Below is a (rough) chronological guide of the highlights from each section we have covered.

Some definitions will be given. In other places it may say "Define:" and then list a series of terms that were given with definitions in the section, usually in bold face in the text. You should try and first define these from memory, then later go back through the section(s) and check to make sure your definition didn't omit anything relevant.

Chapter 2 Review

Section 2.1

$A = \{a_{ij}\}_{i=1,j=1}^{m,n}$ is a representation of an $m \times n$ matrix
The diagonal entries are a_{ii} for $i = 1, \ldots, n$. They form the **main diagonal**
Define: identity matrix, zero matrix, sum of 2 matrices, scalar multiple of a matrix
Matrix multiplication: thought of initially as composition of 2 matrix transformations
For A an $m \times p$ matrix and B a $p \times n$ matrix, their product $AB = [Ab_1 \cdots Ab_p] = \{c_{ij}\}_{i=1,j=1}^{m,n}$, where c_{ij} is the (dot) product of the i^{th} row of A and the j^{th} column of B (if thought of as vectors in $\mathbb{R}^p$).
Properties of matrix multiplication (Linearity). In general matrix multiplication is NOT commutative
Define: A^k-powers of A, transpose of a matrix-rows become columns (columns become rows)
Properties of the transpose: Linearity, $(A^T)^T = A, (AB)^T = B^T A^T$

Section 2.2

Define: Inverses, singular/non-singular (non-invertible/invertible)
2×2 case: formula for the inverse and determinant
If A is a non-singular $n \times n$ matrix, then $Ax = b$ has a unique solution $\forall b \in \mathbb{R}^n$ (more equivalent conditions later, the invertible matrix theorem, IMT)
Properties of the inverse: $(A^{-1})^{-1} = A, (AB)^{-1} = B^{-1}A^{-1}, (A^{-1})^T = (A^T)^{-1}$
Elementary matrices- EA equals $A, m \times n$ after applying some row opration, where E is the $m \times m$ matrix gotten by doing the same row operation to I_m, the $m \times m$ identity matrix
Find A^{-1} by row reducing $[A\ I] \rightsquigarrow [I\ A^{-1}]$
Alternatively, the j^{th} column of A^{-1} is the solution to the systems $Ax = e_j$

Section 2.3

The Invertible Matrix Theorem (IMT)- For an $n \times n$ matrix A, TFAE facts/properties about A: (i) invertible (non-singular), (ii) row equilalent to I_n, (iii) has n pivots, (iv) $Ax = 0$ has a unique solution (the trivial one, $x = 0$), (v) linearly independent columns, (vi) the transformation $x \mapsto Ax$ is 1-1, (vii) $Ax = b$ is consistent $\forall b \in \mathbb{R}^n$, (viii) columns of A span $\mathbb{R}^n$, (ix) the transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$ (has kernel $= \{0\}$), (x) A^T is invertible, (xi) has determinant $\neq 0$

Chapter 3 Review

Section 3.1

The determinant of $A =$ cofactor expansion across any row or down any column
$\text{det}A = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj} = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$, where $C_{ij} = (-1)^{i+j}\text{det}A_{ij}$ with A_{ij} being the cofactor of the (i,j)-entry of A (the $(n-1) \times (n-1)$ matrix left when you blackout the i^{th} row and j^{th} column of A)

$$\text{det}A = \sum_{i=1 \mathbf{or} j=1}^{n} (-1)^{i+j}\text{det}A_{ij}$$

The formula above is for a cofactor expansion across row i when you sum from $j = 1$ to n with i fixed, and it is a cofactor expansion down column j when you sum from $i = 1$ to n with j fixed
Define: upper triangular, lower triangular, and diagonal matrices. What is a **triangular matrix**?
Theorem: A triangular matrix has determinant equal to the product of its entries on the main diagonal

Section 3.2

Effects of row operations on the determinant: What does each type of row op do to $\det A$?

If U is an echelon form of A, then $\det A = \begin{cases} (-1)^r * (\text{product of pivots in} U) & \text{if } A \text{ is non-singular} \\ 0 & \text{if } A \text{ is not invertible} \end{cases}$

Properties of the determinant: $\det A^T = \det A$ (think column operations), $\det(AB)=(\det A)(\det B)$

The determinant function is linear: $T_j(x) = \det([a_1 \cdots a_{j-1}\ x\ a_{j+1} \cdots a_n])$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$. $T_j(cx) = cT_j(x)$ and $T_j(u+v) = T_j(u) + T_j(v)$

HOWEVER: $\det(A+B) \neq \det A + \det B$

Problems

1. Find A^{-1} if $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & a \end{bmatrix}$, where a is any real number except 0.

2. Are the following 4 vectors linearly independent? If so, prove it, if not express one as a linear combination of the others.

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} -3 \\ 4 \\ -5 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix} \quad v_4 = \begin{bmatrix} 13 \\ 1 \\ -8 \\ 4 \end{bmatrix}$$

3. Find the inverse of the following 3×3 matrix, $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$.

4. Find the following determinants: (i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (ii) $\begin{bmatrix} a & b & c \\ 1 & 1 & 0 \\ d & e & f \end{bmatrix}$

(iii) $\begin{bmatrix} -10 & 4 & 1 & -2 & -2 \\ 20 & -8 & -2 & 4 & 4 \\ 3 & -4 & -1 & 0 & 5 \\ 2 & 0 & 0 & -1 & 1 \\ 7 & 12 & 3 & 0 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ b & c & 1 & 0 & 0 \\ d & e & f & 1 & 0 \\ g & h & i & j & 1 \end{bmatrix}$

5. Using a matrix inverse, find the solution(s) to the following system for any three real parameters a, b, and c. Note that your solutions for x, y, and z will depend on the numbers a, b, and c, thus your answer should be formulas for the unknown variables including these real parameters. Hint: First write the system as a matrix equation.

$$\begin{array}{rcrrcl} x & + & 2y & +3z & = & a \\ & & y & +4z & = & b \\ 5x & + & 6y & & = & c \end{array}$$

6. Find the determinant of the following matrix, where a is any real number: $A = \begin{bmatrix} 2 & -3 & 7 \\ 1 & 1 & a \\ 4 & -2 & 2 \end{bmatrix}$.

7. Prove that any 2 $n \times n$ non-singular matrices are row equivalent.

8. Let U be any upper triangular matrix with non-zero diagonal entries. Explain why U must be invertible and explain why U^{-1} must also be upper triangular.