5.3 Diagonalization

using e.vects. + e.vals. we factor a matrix into $A = PDP^{-1}$ for some P invertible and some D diagonal. Thus computing A^k becomes very easy!

what are powers of a diagonal matrix?

__EX1__ let $D = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$ $D^2 = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 25 \end{bmatrix}$ $D^3 = D \cdot D^2 = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -125 \end{bmatrix}$

and in general $D^k = \begin{bmatrix} 2^k & 0 \\ 0 & (-5)^k \end{bmatrix}$ for $k \geq 1$

__EX2__ If $A = PDP^{-1}$ for those matrices given below, compute powers of A.

let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Thus $P^{-1} = -1\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

So $A^2 = (PD\cancel{P^{-1}})(\cancel{P}DP^{-1}) = PDDP^{-1} = PD^2P^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

b/c associativity and $P^{-1}P = I$

$A^3 = (PD^2\cancel{P^{-1}})(\cancel{P}DP^{-1}) = PD^3P^{-1}$

and in general for $k \geq 1$

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2^k & 2\cdot 3^k \\ 2^k & 3^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 3^k - 2^k & 2\cdot 2^k - 2\cdot 3^k \\ 3^k - 2^k & 2\cdot 2^k - 3^k \end{bmatrix}$$

so $A^2 = \begin{bmatrix} 2\cdot 9-4 & 2\cdot 4-2\cdot 9 \\ 9-4 & 2\cdot 4-9 \end{bmatrix} = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 2\cdot 27-8 & 2\cdot 8-2\cdot 27 \\ 27-8 & 2\cdot 8-27 \end{bmatrix} = \begin{bmatrix} 46 & -38 \\ 19 & -11 \end{bmatrix}$

verify $A^2 = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$ verify $A^3 = A^2 \cdot A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 46 & -38 \\ 19 & -11 \end{bmatrix}$

When A can be factored like this w/ an invertible P and diagonal D, we say A is __diagonalizable__

__Thm5__ (Diagonalization Thm.)
An $n \times n$ matrix A is diagonalizable iff A has n lin. indep. eigenvectors.

In fact $A = PDP^{-1}$ w/ D diagonal iff the cols. of P are n linearly independent eigenvectors of A, in which case the entries in D are the corresponding eigenvalues along the diag.

__Note:__ that EX2 above was exs 2,3,3.5 in sec. 5.1 w/ $\lambda = 2, 3$ and e.vects. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, resp.

The n lin. indep. eigenvectors form a basis for $\mathbb{R}^n$, we call them, the cols. of P, an __eigenvector basis.__

__pf of thm5__ If P has cols $v_1, \dots, v_n$ and $D = \begin{cases} d_{ii} = \lambda_i & \text{for } i = 1, \dots, n \\ d_{ij} = 0 & \text{for } i \neq j \end{cases}$ Then $AP = A[v_1\ v_2 \cdots v_n] = [Av_1\ Av_2 \cdots Av_n]$

and $PD = P\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix} = [\lambda_1 v_1\ \lambda_2 v_2 \cdots \lambda_n v_n]$

So, if A diagonalizable w/ $A = PDP^{-1}$ then mult. by P on right
$\Rightarrow AP = PD$ so $[Av_1\ Av_2 \cdots Av_n] = [\lambda_1 v_1\ \lambda_2 v_2 \cdots \lambda_n v_n]$

So $Av_i = \lambda_i v_i$ for $i = 1, \dots, n$. P invertible $\Rightarrow \{v_1, \dots, v_n\}$ are lin. indep. and since cols are non zero they must be eigenvectors $v_1, \dots, v_n$ w/ eigenvalues $\lambda_1, \dots, \lambda_n$ of A.

Shows the "only if" part (A diag.) $\Rightarrow$ rest of thm.

Now show by construction given n lin. indep. eigenvectors for A and assoc. eigenvalues, that A is diagonalizable.

let $P = [v_1 \cdots v_n]$ $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots\ \lambda_n \end{bmatrix}$ then $AP = PD$ as before $\Rightarrow A = PDP^{-1}$ by using assumption that the eigenvectors are linearly indep. and thus P is invertible.

__EX3__ Diagonalizing a matrix 4 steps ① find e.vals. ② find n lin. indep. e.vects. ③ Construct P ④ Construct D

$A = \begin{bmatrix} -1 & 0 & 3 \\ -2 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ __Step1__ here setting $\det(A - \lambda I) = 0$
gives char. eqn. $-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$
which factors $(\lambda^2 - 1)(\lambda - 2) = (\lambda + 1)(\lambda - 1)(\lambda - 2) = 0$
So A has e.vals. $-1, 2, 1$

__Step2__ look at $(A - \lambda I)x = 0$

for $\lambda = -1$
$\begin{bmatrix} 0 & 0 & 3 & 0 \\ -2 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_1 - x_2 - x_3 = 0$, $x_3 = 0$ $x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \leftarrow v_{-1}$

for $\lambda = 2$
$\begin{bmatrix} -3 & 0 & 3 & 0 \\ -2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_1 - x_3 = 0$, $x_2 = 0$ $x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \leftarrow v_2$

for $\lambda = 1$
$\begin{bmatrix} -2 & 0 & 3 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_1 = 0$ $x_3 = 0$ x_2 free $x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow v_1$

__Step3__ $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

__Step4__ $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

find P^{-1} and verify that $A = PDP^{-1}$

$P^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\begin{bmatrix} -1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ -2 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = A$

EX4

let $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ w/ char. poly $-(\lambda-1)(\lambda+2)^2$ find the e.vects.

So $\lambda = 1, -2$ ← mult. 2

$[A - I \; 0] = \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & -9 & -9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 - x_3 = 0$
$x_2 + x_3 = 0$

$x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ ← V_1 basis for $\lambda = 1$

$[A + 2I \; 0]$

$\left[\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + x_2 = 0$
$x_3 = 0$

$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ ← V_{-2} basis for $\lambda = -2$

This matrix only has 2 eigenvectors so this matrix A is NOT diagonalizable!

Thm 6 An $n \times n$ matrix w/ n distinct e.values is diagonalizable

This is a sufficient condition, but not necessary, see ex 3 in book.

pf see thm. 2 in sec. 5.1, direct consequence.

EX5 Is $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ diagonalizable?

soln. triangular
$\Rightarrow$ e.values are $1, 0, -1, -2$
distinct $\Rightarrow$ diagonalizable.

EX6 diagonalize, if possible.

$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ 2 e.vals. $3, -1$

$[A - 3I \; 0]$

$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 \end{array}\right]$

$x_1 - 4x_4 = 0$
$-4x_3 = 0$

x_2, x_4 free

$x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

form basis for $\lambda = 3$ e.sp.

$[A + I \; 0]$

$\left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$

x_3, x_4 free

$x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So $P = \begin{bmatrix} 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

How to diagonalize w/o distinct eigenvalues...

Thm 7 let A be $n \times n$ matrix w/ dist. e.vals. $\lambda_1, \ldots, \lambda_p$ $(p \le n)$

(a) for $1 \le k \le p$, the dimension of the eigenspace for λ_k is $\le$ mult. of e.val. λ_k.

(b) A is diagonalizable iff sum of dimensions of distinct eigenspaces is n. Happens iff dimension = mult in (a) for all e.vals.

(c) If A diagonalizable and $\mathcal{B}_k$ a basis for e.sp. for λ_k, then total collection of all vectors in $\mathcal{B}_1, \ldots, \mathcal{B}_p$ form an eigenbasis for $\mathbb{R}^n$.