Below is a (rough) chronological guide of the highlights from each section we have covered.

Some definitions will be given. In other places it may say "Define:" and then list a series of terms that were given with definitions in the section, usually in bold face in the text. You should try and first define these from memory, then later go back through the section(s) and check to make sure your definition didn't omit anything relevant.

Chapter 4 Review

Section 4.1

A **Vector Space** is a non-empty set V with 2 defined operations: addition of elements in V and scalar multiplication of elements in V by real numbers in $\mathbb{R}$, subject to the following axioms or restrictions about how these operations should work and what elements need to be in V:

$\forall u, v, w \in V, \forall c, d \in \mathbb{R}$

$u + v \in V$ - addition is closed

$u + v =$??? - addition is commutative

$(u + v) + w = u + (v + w)$ - addition is associative

$\exists 0 \in V$ such that ??? $=$??? $= u$ - existence of an additive identity

$\exists -u \in V$ such that ??? $= 0$ - existence of additive inverses

$cu \in V$ - ??? is closed

$c(u + v) =$??? + ??? - 1st distributive law for the operations

$(c + d)u =$??? + ??? - 2nd distributive law for the operations

$c(du) = (cd)u$ - scalar multiplication is associative

$1u = u$ - $1 \in \mathbb{R}$ is the identity for scalar multiplication

For each of the axioms above, wherever you see "???" fill in that axiom's missing condition.

The above axioms imply a few other properties about the elements in a vector space, like $(-1)u = -u, 0u = 0, c0 = 0$. Can you find other universal conditions that are true for certain special elements like $0, 1, -1$ perhaps, and other arbitrary elements like u, v in V and arbitrary scalars like c, d in $\mathbb{R}$ in any vector space V?? (Examples: $u + u = 2u$ or $(-1)(-u) = u$)

Some examples of vector spaces we have looked at are $\mathbb{R}^n, \mathbb{P}_n$, certain function spaces, the space of all rotations in the plane, symmetric matrices, etc. What other examples can you come up with?

Define: Subspace (subset of a vector space that satisfies which axioms of the parent superspace?)

Examples: V and $\{0\}$ are always subspaces of V.$\mathbb{P}_n$ is a subspace of $\mathbb{P}$. Planes through the origin in $\mathbb{R}^3$ are subspaces that are isomorphic to $\mathbb{R}^2$. Lines through the origin in $\mathbb{R}^2$ are subspaces that are isomorphic to $\mathbb{R}$. Spans or generated sets are subspaces of any vector space that contains each of the elements you have taken the span of (i.e. the generating set).

Theorem: The span of any number of vectors forms a subspace of a vector space that contains all of these vectors. How is this theorem proved?

Section 4.2

Define: Null space (solutions to the homogeneous system) and Column space (all linear combinations of the columns) of a matrix.

Theorems: The null space and column space of a matrix are subspaces. How are these proven?

How does one find both Nul A and Col A?

Give the general definition of a linear transformation (without matrices) between vector spaces.

Define: Kernel and Range of a linear transformation between vector spaces.

If the transformation is between real vector spaces, then a standard matrix exists, and then the kernel is the null space of the standard matrix and the range is the column space of the standard matrix.

Section 4.3

Redefine linearly independent and dependent for an arbitrary vector space (possibly not $\mathbb{R}^n$).
A **Basis** for a vector space V is a linearly independent set of vectors in V whose span is V.
Examples: What are the standard bases for $\mathbb{R}^n$ and $\mathbb{P}_n$?
Spanning set theorem: removing linearly dependent elements from a set doesn't that set's span.
How do you find bases for the Null and Column spaces?? The pivot columns of A form a basis for $\mathrm{Col\,A}$ and the parametric vector form of solutions to $Ax = 0$ yields a basis for $\mathrm{Nul\,A}$.

Section 4.4

Unique representation in a basis implies the basis coordinate map is an isomorphism. Why?
$P_{\mathcal{B}} = [b_1 \cdots b_n]$ changes the $\mathcal{B}$-coordinates of any $x \in \mathbb{R}^n$ into the standard coordinates. i.e. $x = P_{\mathcal{B}}[x]_{\mathcal{B}}$. Since $\mathcal{B}$ is a basis, $P_{\mathcal{B}}^{-1}$ exists and defines an isomorphism from $\mathbb{R}^n$ to itself (changing the coordinates), by the Invertible Matrix Theorem.
Examples: Find bases, basis-coordinates of certain elements, and elements with given basis-coordinates for the following examples: polynomials in $\mathbb{P}_n$, symmetric matrices, subspaces of $\mathbb{R}^n$.

Section 4.5

Every basis has the same size. That intrinsic size corresponding to the vector space is define to be the **dimension**. Sets of more than this size are necessarily linearly dependent.
What is the difference between finite and infinite dimensional vector spaces?
Classify all subspaces of $\mathbb{R}^3$.
$\dim H \leq \dim V$ if H is a subspace of V, leads us to the basis theorem.
Basis theorem: Given a p dimensional vector space V, (i) any linearly independent set of p vectors forms a basis for V (maximally independent) (ii) any set of p elements that spans V is a basis for V (minimally spanning).
$\dim(\mathrm{Nul\,A})$ and $\dim(\mathrm{Col\,A})$ are given in terms of the number of free and basic variables in the homogeneous system $Ax = 0$, i.e. pivot and non-pivot columns of A. Given A and an echelon form of A, we can quickly find $\mathrm{Nul\,A}$ and $\mathrm{Col\,A}$ (in terms of a basis for each).

Section 4.6

Define: Row space of a matrix.
Note that row operations don't change the row space, unlike the column space. What does this mean about linear dependence relations among rows of row-equivalent matrices? What about linear dependence relations among columns of row-equivalent matrices?
The above fact implies that $\mathrm{Row}A$ is also easily found from any echelon form of A (This time taken from U the echelon form, not A itself as was done when finding $\mathrm{Col\,A}$).
Despite these differences between the row and column spaces, their dimensions are both the same as the number of pivots in A. We call this number the **rank** of A.
Since the dimension of the null space is given by the free variables in $Ax = 0$ (i.e. the number of non-pivots of A), we have the Rank Theorem: For A an $m \times n$ matrix, $\mathrm{rank}A + \mathrm{nullity}A = n$.
For $Am \times n, \mathrm{Col\,A}$ is a subspace of $\mathbb{R}^m$ and $\mathrm{Row}A$ and $\mathrm{Nul\,A}$ are subspaces of $\mathbb{R}^n$.
IMT(continued), TFAE: A invertible, Columns of A form a basis for $\mathbb{R}^n, \mathrm{Col\,A} = \mathbb{R}^{\mathrm{n}}$, $\mathrm{rank}A = n, \mathrm{Nul\,A} = \{0\}$ (etc.??)

Problems

1. Find Nul A, where $A = \begin{bmatrix} -1 & 0 & 3 & -2 \\ 6 & 0 & 2 & -8 \\ 4 & 0 & -8 & 4 \end{bmatrix}$

2. Let $A = \begin{bmatrix} -4 & -3 & 0 & 4 \\ 7 & 5 & 1 & 2 \\ 2 & 3 & -6 & 3 \\ 1 & 1 & -1 & -1 \end{bmatrix}$

   (a) Find a basis for the column space of A.

   (b) Find a basis for the row space of A.

   (c) What is the rank of A.

3. Consider the set of vectors of the form $\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{bmatrix}$, where a, b, and c are any real numbers. Find a basis of vectors in $\mathbb{R}^4$ that spans this subspace. Make sure your answer is a BASIS!

4. Let $W = \left\{ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} : \begin{array}{c} 2w-3y+z=0 \\ x-3y+4z=0 \end{array} \right\}$, where $w, x, y,$ and z are real numbers. Show W is a subspace in 2 different ways, by finding a matrix whose column space is W and another matrix whose null space is W.

5. Explain what is wrong with the following logic: $f(t) = 2+t$ and $g(t) = wt+t^2$, thus $g(t) = t * f(t)$ and so $\{f, g\}$ is linearly dependent since g is a multiple of f.

6. Consider a linear transformation, T from $\mathbb{P}_2$ to $\mathbb{R}^2$ defined by $T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$. For example, if $p(t) = 3+5t+6t^2$ then $T(p) = \begin{bmatrix} 3 \\ 14 \end{bmatrix}$.

   (a) Show T is a linear transformation. Hint: Consider $T(p+q)$ and $T(cp)$ for arbitrary polynomials $p, q \in \mathbb{P}_2$ and an arbitrary scalar c, to show the linearity properties are satisfied by T.

   (b) Find a polynomial p in $\mathbb{P}_2$ that spans the kernel of T.

   (c) Describe the range of T.

7. Show that is P is an invertible $m \times m$ matrix, then the rankAP=rank$A$.