6.1 Inner Product, Length, orthogonality

generalize well known notions from $\mathbb{R}^2 + \mathbb{R}^3$ to $\mathbb{R}^n$

Inner Product

given $u,v \in \mathbb{R}^n$ u,v are $n\times 1$ matrices (vectors).

So u^T is a $1\times n$ matrix and u^Tv a 1×1 matrix, a scalar.

we call u^Tv the <u>inner product</u> of u and v, sometimes written $u\cdot v$ or called the <u>dot product</u> of u and v or written $\langle u,v\rangle$

if $u=\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}$ $v=\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}$ Then $u^Tv=[u_1 \cdots u_n]\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}=u_1v_1+\cdots+u_nv_n=u\cdot v$

<u>Ex</u> find $u\cdot v$ and $v\cdot u$ for $u=\begin{bmatrix}2\\-1\\3\end{bmatrix}$ $v=\begin{bmatrix}4\\1\\-2\end{bmatrix}$

$u\cdot v=[2\ -1\ 3]\begin{bmatrix}4\\1\\-2\end{bmatrix}=2(4)+(-1)(1)+(3)(-2)=1$

$v\cdot u=[4\ 1\ -2]\begin{bmatrix}2\\-1\\3\end{bmatrix}=4(2)+(1)(-1)+(-2)(3)=1$

the dot product is commutative!

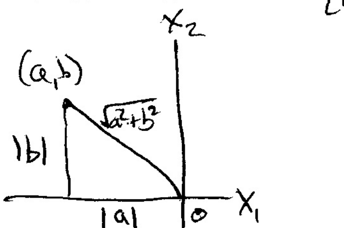
<u>Thm 1</u> (properties of the dot product)
let $u,v,w \in \mathbb{R}^n$ (vectors in $\mathbb{R}^n$) $c\in\mathbb{R}$ (scalar)

a) $u\cdot v=v\cdot u$ commutative
b) $(u+v)\cdot w=u\cdot w+v\cdot w$ distributive
c) $(cu)\cdot v=c(u\cdot v)=u\cdot(cv)$ scalar mult. commutes
d) $u\cdot u\geq 0$ and $u\cdot u=0$ iff $u=0$
e) using b) + c) repeatedly
$(c_1u_1+\cdots+c_pu_p)\cdot w=c_1(u_1\cdot w)+\cdots+c_p(u_p\cdot w)$
dot prod. of a lin. comb. and w is same lin. comb. of dot prods. w/ w.

<u>Length</u> since $v\cdot v\geq 0$ can take it's square root!

<u>Defn</u> the <u>length</u> (or <u>norm</u>) of a vector v, is the nonnegative scalar $\|v\|$
s.t. $\|v\|=\sqrt{v\cdot v}=\sqrt{v_1^2+v_2^2+\cdots+v_n^2}$ and $\|v\|^2=v\cdot v$ where $v^T=[v_1 \cdots v_n]$

<u>ex</u> $v\in\mathbb{R}^2$ $v=\begin{bmatrix}a\\b\end{bmatrix}$



pythagorean thm. in $\mathbb{R}^2$

for any scalar $c\in\mathbb{R}$ $\|cv\|=|c|\cdot\|v\|$
b/c $\|cv\|^2=(cv)\cdot(cv)=c^2(v\cdot v)=c^2\|v\|$, take square roots.

if $\|u\|=1$, we call u a unit vector.
For any vector $v\in\mathbb{R}^n$, we can divide v by it's length (mult. by $\frac{1}{\|v\|}$) to obtain a unit vector, u which is in the same direction as v, or on the line spanned by v.
This process is called <u>normalization</u> or <u>normalizing</u> a vector.

EXS ②

let $V = (2,-1,-2,0)$, find a unit vector u in the same direction as V.

find $\|V\|$,

$\|V\|^2 = V\cdot V = 2^2 + (-1)^2 + (-2)^2 + 0^2 = 9$ so $\|V\| = \sqrt{9} = 3$.

so $\frac{1}{3}V = \frac{1}{3}\begin{bmatrix}2\\-1\\-2\\0\end{bmatrix} = \begin{bmatrix}2/3\\-1/3\\-2/3\\0\end{bmatrix} = \frac{1}{\|V\|}V = \frac{V}{\|V\|} = u$ is a unit vector

check that $\|u\| = 1$, suffices to show $\|u\|^2 = 1$

$\|u\|^2 = u\cdot u = (\frac{2}{3})^2 + (\frac{-1}{3})^2 + (\frac{-2}{3})^2 + 0^2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$

EX ③

let $W = \text{Span}\left\{ [redacted] \begin{bmatrix}1\\3/4\end{bmatrix} \right\}$

find a unit vector z that is a basis for W.

$W = \{k\vec{x} : k \in \mathbb{R}\}$ since $\vec{x} = \begin{bmatrix}1\\3/4\end{bmatrix}$ W is also the span of any nonzero vector in W

thus, if $\vec{y} = 4\vec{x} = \begin{bmatrix}4\\3\end{bmatrix}$, then $W = \text{Span}\{\vec{y}\}$

$W = \{k\vec{y} : k \in \mathbb{R}\}$

and $\|y\| = y\cdot y = 4^2 + 3^2 = 25$ and $\|y\| = 5$, so $z = \frac{y}{5} = \begin{bmatrix}4/5\\3/5\end{bmatrix}$

Note that $\tilde{z} = \begin{bmatrix}-4/5\\-3/5\end{bmatrix}$ is another unit vector in W.

Distance in $\mathbb{R}^n$

for 2 real numbers, $a, b \in \mathbb{R}$ dist. from a to b is $|a-b|$

Defn for $u, v \in \mathbb{R}^n$, dist. between u and v, written $\text{dist}(u,v)$ is the length of $u-v$.

i.e. $\text{dist}(u,v) = \|u-v\|$

EXS ④ in $\mathbb{R}^2$ $u = \begin{bmatrix}1\\5\end{bmatrix}$ $v = \begin{bmatrix}4\\3\end{bmatrix}$ so $u - v = \begin{bmatrix}1\\5\end{bmatrix} - \begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}-3\\2\end{bmatrix}$ and $\|u-v\| = \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$

see fig. 4 on pg 378

check $v - u = \begin{bmatrix}3\\-2\end{bmatrix}$ and $\|v-u\| = \sqrt{9+4} = \sqrt{13} = \|u-v\|$

So $\text{dist}(u,v) = \text{dist}(v,u)$

at least in $\mathbb{R}^2$.

⑤ for $u, v \in \mathbb{R}^3$ arbitrary $u = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$ $v = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ $\text{dist}(u,v) = \|u-v\| = \sqrt{(u-v)\cdot(u-v)}$

$= \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2}$

very similar for $\mathbb{R}^n$.

(replace u_2, v_2, and $(u_2-v_2)^2$ by $\cdots$ or $\vdots$ and all 3's by n's)

Orthogonal Vectors

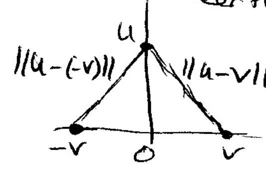
Orthogonal is a generalization of perpendicular in $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^n$.

Euclids axioms for $\mathbb{R}^2$ say

Span{u} ⊥ Span{v} (2 lines) iff dist. from u to v = dist from u to -v. consider their squares

So $[\text{dist}(u,-v)]^2 = \|u+v\| = (u+v)\cdot(u+v) \overset{\text{thm (1b)}}{=} u\cdot(u+v) + v\cdot(u+v)$

$= u\cdot u + u\cdot v + v\cdot u + v\cdot v = \|u\|^2 + \|v\|^2 + 2u\cdot v$



and some calc. replacing $-v$ by v yields $[\text{dist}(u,v)]^2 = \|u\|^2 + \|v\|^2 + 2u\cdot(-v) = \|u\|^2 + \|v\|^2 - 2u\cdot v$

If these are to be equal when $u \perp v$ then

$2u\cdot v = -2u\cdot v \Leftrightarrow \boxed{u\cdot v = 0}$ condition for orthogonality of 2 vectors.

Defn for any $u, v \in \mathbb{R}^n$, $u \cdot v = 0$ iff u and v are <u>orthogonal</u> (to each other).
$\vec{0}$ is orthogonal to everything b/c $0^T v = 0$ $\forall v \in \mathbb{R}^n, \forall n$.

Thm 2 pythagorean thm.
2 vectors u and v are orthogonal iff $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Orthogonal Complements

if $z \in \mathbb{R}^n$ is orthogonal to every $w \in W$ a supspace of $\mathbb{R}^n$, then z is <u>orthogonal to W</u>
the set of all z that are orthogonal to W is called the <u>orthogonal complement</u> of W and is denoted $W^\perp$.
(read as "W perpendicular" or "W perp").

Ex6 let W be a plane through $\vec{0}$ in $\mathbb{R}^3$, and L a line through $\vec{0}$ perpendicular to W.
for z and w nonzero, z is on L and w is in W, then segment from 0 to z, and segment from 0 to w are perpendicular, i.e. $z \cdot w = 0$
Since z and w are arbitrary, all pts. on L are $\perp$ to all $w \in W$.
thus L and W are orthogonal complements of each other, i.e. $W = L^\perp$ and $L = W^\perp$

Facts ① A vector $x \in W^\perp$ iff x is orthogonal to every vector in a set that spans W.
② $W^\perp$ is a subspace of $\mathbb{R}^n$.

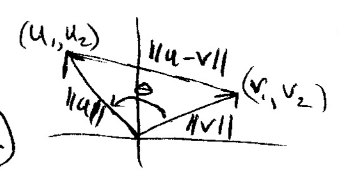
recall figure from sec. 4.6 proved by next thm.

Thm 3 Let A be an $m \times n$ matrix.
$(\text{Row} A)^\perp = \text{Nul} A$ and $(\text{Col} A)^\perp = \text{Nul} A^T$

pf
if $x \in \text{Nul} A$ then $x \cdot r_i = 0$ $\forall i$ if r_i = i^{th} row of A by row col. rule
since $\{r_1, \ldots, r_n\}$ span row A, $x \perp \text{row} A$.
conversely, $x \perp \text{row} A$, then $x \cdot r_i = 0$ $\forall i$, so $Ax = 0 \Rightarrow x \in \text{Nul} A$
so $(\text{row} A)^\perp = \text{Nul} A$. consider A^T so $(\text{row} A^T)^\perp = \text{Nul} A^T$
but row A^T = col A and the thm. is proved.

Angles in $\mathbb{R}^2$ + $\mathbb{R}^3$

let θ be the angle between 2 vectors u and v
then
$$u \cdot v = \|u\| \cdot \|v\| \cdot \cos\theta$$
(generalizes to $\mathbb{R}^3$ and $\mathbb{R}^n$ also)



apply the law of cosines to the triangle below.
$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\| \cdot \|v\| \cos\theta$$
so rearranging
$$2\|u\| \cdot \|v\| \cos\theta = u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2$$
$$= 2(u_1 v_1 + u_2 v_2) = 2 u \cdot v$$