

Aztec Diamonds

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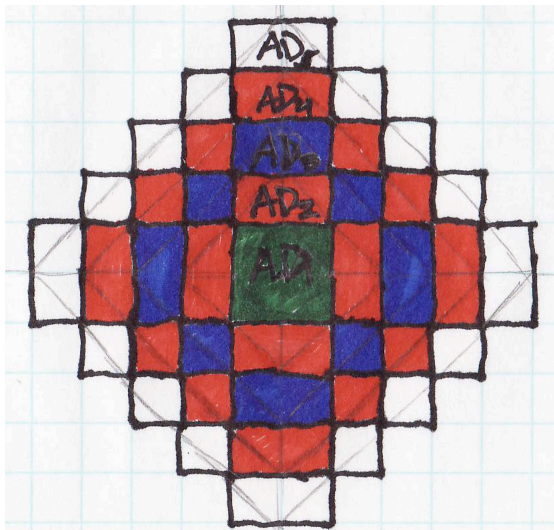
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29 Aug 2007

The Aztec Diamond

The Aztec Diamond of order n is defined as the union of all unit squares in the plane whose coordinates lie on the integer lattice and whose *centers* satisfy the relationship $|x| + |y| \leq n$.

Aztec Diamonds of orders 1-5



A Little Graph Theory

G is a graph if $G = \{V, E\}$ such that,

- ▶ V is a collection of vertices
- ▶ $E = \{(v_i, v_j) : v_i, v_j \in V\}$ is the set of edges of the graph

A graph $G = \{V, E\}$ is bi-partite if

- ▶ V can be partitioned into 2 non-empty sets B and R
- ▶ For every edge $e = (v_i, v_j) \in E$, either $v_i \in B$ and $v_j \in R$ or $v_j \in B$ and $v_i \in R$

Given a bi-partite graph, we may color the vertices in B black and the ones in R red, so that every edge of the graph joins a black vertex to a red one.

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Aztec Duals

Given any Aztec Diamond of order n , we may construct a graph as follows:

- ▶ Place a vertex in the center of every square in the Aztec Diamond.
- ▶ Place an edge between any 2 vertices whose corresponding squares are adjacent.

Because the tiling of the entire plane with unit squares whose corners lie on the integer lattice can be colored like a checkerboard, the dual graph of an Aztec Diamond as described above, is a bi-partite graph.

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Perfect Matchings

A perfect matching of a graph G , is a subgraph of G consisting of all of the vertices in V and a subset of edges such that each vertex is incident to exactly one edge. This is a pairing off of all the vertices of G .

Note: A graph must have an even number of vertices in order to have a perfect matching.

We will now see an example of the dual graph of an order 5 Aztec Diamond and a perfect matching of that graph.

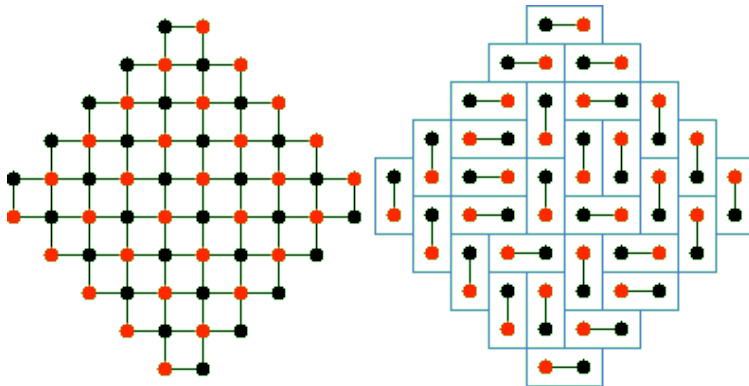
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Perfect Matching = Domino Tiling



North, East, South, and West Dominos

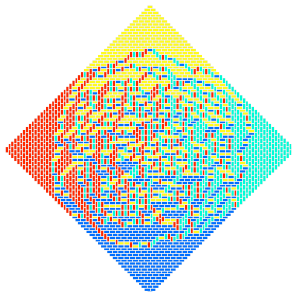
Color the squares of the Aztec Diamond black and white, so that the leftmost square of every row is white. A horizontal

domino whose left square is white is called a North domino and one whose left square is black is a South domino. A vertical

domino whose top square is white is a West domino and one whose top square is black is an East domino.

Pure Phases of a Tiling

A tiling of an Aztec Diamond permits (potentially) 4 different types of pure phases. One in each corner of the diamond, with each “direction” of domino.

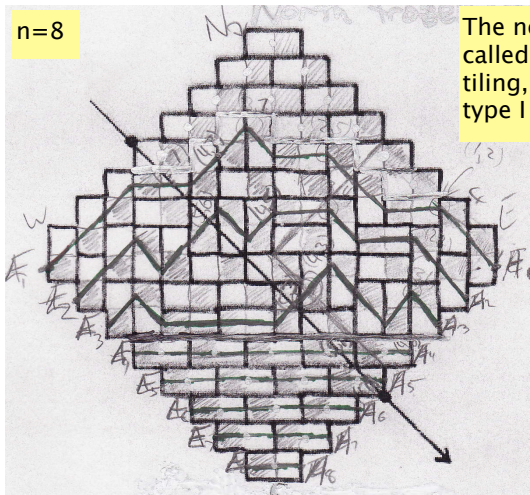


Non-Intersecting Paths

- ▶ For any order n Aztec Diamond there are a number of different sets of n non-intersecting paths which completely describe the tiling.
- ▶ The region above the north most “Type 1 DR-path” is called the *North Polar Region* or NPR for short. It consists of all North dominos in a brick like pattern, stemming from the northern tip of the diamond.
- ▶ The region below the south most “Type 2 DR-path” is called the *South Polar Region*.
- ▶ We could analogously define sets of non-intersecting paths which describe the boundary of the East and West frozen regions.

Type 1 Non-Intersecting Paths

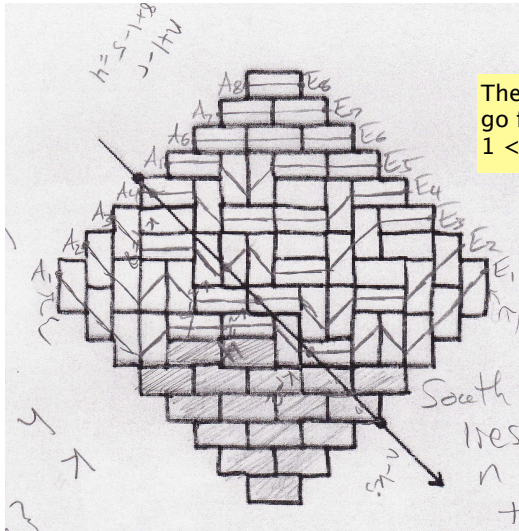
$n=8$



The north frozen region is called a pure phase of the tiling, it lies above the first type I path from A_1 to E_1

Type I
DR paths
From
 $A_i \rightarrow E_i$
 $1 \leq i \leq n$

Type 2 Non-Intersecting Paths



The n type II paths go from A_i to E_i for $1 \leq i \leq n$

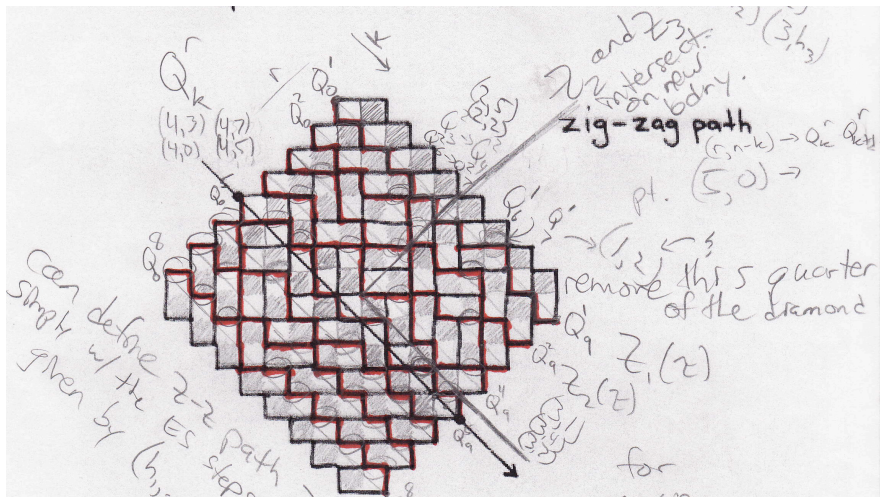
Creating Random Tilings

The following website, developed by Jim Propp, creates random tilings of many different regions. The software uses the Propp-Wilsons algorithm to construct a tiling of a given region with uniform probability (i.e. every tiling is equally likely).

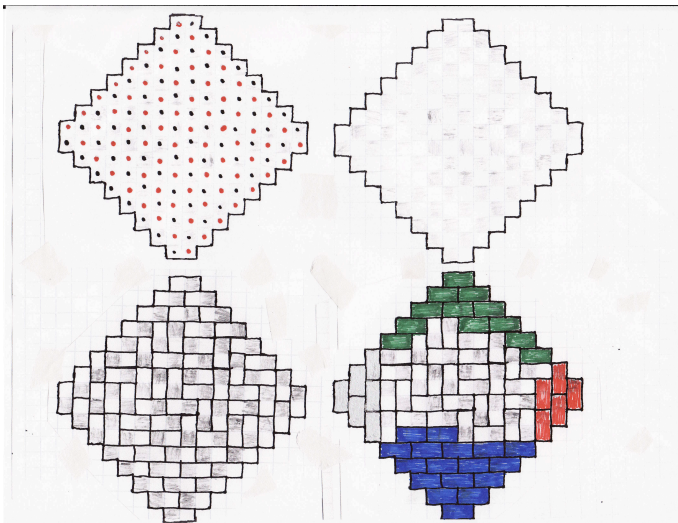
<http://www.math.wisc.edu/~propp/tiling/www/applets/>

The algorithm works by flipping a coin to decide whether to increase or decrease (the heights of) alternating cycles of both the left and right graphs, thus making them more and more similar at each step of the algorithm.

Zig-Zag paths and Particle Configurations



Handout1



Handout2

