The Lindstrom-Gessel-Viennot Method

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k-vertices and k-paths

- Let D be an acyclic digraph and fix a positive integer, k.
- A *k-vertex* is a k-tuple of vertices of D.
- Given 2 k-vertices of D, $\mathbf{u} = (u_1, \ldots, u_k)$ and $\mathbf{v} = (v_1, \ldots, v_k)$, a *$k$-path* from $\mathbf{u}$ to $\mathbf{v}$ is a k-tuple $\mathbf{A} = (A_1, \ldots, A_k)$ such that A_i is a path from u_i to v_i consisting of directed edges in the graph D.

Disjoint k-paths

- We say that a k-path is *disjoint* if all of the paths A_i are vertex-disjoint.
- If $\pi \in S_k$ is a permutation in the permutation group on $\{1, \ldots, k\}$, then we use $\pi(\mathbf{v})$ to denote the k-vertex $(v_{\pi(1)}, \ldots, v_{\pi(k)})$.

Weightings

- Let us assign a weight to every edge in the digraph D.
- As we did last week, we define the weight of a path to be the product of the weights of its edges.
- The weight of a k-path is then the product of the weights of its components.

Collections of k-paths and their Weights

- Define $\mathrm{P}(u_i, v_j)$ to be the set of paths from u_i to v_j and $P(u_i, v_j)$ to be the **sum** of their weights. We can then define $\mathrm{P}(\mathbf{u}, \mathbf{v})$ and $P(\mathbf{u}, \mathbf{v})$ to be the set of all k-paths from $\mathbf{u}$ to $\mathbf{v}$ and the sum of their weights.
- Let $\mathrm{N}(\mathbf{u}, \mathbf{v})$ be the subset of $\mathrm{P}(\mathbf{u}, \mathbf{v})$ of all disjoint k-paths from $\mathbf{u}$ to $\mathbf{v}$ and $N(\mathbf{u}, \mathbf{v})$ to be the sum of their weights.

Lindström's Theorem

Given a permutation $\pi \in S_k$ we see that

$$P\left(\mathbf{u}, \pi(\mathbf{v})\right) = \prod_{i=1}^{k} P\left(u_i, v_{\pi(i)}\right).$$

Theorem (Lindström)

$$\sum_{\pi \in S_k} (\operatorname{sgn} \pi) N\left(\mathbf{u}, \pi(\mathbf{v})\right) = \left|P(u_i, v_j)\right|_1^k,$$

where we use $|m_{ij}|_r^s$ to denote the determinant of a matrix $M = (m_{ij})_{i,j=r,\ldots,s}$.

Proof

Expanding the determinant on the right hand side,

$$\sum_{\pi \in S_k} (\text{sgn } \pi) N(\mathbf{u}, \pi(\mathbf{v})) = \sum_{\pi \in S_k} (\text{sgn } \pi) P(\mathbf{u}, \pi(\mathbf{v})), \qquad (1)$$

is equivalent to the theorem. We will prove this by constructing a bijection $\mathbf{A} \to \mathbf{A}^{\star}$ from

$$\Omega = \cup_{\pi \in S_k} [\text{P}(\mathbf{u}, \pi(\mathbf{v})) - \text{N}(\mathbf{u}, \pi(\mathbf{v}))]$$

to itself, which satisfies the following three properties:

i $\mathbf{A}^{\star\star} = \mathbf{A}$.

ii The weight of $\mathbf{A}^{\star}$ equals the weight of $\mathbf{A}$.

iii For $\mathbf{A} \in P(\mathbf{u}, \pi(\mathbf{v}))$ and $\mathbf{A}^{\star} \in P(\mathbf{u}, \sigma(\mathbf{v}))$, (sgn σ) = -(sgn π).

Proof continued

- First note that the k-paths in Ω are all of the non-disjoint k-paths in $\mathrm{P}(\mathbf{u}, \mathbf{v})$.
- Given that this supposed bijection exists, we can then pair up the k-paths which appear in Ω as $\{\mathbf{A}, \mathbf{A}^{\star}\}$.
- Since $\{\mathbf{A}$ and $\mathbf{A}^{\star}\}$ have the same weight and the signs of π and σ differ by a minus, these two terms on the right hand side of (1) cancel with each other.
- This leaves only the terms involving disjoint k-paths from the determinant, which is the left hand side of (1).
- So the existence of this bijection would prove the theorem.

Constructing the Bijection

- Let $\mathbf{A} = (A_1, \ldots, A_k) \in \Omega$ be a non-disjoint k-path.
- Let i be minimal such that A_i intersects another path.
- Let w be the first vertex along the path A_i which is contained in another path.
- Let $j > i$ be minimal such that A_j contains the vertex w.
- We construct $A_i^\star$ by following A_i up to w and then following A_j to the end.
- Construct $A_j^\star$ by following A_j up to w and then following A_i to the end.
- Let $A_l^\star = A_l$ when $l \neq i, j$.

Verifying Properties i. ii. and iii.

- By construction (the use of the minimal *i* and *j* and choosing the first vertex *w*) property i., $\mathbf{A}^{\star\star} = \mathbf{A}$ is completely clear.
- Since both $\mathbf{A}$ and $\mathbf{A}^{\star}$ contain exactly the same edges they must have the same weights, so ii. is obvious as well.
- In order to show iii. we need to relate π and σ to each other when $\mathbf{A} \in P(\mathbf{u}, \pi(\mathbf{v}))$ and $\mathbf{A}^{\star} \in P(\mathbf{u}, \sigma(\mathbf{v}))$. From the construction it is clear that $\pi = \sigma \circ (ij)$, where (ij) denotes the transposition which flips *i* and *j* and leaves all others fixed. Thus (sgn π) = -(sgn σ).

Gessel and Viennot's Corollary

We call a pair of k-vertices *non-permutable* if $N(\mathbf{u}, \pi(\mathbf{v})) = \emptyset$ whenever π is not the identity permutation.

Corollary (Gessel and Viennot)

If $(\mathbf{u}, \mathbf{v})$ is non-permutable, then Lindström's theorem says

$$N(\mathbf{u}, \mathbf{v}) = |P(u_i, v_j)|_1^k. \tag{2}$$

It is this corollary which can be applied to our Aztec diamond.