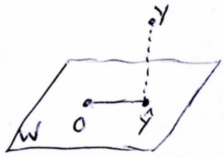
6.3 Orthogonal Projections



$\hat{y}$ is the unique vector in W such that (i) $y-\hat{y}$ is $\perp$ to W
and (ii) $\hat{y}$ is the closest vector to y in W

Suppose we have an orthogonal basis for $\mathbb{R}^n$ $\{u_1,\ldots,u_n\}$ it can be divided into two sets, one that spans W and the rest which span $W^\perp$.

Then a lin. comb. for y in the basis has 2 parts, 1 in W and 1 in $W^\perp$.

EX let $\{u_1,u_2,u_3,u_4,u_5\}$ be an orthogonal basis for $\mathbb{R}^5$

$y=c_1u_1+c_2u_2+\cdots+c_5u_5$ $\quad W=\text{Span}\{u_1,u_4\}$

so $y=z_1+z_2$ w/ $z_1=c_1u_1+c_4u_4$ $\quad z_2=c_2u_2+c_3u_3+c_5u_5$

where $z_1\in W$ and $z_2\in W^\perp$ (why is z_2 in $W^\perp$?)

Thm 8 Only need a basis for W (not all of $\mathbb{R}^n$)

If W is a subspace of $\mathbb{R}^n$, every $y\in\mathbb{R}^n$ can be written uniquely as $y=\hat{y}+z$ where $\hat{y}\in W$ and $z\in W^\perp$

In fact, if $\{u_1,\ldots,u_p\}$ is an orthogonal basis for W, then

$$\hat{y}=\frac{y\cdot u_1}{u_1\cdot u_1}u_1+\cdots+\frac{y\cdot u_p}{u_p\cdot u_p}u_p \quad\text{and}\quad z=y-\hat{y}.$$

$\hat{y}$ is the orthogonal projection of y onto W. written $\text{proj}_W y$.

pf similar to pf in 6.2 for existence. To see uniqueness suppose $y=\hat{y}+z$ and $y=\hat{y}_1+z_1$ w/ $\hat{y},\hat{y}_1\in W$ and $z,z_1\in W^\perp$ then $\hat{y}+z=\hat{y}_1+z_1 \Rightarrow \hat{y}-\hat{y}_1=z_1-z=v$ some $v\in\mathbb{R}^n$
but $v\in W$ and $v\in W^\perp \Rightarrow v\cdot v=0 \Rightarrow v=0 \Rightarrow \hat{y}=\hat{y}_1$ and $z_1=z$ so it is unique.

Thus $\text{proj}_W y$ depends only on W and its basis (not on the basis for $W^\perp$, the rest of $\mathbb{R}^n$)

EX2 recall $u_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $u_2=\begin{bmatrix}-2\\1\\1\end{bmatrix}$ $u_3=\begin{bmatrix}0\\1\\-1\end{bmatrix}$ was an orth. basis for $\mathbb{R}^3$

let $W=\text{Span}\{u_1,u_3\}$ write $y=\begin{bmatrix}1\\2\\3\end{bmatrix}$ as the sum of a vector in W and one in $W^\perp$

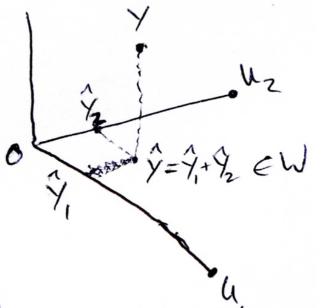
$$\hat{y}=\frac{y\cdot u_1}{u_1\cdot u_1}u_1+\frac{y\cdot u_3}{u_3\cdot u_3}u_3=\frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}+\frac{-1}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix}=\begin{bmatrix}2\\3/2\\5/2\end{bmatrix} \quad\text{and}\quad z=y-\hat{y}=\begin{bmatrix}1\\2\\3\end{bmatrix}-\begin{bmatrix}2\\3/2\\5/2\end{bmatrix}=\begin{bmatrix}-1\\1/2\\1/2\end{bmatrix}$$

check that $z\perp W$ (can show $z\cdot u_1=z\cdot u_3=0$ or that $z\in\text{Span}\{u_2\}=W^\perp$)

$$\text{thus}\quad \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{y}=\underbrace{\begin{bmatrix}2\\3/2\\5/2\end{bmatrix}}_{\hat{y}}+\underbrace{\begin{bmatrix}-1\\1/2\\1/2\end{bmatrix}}_{z}$$

Geometrically if $W = \mathrm{Span}\{u_1, \dots, u_p\}$ an orth. basis each is like an axis for the space W.

consider $W = \mathrm{Span}\{u_1, u_2\}$ in $\mathbb{R}^3$



Note this pf will show that $\hat{y}$ is actually independent of what choice of orth. basis for W you've made.

<u>properties of orth. projs.</u>

if $y \in W = \mathrm{Span}\{u_1, \dots, u_p\}$ then $\mathrm{proj}_W y = y$

<u>Thm 9</u> Best approximation thm.

Suppose W subsp. of $\mathbb{R}^n$, $y \in \mathbb{R}^n$, $\hat{y} = \mathrm{proj}_W y$

Then $\hat{y}$ is the closest pt. in W to y, in the sence that

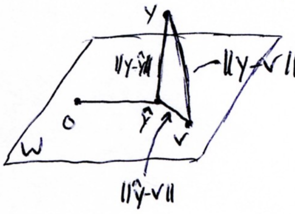
$$\|y - \hat{y}\| < \|y - v\| \quad \forall v \neq \hat{y} \text{ in } W$$

$\hat{y}$ is called the <u>best approx. to y by elements of W.</u>

pf let $v \in W$ then $\hat{y} - v \in W$ and $y - \hat{y} \in W^\perp$ by thm 8. so $y - \hat{y}$ and $\hat{y} - v$ are orthogonal

thus $y - v = (y - \hat{y}) + (\hat{y} - v)$ and by pyth. thm. $\|y - v\|^2 = \|y - \hat{y}\|^2 + \|\hat{y} - v\|^2$

so if $v \neq \hat{y}$ then $\hat{y} - v \neq 0 \Rightarrow \|\hat{y} - v\|^2 > 0 \Rightarrow \|y - \hat{y}\| < \|y - v\|$ as was to be shown.



<u>EX 3</u> if $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $W = \mathrm{Span}\{u_1, u_3\}$

then the closest pt. in W to $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\hat{y} = \begin{bmatrix} 2 \\ 3/2 \\ 5/2 \end{bmatrix}$ (as in ex 2)

<u>EX 4</u>

The distance from y to a subspace W is given as the min. of the dist. from y to $w \in W$ over all $w \in W$, thus since $\hat{y}$ is the minimizer

$\mathrm{dist}(y, W) = \mathrm{dist}(y, \hat{y}) = \|y - \hat{y}\|$

suppose the basis was o-n instead of just orthogonal, what changes in the thms, we've had in the section so far?

if $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ find $\mathrm{dist}(y, W)$ where $W = \mathrm{Span}\{u_1, u_2\}$

<u>soln</u>

first find $\hat{y} = \mathrm{proj}_W y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{12}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9/2 \\ 9/2 \end{bmatrix}$

and $y - \hat{y} = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix}$ has length $\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{1}{\sqrt{2}}$

Thm 10 If $\{u_1, \dots, u_p\}$ is an o-n basis for W a subspace of $\mathbb{R}^n$, then

$\mathrm{proj}_W y = (y \cdot u_1) u_1 + \dots + (y \cdot u_p) u_p$ (4)

and if $U = [u_1 \cdots u_p]$

then $\mathrm{proj}_W y = U U^T y$ (5)

pf (4) is immediately clear from thm. 8 b/c $u_j \cdot u_j = 1 \ \forall j$

we also see $\mathrm{proj}_W y$ is a lin. comb. of cols of U in (4) w/ wts. $(y \cdot u_j) = u_j^T y \ \forall j = 1, \dots, p$ which are entries in $U^T y \Rightarrow (5)$