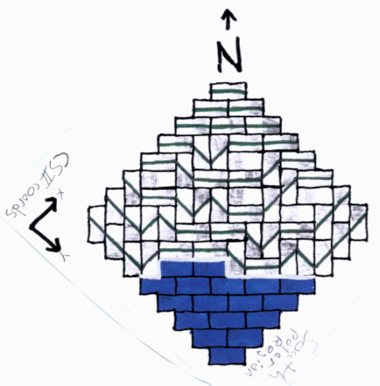
# More Non-Intersecting Paths

- Drawing a different set of line segments inside every **North**, East, and West domino in a tiling $\tau$ leads us to $n$ non-intersecting type 2 paths.



# Type 2 Paths

- These paths start at $A_j = (-n-1+j, j-\frac{1}{2})$ and end at $E_j = (n+1-j, j-\frac{1}{2})$ for $1 \leq j \leq n$. They are called type 2 DR-paths.



# Weighting of Paths

- The probability measure defined earlier on tilings can be viewed in another way as a probability measure on the DR-paths associated with a tiling.
- To do this we put a weight of 1 on any step passing through a horizontal domino and a weight of $\omega$ on any step passing through a vertical domino.
- The weight of a *single* path is then defined as the product of the weights of all of its steps.

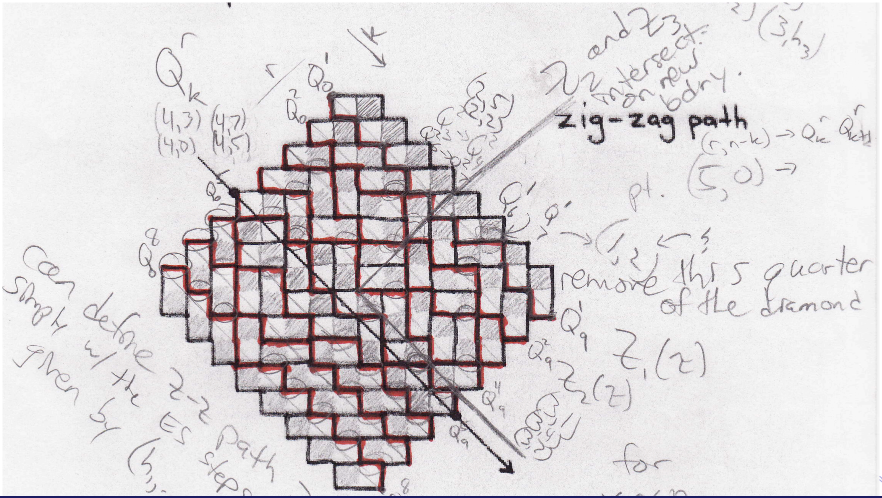
# Weighting of Collections of Paths

- The weight of a collection of $n$ non-intersecting DR-paths (type 1 or type 2) is the product of the weights of each of the $n$ paths.
- For any tiling $\tau$, the weight of both the $n$ type 1 DR-paths and $n$ type 2 DR-paths is $\omega^{v(\tau)}$, when the paths are obtained from the tiling.
- The weight of any set of non-intersecting DR-paths (generally a subset of the $n$ type 1 or $n$ type 2 paths) is the product of the weights of all of the steps on all of the paths.

# Zig-Zag Paths

- We can also define another collection of $n$ non-intersecting paths associated with any tiling, called zig-zag paths.
- These paths go from the upper left side of the diamond to the lower right side of the diamond, weaving a diagonal slice of white squares, so that they never intersect a domino.
- We denote the $r^{th}$ zig-zag path by $Z_r$, and it starts from the point $Q_0^r = (-r, n+1-r)$ and ends at the point $Q_{n+1}^r = (n+1-r, -k)$.
- $Z_r$ it consists of exactly $r$ East-South steps and $n+1-r$ South-East steps.

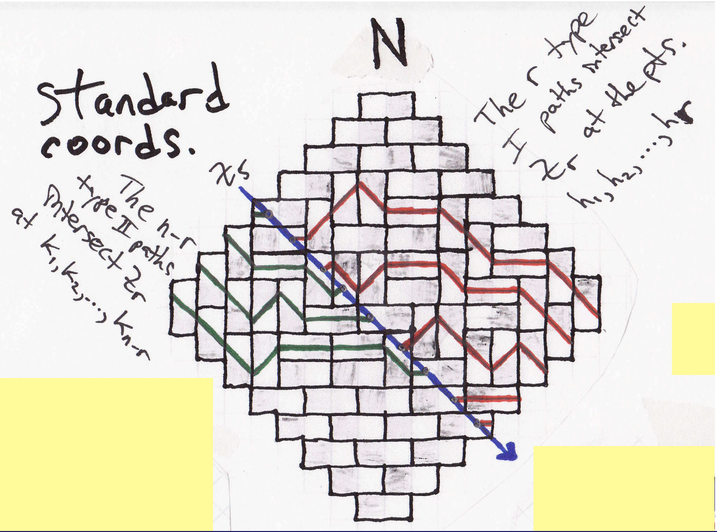
# Zig-Zag Paths - drawn in red



# Particle and Hole Configurations

- Given a zig-zag path, $Z_r$ we associate the $r$ ES-steps with locations of particles and the $n+1-r$ SE-steps with locations of holes.
- In an alternative coordinate system, the *particles* coordinates are $(r, h_j)$ for $1 \leq j \leq r$.
- We say that the particle configuration of the $r^{th}$ zig-zag path is $p(Z_r) = (h_1, \ldots, h_r)$, where $0 \leq h_1 < \ldots < h_r \leq n$.
- Similarly, in yet another coordinate system, we can place the *holes* at coordinates $(n+1-r, k_j)$ for $1 \leq j \leq n+1-r$.
- We call $h(Z_r) = (k_1, \ldots, k_{n+1-r})$ the hole configuration of the $r^{th}$ zig-zag path, where $0 \leq k_1 < \ldots < k_{n+1-r} \leq n$.

# Adding a Zig-Zag Path to an Aztec Diamond



# Johansson's Lemma

A lemma of Kurt Johansson says the following three useful things:

- The points $(r, h_j)$ for $1 \leq j \leq r$, are the intersection points of the first $r$ type 1 DR-paths and the $r^{th}$ zig-zag path, $Z_r$.
- The points $(n+1-r, k_j)$ for $1 \leq j \leq n+1-r$, are the intersection points of the first $n+1-r$ type 2 DR-paths and $Z_r$.
- $\{h_1, \ldots, h_r\} \cup \{k_1, \ldots, k_{n+1-r}\} = \{0, \ldots, n\}$.

# Question?

Given a fixed $r, 1 \leq r \leq n$ and some $h = (h_1, \ldots, h_r)$, where $0 \leq h_1 < \ldots < h_r \leq n$, what is the probability that a randomly selected tiling $\tau \in \mathrm{T}(A_n)$ has particle configuration $p(Z_r(\tau)) = h$?

$$\mathbb{P}[p(Z_r(\tau)) = h] = ?$$

# Answer

$$\mathbb{P}[p(Z_r(\tau)) = h] = \frac{r!}{Z_{r,n,q}} \Delta_r(h)^2 \prod_{j=1}^{r} \binom{n}{h_j} q^{h_j} p^{n-h_j},$$

where $q = \frac{\omega^2}{1+\omega^2}$, $p = 1 - q$, and $\Delta_r(h)$ is a van der Monde determinant.

The normalization constant $Z_{n,r,q}$ is given by the following formula;

$$Z_{n,r,q} = n! \left( \prod_{j=0}^{n-1} \frac{j!}{(r-j)!} \right) (r!)^n (pq)^{n(n-1)/2}.$$

# Next Time

- The Lindstrom-Gessel-Viennot theorem, corollary, and method for computing weights of collections of non-intersecting paths using determinants.
- An application of the L-G-V method to the Aztec diamond to prove the result on the previous slide.
- Corollaries of the result on the previous slide.
- The previous formulas relation to the Krawtchouk ensemble.