On ideals defining monomial curves that have arbitrarily large minimal sets of generators

Casey Warmbrand

April 27, 2006

This is paper is based on H. Bersinsky's article, "On prime ideals with generic zeros $x_i = t^{n_i}$ " [1]. Our goal will be to understand the proof laid out in this paper.

Let $n_i, 1 \leq i \leq r, r \geq 3$ be coprime natural numbers. We will consider the homomorphism $\theta : K[x_1, x_2, \ldots, x_r] \to K[t]$ defined by $x_i \mapsto t^{n_i}$, for an arbitrary field K. Let $P = ker(\theta)$. We will be considering the minimum number of generators for P as we vary the n_i's over the set of valid natural numbers. It is shown in Ernst Kunz's book entitled "Introduction to Commutative Algebra and Algebraic Geometry" [2], that for $r = 3$ the minimum number of generators is either 2 or 3. Here, we will first consider $r = 4$ and show that the minimum number of generators is unbounded. Finally, by induction, we will conclude that the minimum number of generators is unbounded for all $r \geq 4$.

1 Preliminaries

For a given r-tuple of natural numbers $\{n_i\}_{i=1}^r$ we denote the additive semigroup $\{z | z = \sum_{i=1}^r z_i n_i, z_i$ a non-negative integer $\}$ by $\langle n_1, n_2, \ldots, n_r \rangle$. We will also need that $(n_1, n_2, \ldots, n_r) = 1$ and $n_i \notin \langle n_1, \ldots, \hat{n_i}, \ldots, n_r \rangle$.

For K an arbitrary field, let $P = P(n_1, \ldots, n_r) = ker(\theta)$ for $\theta : K[x_1, x_2, \ldots, x_r] \to K[t]$, for t transcedental over K. We let m =the minimum number of generators for P denoted by $|P(n_1, \ldots, n_r)|$. $P(r)$ = the least upper bound of m over all r-tuples $(n_1, \ldots, n_r)$ satisfying the conditions above.

2 The Theorem

Theorem 2.1. *For* $r \geq 4$, $P(r) = \infty$.

The proof will follow by induction on r.

2.1 The Base Case

Initially let $r = 4$. Set $q_1 = q_2 + 1, q_2$ even, $q_2 \geq 4, d_1 = q_2 - 1$. Let:
$n_1 = q_1 q_2$
$n_2 = q_1 d_1$
$n_3 = q_1 q_2 + d_1$
$n_4 = q_2 d_1$.

Using Macaulay, for $q_2 = 4$, it is easily verified that these n_i's are co-prime and none are conatined in the ideal generated by the rest. Further, for $q_2 = 4$, Macaulay tells us that $m = 8$, and for $q_2 = 6, m = 12$. In fact this pattern continues for all $q_2 \leq 10$, where $m = 2q_2$. For $q_2 \geq 12$, Macaulay cannot compute the map θ.

$$A_1 = \{f_\mu | f_\mu = x_1^{\mu-1} x_3^{q_2-\mu} - x_2^{q_2-\mu} x_4^{\mu+1}, 1 \leq \mu \leq q_2\} \subset P$$

$$\{g_1 = x_1^{d_1} - x_2^{q_2}, g_2 = x_3 x_4 - x_2 x_1\} \subset P$$

Remark. It should be noted that g_1 and g_2 will always be two of the generators for P that Macaulay gives.

Lemma 2.2. $a_l n_l \in \langle \hat{n}_l, n_i, n_j, n_k \rangle$ *with* a_l *minimal* $\implies a_1 = d_1, a_2 = q_2, a_3 = d_1,$ *and* $a_4 = q_1$.

Proof. The idea here, is to express one n_i, as a linear combination of the other n_j's, and given the way these are defined in terms of q_1, q_2, and d_1, we can infer the necessary relations on their coefficients. For example, if $a_1 n_1 = c_2 n_2 + c_3 n_3 + c_4 n_4$, then $a_1 > c_3$ since $n_3 > n_1$, we can also see that $d_1 | (a_1 - c_3) \implies a_1 \geq d_1$. Thus a_1 minimal, gives us $a_1 = d_1$. □

$$A_2 = \{f | f = x_1^{b_1} x_4^{b_4} - x_2^{c_2} x_3^{c_3}, b_1, c_3 < d_1, f \in P\}$$

$$A = A_1 \cup A_2 \cup \{g_1, g_2\}, A' = A \cup \{f | f = x_1^{b_1} x_4^{b_4} - x_2^{c_2} x_3^{c_3}, f \in P\}.$$

Let (A) and (A') be the ideals generated by these sets. From Lemma 2 in [1] it is shown that $(A') = P \implies (A) = P$ and then in Lemma 3 also in [1] that $(A') = P$, thus $(A) = P$.

Next we must consider P^e, the extension of P in $K\{x_1, x_2, x_3, x_4\}$, the formal ring of power series.

Lemma 2.3. $\bar{f}_\mu \in P^e/P^e(x_1, x_2, x_3, x_4), 1 \le \mu \le q_2$, *are linearly independent over* K.

Proof. Let ϕ be the substitution homomorphism:
$x_1 \to T, x_2 \to S^{q_1}, x_3 \to TS, x_4 \to S^{q_2}$.

Then $\phi(g_1) = T^{d_1} - S^{q_1 q_2} = w, \phi(f_\mu) = S^{q_2-\mu}w, 1 \le \mu \le q_2$, and $\phi(g_2) = 0$.

$$f = x_1^{b_1} x_4^{b_4} - x_2^{c_2} x_3^{c_3} \in A_2 \implies \phi(f) = T^{b_1} S^{b_4 q_2} - S^{c_2 q_1 + c_3} T^{c_3}.$$

Since $d_1 \nmid q_1 q_2$, we have that:

$$b_1 n_1 + b_4 n_4 = c_2 n_2 + c_3 n_3$$
$$\iff$$
$$(b_1 - c_3) q_1 q_2 = (c_2 q_1 + c_3 - b_4 q_2) d_1$$
$$\iff$$
$$b_1 = c_3 \text{ and } b_4 q_2 = c_2 q_1 + c_3.$$

Substituting above we get $\phi(f) = 0$.

Applying ϕ to $\sum_{\mu=1}^{q_2} c_\mu f_\mu \in P^e(x_1, x_2, x_3, x_4), c_\mu \in K$, yields $\sum_{\mu=1}^{q_2} c_\mu S^{q_2-\mu} \in \phi(x_1, x_2, x_3, x_4)$. Since any non-zero polynomial $f \in K[S] \cap \phi(x_1, x_2, x_3, x_4)$ has degree $\ge q_2$, $c_\mu = 0 for 1 \le \mu \le q_2$. Thus we have shown linear independence over K. □

Corollary 2.4. $|P(n_1, n_2, n_3, n_4)| \ge q_2$.

Proof. If we have found q_2 many linear independent elements, then clearly any generating set must be at least this large. □

Corollary 2.5. $P(4) = \infty$.

Proof. Since all the n_i's are determined by q_2, just allow $q_2 \to \infty$ and we see that $P(r) = \infty$. □

2.2 The Induction Step

This is a very natural induction step. It seems intuitive to expect that if $\exists$ (n_1, n_2, n_3, n_4) s.t. $|P(n_1, n_2, n_3, n_4)| = m_0$, then $\exists$ $(\tilde{n_1}, \tilde{n_2}, \tilde{n_3}, \tilde{n_4}, \tilde{n_5})$ s.t. $|P(\tilde{n_1}, \tilde{n_2}, \tilde{n_3}, \tilde{n_4}, \tilde{n_5})| \ge m_0$. And in fact that is exactly what is shown by Bresinsky in [1].

If $P(r) = \infty$, for some $r \ge 4$. Let $|P(n_1, \ldots, n_r)| = m$. Set $n_i' = 2n_i, 1 \le i \le r$, and let n_{r+1}' be a sufficiently large prime. It follows that $|P(n_1', \ldots, n_{r+1}')| \ge m$. Thus, $P(r+1) = \infty$ also.

3 Geometric Interpretation

One geometric interpretation of $P = ker(\theta)$, is that it is a vanishing ideal of the curve with parametrization
$x_1 = t^{n_1}, x_2 = t^{n_2}, x_3 = t^{n_3}, x_4 = t^{n_4}$.

4 Examples

For $K = \mathbb{Q}$, we can use Macaulay to calculate the genrators of P exactly.

Using the construction from the proof and starting as small as possible, $q_2 = 4$. Thus $q_1 = 5, d_1 = 3$, and $(n_1, n_2, n_3, n_4) = (20, 15, 23, 12)$.

The generators of $P(20, 15, 23, 12)$ are:

1. $-x_4^5 + x_1^3$
2. $x_1x_2 - x_3x_4$
3. $-x_4^5 + x_2^4$
4. $-x_2x_4^4 + x_1^2x_3$
5. $-x_2^2x_4^3 + x_1x_3^2$
6. $-x_1x_4^4 + x_2^3x_3$
7. $-x_2^3x_4^2 + x_3^3$
8. $-x_1^2x_4^3 + x_2^2x_3^2$

Note that there are 8 generators for $q_2 = 4$ and that $1. = g_1$ and $2. = g_2$.

If we try $q_2 = 6$, then $q_1 = 7, d_1 = 5$ and $(n_1, n_2, n_3, n_4) = (42, 35, 47, 30)$.

Macaulay tells us that the generators of $P(42, 35, 47, 30)$ are:

1. $-x_4^7 + x_1^5$
2. $x_1x_2 - x_3x_4$
3. $-x_4^7 + x_2^6$
4. $-x_2x_4^6 + x_1^4x_3$
5. $-x_2^2x_4^5 + x_1^3x_3^2$

6. $-x_1x_4^6 + x_2^5x_3$
7. $-x_2^3x_4^4 + x_1^2x_3^3$
8. $-x_1^2x_4^5 + x_2^4x_3^2$
9. $\{-x_2^4x_4^3 + x_1x_3^4\}$
10. $\{-x_2^5x_4^2 + x_3^5\}$
11. $\{-x_1^3x_4^4 + x_2^3x_3^3\}$
12. $\{-x_1^4x_4^3 + x_2^2x_3^4\}$

Here we have 12 generators.

Again note that 1. $= g_1$ and 2. $= g_2$. Further, we notice the similarities between 3. $-$ 8. in the cases $q_2 = 4$ and $q_2 = 6$. Each term has one of the exponents raised by exactly 2. The last four generators (shown in curly braces) appear to be of a different form than the generators obtained when $q_2 = 4$.

Based on the calculations for $q_2 \leq 10$ it seems as though the number of generators is always $2q_2$, but this result was not proved in my paper, and to my knowledge there is no such proof. But it seems to be a reasonable conjecture, worthy of further investigation.

References

[1] H. Bersinsky. On prime ideals with generic zero. *Proceedings of the American Mathematical Society*, 47(2):329–332, 1975.

[2] Ernst Kunz. *Introduction to Commutative Algebra and Algebraic Geometry.* Birkhauser, 1933.