Below is a (rough) chronological guide of the highlights from each section we have covered.

Some definitions will be given. In other places it may say "Define:" and then list a series of terms that were given with definitions in the section, usually in bold face in the text. You should try and first define these from memory, then later go back through the section(s) and check to make sure your definition didn't omit anything relevant.

Chapter 1 Review

Section 1.1

Linear Equation:
Write in the formula for a generic linear equation above, with the variables $x_1, x_2, \ldots, x_n$, with scalar coefficients $a_1, a_2, \ldots, a_n$ and another scalar (real number) b.
A **linear system** or **system of linear equations** is a collection of liner equations of the form above, each having the *same* variables and (possibly) *different* sets of scalars.
Solution sets: What are the three types of solution sets a linear system can have?
Define: coefficient matrix, augmented matrix
Elementary Row Operations: What are the three types? (Names, Notation, and Examples)
Existence and Uniqueness: What do these terms mean in the context of solutions to a linear system?

Section 1.2

What are Echelon Form and Reduced Echelon Form? Which is unique? (why?)
Row Reduction Algorithm (forward and backward phases) - uses concepts of a **pivot**, both position and column, and **leading entry**, define these along with basic variable and free variable
How do we tell which variables in a linear system are **basic** and which are **free**?
Solution sets: What is the general solution to a linear system and how is it found?
Existence and Uniqueness Theorem: Classifies when each of the three types of solutions occur
Solving linear systems using row reduction (finding echelon or reduced echelon forms, perhaps)

Section 1.3

Introduction to (column) **vectors**. A vector in $\mathbb{R}^n$ is an ordered n-tuple of real numbers.
How do we represent vectors in $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$?
Define: Sum and scalar multiple (for vectors) -
Given vectors $u, v \in \mathbb{R}^n$ and a scalar $c \in \mathbb{R}$, what are $u + v$ and cu?
Geometrically how can we interpret these two operations?
for addition think parallelogram, for scalar multiplication think line through the origin
Algebraic properties of vectors in $\mathbb{R}^n$ - Note how these relate to the axioms we learned later in the definition of a vector space from chapter 4!
Define: **LINEAR COMBINATION** and **SPAN** (2 of the most important concepts in the math215)

Section 1.4

Matrix-vector multiplication: Ax is defined as a linear combination of the columns of the matrix A with weights being the entries of the vector x
The matrix equation $Ax = b$: an organized method of representing a system of linear equations, can also be denoted by the augmented matrix $[A\ b]$
Theorem 4 - for an $m \times n$ matrix A, TFAE (the following are equivalent): (i) columns of A span $\mathbb{R}^m$, (ii) consistency of $Ax = b, \forall b \in \mathbb{R}^m$, (iii) A has m pivots, one in every row
Row-vector rule for matrix-vector products - Ax has i^{th} entry given by the dot product of the i^{th} row of A with the vector x. Write this out more formally in general, then use it for a few examples
Properties of the matrix-vector product Ax - Write down these oft repeated **Linearity Properties**!
What are the definitions for I_n, the $n \times n$ identity matrix, and $0_{m\times n}$, the zero matrix of size $m \times n$?

Section 1.5

Define: Homogeneous linear system, trivial and non-trivial solutions

$Ax = 0$ has infinitely many solutions (i.e. a non-trivial solution) if and only if free variable(s)

What is **Parametric Vector Form** (PVF)? How does one get a basis for the solution set?

Non-homogeneous solutions - The system $Ax = b$ has solutions $x = p + x_h$, where p is any particular solution (i.e. $Ap = b$) and x_h is the general solution to the associated homogeneous system

Section 1.6

Applications of Linear Systems - What kind of real-world problems can linear algebra help us to solve?

Section 1.7

Define: Linearly independent and linearly dependent, linear dependence relation

$Ax = 0$ has only the trivial solution if and only if the columns of A are linearly independent

When are sets of 1 and 2 vectors independent/dependent?

Theorem: An ordered set of vectors is linearly dependent if and only if 1 vector is a linear combination of all the others...in fact, 1 vector is necessarily a linear combination of the *preceeding* vectors

Theorem: A set with more vectors than the dimension of the vectors is linearly dependent

i.e. any p vectors in $\mathbb{R}^n$ with $p > n$ must satisfy some linear dependence relation

A set containing the zero vector is: (i) independent, (ii) dependent, or (iii) can't be determined?

Section 1.8

A **transformation** (also called a **function** or **mapping**) T from a set D to a set C, is a rule that assigns a single element of C to every element of D (i.e. for each $x \in D$, there is a single element of C that T picks out, namely $T(x)$)

Define: Domain-D, Codomain-C, Range-$T(D)$, image-$T(x)$, and preimage-x

Matrix transformations: The domain D and codomain C are real vector spaces, the transformation is given by $x \mapsto Ax$, where A, the **standard matrix**, is $m \times n$. What size dimensions are the domain and codomain? (i.e. D and C are both $\mathbb{R}^k$ for what two values of k?)

What has to be true for a transformation to be **Linear**? Show that for a linear transformation $T : V \to W$, (i) $T(0) = 0$ and (ii) $T(cu + v) = cT(u) + T(v), \forall u, v \in V, \forall c \in \mathbb{R}$

Generalize: what does a linear transformation do to an arbitrary linear combination in V

Matrix transformations are necessarily linear transformations (linearity properties of the product AX)

Section 1.9

The standard matrix A is given by what it does to the standard basis elements of the domain, $\mathbb{R}^n$ (i.e. $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ linear $\implies T(x) = Ax$, where $A = [T(e_1) \cdots T(e_n)]$

Classify common geometric linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. 5 types of reflections; horizontal and vertical versions of each of the following: contractions/expansions, shears, and projections; counter-clockwise and clockwise rotations by an angle θ

Define: One to one (1-1) and onto, for any function (not just between real vector spaces)

Theorem: A linear transformation is 1-1 if and only if $T(x) = 0$ has only the trivial solution

Theorem: Let $T(x) = Ax$, with $A, m \times n$ then: (i) T is onto if and only if the columns of A span $\mathbb{R}^m$, (ii) T is 1-1 if and only if the columns of A are linearly independent

Given 2 functions, f and g, let $f \circ g$, called the composition of f and g, be the function that first maps x to $g(x)$ and then maps $g(x)$ to $f(g(x))$. Thus, in order for $f \circ g$ to be well-defined we must have that the domain of f contains the range of g

Given 2 linear transformations, T_1 and T_2 with standard matrices A_1 and A_2, respectively, if A_1 is $m \times p$ and A_2 is $p \times n$, then $T_2 : \mathbb{R}^n \to \mathbb{R}^p$ and $T_1 : \mathbb{R}^p \to \mathbb{R}^m$, thus $T_1 \circ T_2 : \mathbb{R}^n \to \mathbb{R}^m$ is well-defined and is given by $T_1 \circ T_2(x) = T_1(T_2(x)) = T_1(A_2x) = A_1A_2x$

Problems

1. Consider the problem of determining whether the following linear system is consistent:

$$\begin{array}{rcrcrcr} 4x_1 & - & 2x_2 & + & 7x_3 & = & -5 \\ 8x_1 & - & 3x_2 & + & 10x_3 & = & -3 \end{array}$$

   (a) Define appropriate vectors, and restate the problem in terms of linear combinations. Then solve the problem.

   (b) Define an appropriate matrix, and restate the problem using the phrase "columns of A."

   (c) Deefine an appropriate linear transformation T using the matrix from part (b), and restate the problem in terms of T.

2. Describe the possible echelon forms of the matrix A. Use the notation from section 1.2 or your 1st quiz.

   (a) A is a 3×2 matrix.

   (b) A is a 3×3 matrix whose columns span $\mathbb{R}^3$.

3. Consider the system:

$$\begin{array}{rcrcr} x & + & hy & = & 3 \\ 2x & + & 4y & = & k \end{array}$$

   (a) Find *all* h and k such that the system has exactly one solution.

   (b) Find *all* h and k such that the system is inconsistent.

   (c) Find a specific h and k such that the system has infinitely many solutions.

4. Write the *reduced* echelon form of a 3×3 matrix A such that the first 2 columns of A are pivot columns and

$$A \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

5. Determine the value(s) of a such that $\left\{ \begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ 2a+3 \end{bmatrix} \right\}$ is linearly independent.

6. Suppose $\{v_1, v_2\}$ is linearly independent in $\mathbb{R}^n$. Show that $\{v_1, v_1 - v_2\}$ is also linearly independent.

7. Explain why a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is the composition of 2 transformations which have standard matrices cannot be 1-1 if either one of the transformations is a projection onto the y-axis. Also explain why it cannot be onto.