5.2 The Characteristic Equation

EX1 let $A = \begin{bmatrix} 3 & 8 \\ 1 & -4 \end{bmatrix}$ find its eigenvalues.
find scalars λ s.t. $(A-\lambda I)x = 0$ has non triv. solns.

we want $A-\lambda I$ to be singular (not invertible) i.e. $\det(A-\lambda I) = 0$!

$$\det[A-\lambda I] = \det\left(\begin{bmatrix} 3 & 8 \\ 1 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 3-\lambda & 8 \\ 1 & -4-\lambda \end{vmatrix} = (3-\lambda)(-4-\lambda)-(8)(1) = 0$$

$$= \lambda^2 + 4\lambda - 3\lambda - 12 - 8 = 0$$

so

$$0 = \lambda^2 + \lambda - 20 = (\lambda-4)(\lambda+5)$$

$\therefore$ $\lambda = 4, -5$ solve the eqn. $\det(A-\lambda I) = 0$ and are the eigenvalues of A.

Instead of thinking about $(A-\lambda I)x = 0$, and figuring out for which $\lambda \in \mathbb{R}$ this has nontrivial solns. $x \in \mathbb{R}^n$
we can ask when $\det(A-\lambda I) = 0$, i.e. for what $\lambda \in \mathbb{R}$. Instead of 2 unknowns, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$ we only have 1 unknown $\lambda \in \mathbb{R}$.

This generalizes for A being any $n \times n$ matrix.
But first recall some facts about determinants we will be needing.

Recall

for A $n \times n$ matrix, let U be any E.F. of A obtained w/ row replacements $cR_p + R_q$ and row swaps $R_p \leftrightarrow R_q$, ONLY!
No scalings allowed. cR_p, b/c they change the determinant. Let r = # of row swaps used in going from $A \sim U$. Then U is upper triangular w/ diagonal entries u_{ii} for $i = 1, \dots, n$

and so $\det A = \begin{cases} (-1)^r \prod_{i=1}^{n} u_{ii} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is singular} \end{cases}$

recall A invertible $\Rightarrow$ u_{ii} nonzero $\forall i = 1, \dots, n$
o.w. A singular $\Rightarrow$ row of zeros, so $u_{ii} = 0$ for some i.

Thm IMT (cont.)
Let A be an $n \times n$ matrix
A is invertible iff
s. $\lambda = 0$ is <u>NOT</u> an eigenvalue of A from sec. 5.1
t. $\det A \neq 0$ Thm 4 in sec. 3.2

for A 3×3, $|\det A|$ abs. value = Volume of parallelepiped determined by the columns of A.
see fig. 1 on pg. 312 in sec. 5.2
a_1, a_2, a_3 lin. indep. $\Rightarrow$ Volume $\neq 0$, if dependent, vectors $a_1, a_2, a_3 \in \mathbb{R}^3$ lie in a plane or on a line.
The parallelepiped would then be degenerate. i.e. have Vol=0

Thm 3 from secs. 3.1+3.2 props. of determinants
Let A and B be $n \times n$ matrices
a. A is invertible iff $\det A \neq 0$
b. $\det AB = (\det A)(\det B)$
c. $\det A^T = \det A$
d. If A is triangular, $\det A = \prod_{i=1}^{n} a_{ii}$
e. Row Replacement, $cR_p + R_q$ doesn't change the determinant
Row swap, $R_p \leftrightarrow R_q$ negates the determinant
row Scaling, cR_p scales the det. by the same factor.

The Characteristic Equation
The scalar eqn. $\det(A-\lambda I) = 0$ in the variable λ, is called the <u>characteristic equation of A</u>.

Fact: (lemma) A scalar λ is an eigenvalue of A an $n \times n$ matrix iff λ satisfies the characteristic equation, $\det(A-\lambda I) = 0$.
Notice that when A is $n \times n$, the LHS of the characteristic eqn $\det(A-\lambda I)$ is always a polynomial of degree n in the variable λ.
We call this the <u>characteristic polynomial</u> of A.
The multiplicity of a root λ of the char. poly. is called the <u>(algebraic) multiplicity</u> of the eigenvalue λ.

EX3 Find the char. eqn. of $A = \begin{bmatrix} -1 & -1 & 2 & 4 \\ 0 & 3 & 2 & -5 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

look at $A - \lambda I = \begin{bmatrix} -1-\lambda & -1 & 2 & 4 \\ 0 & 3-\lambda & 2 & -5 \\ 0 & 0 & -1-\lambda & 6 \\ 0 & 0 & 0 & -2-\lambda \end{bmatrix}$ it's upper triangular i.e. in E.F. already

so

$\det(A-\lambda I) = (-1-\lambda)(3-\lambda)(-1-\lambda)(-2-\lambda)$

and the char. eqn. is $(-1-\lambda)^2(3-\lambda)(-2-\lambda) = 0$ or

$\boxed{(\lambda+1)^2(\lambda-3)(\lambda+2) = 0}$ or $\lambda^4 + \lambda^3 - 7\lambda^2 - 13\lambda - 6 = 0$

the eigenvalue -1 has multiplicity 2 here

EX4 char. poly of a 6×6 matrix is $\lambda^5 - 4\lambda^3 + 4\lambda$
Find its eigenvalues and their multiplicities.

recall a root λ of a polynomial $p(t)$ is a value $t = \lambda$ s.t. $p(\lambda) = 0$

Factor the char. poly

$$\lambda^5 - 4\lambda^3 + 4\lambda = \lambda(\lambda^4 - 4\lambda^2 + 4) =$$
$$= \lambda(\lambda^2 - 2)^2 = \lambda[(\lambda-\sqrt{2})(\lambda+\sqrt{2})]^2$$

so the roots of this char. poly are $\lambda = 0$, $\sqrt{2}$ and $-\sqrt{2}$
w/ multiplicity equal to 1 for $\lambda = 0$ and 2 for both $\lambda = \pm\sqrt{2}$

for an $n \times n$ matrix, the char. poly. is in $\mathbb{P}_n$, i.e. is a degree n polynomial. If we allow complex roots then we can say that every $n \times n$ matrix has n eigenvalues, not necessarily distinct if any have multiplicity >1.
For now, only deal w/ real eigenvalues until sec. 5.5 (after thanksgiving?)

Similarity for A and B $n \times n$ matrices

A is said to be similar to B if $\exists$ an invertible matrix P, s.t. $P^{-1}AP = B \Leftrightarrow A = PBP^{-1}$ (iff $AP = PB$)

if we let $Q = P^{-1}$ then $Q^{-1}BQ = A \Rightarrow B$ is similar to A

so the property of similarity is reflexive. Thus we usually say A and B are similar.

changing A into $P^{-1}AP$ is called a similarity transform.

Thm 4 if A, B are similar $n \times n$ matrices, then they have the same char. poly, and thus the same eigenvalues (and same multiplicities)

pf $B = P^{-1}AP \Rightarrow PB = AP$ subtract λP from both sides,

$PB - \lambda P = AP - \lambda P \Rightarrow P(B - \lambda I) = (A - \lambda I)P$ taking determinants and applying thm 3b.

$\cancel{\det P} \cdot \det(B - \lambda I) = \det(A - \lambda I) \cdot \cancel{\det P} \Rightarrow$ they have the same char. poly by dividing by $\det P$

why can I divide by $\det P$? (invertible $\Rightarrow \det P \neq 0$)

Warning: Similarity is not the same as row equivalence.
A similar to $B \Rightarrow \exists$ invertible P s.t. $B = P^{-1}AP$; A row equivalent to $B \Rightarrow \exists$ inv. E s.t. $B = EA$

Dynamical Systems Application

EX5 let $A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$ analyze long term behavior of the system $x_{k+1} = Ax_k$ $(k = 0, 1, 2, \ldots)$ w/ $x_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$

note A is a perturbation of the identity I_2.

soln 1st find e.vals. of A and a basis for each eigenspace

look at char. poly for A

$0 = \begin{vmatrix} .95-\lambda & .03 \\ .05 & .97-\lambda \end{vmatrix} = (.95-\lambda)(.97-\lambda) - (.03)(.05)$

$\lambda^2 - 1.92\lambda + .92$

using the quadratic formula

$\lambda = \dfrac{1.92 \pm \sqrt{(1.92)^2 - 4(.92)}}{2} = \dfrac{1.92 \pm \sqrt{.0064}}{2} = \dfrac{1.92 \pm .08}{2}$

so $\lambda_+ = 1$ and $\lambda_- = .92$

you can check that $v_+ = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $v_- = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are corresponding e.vectors which form bases for the eigenspaces, respectively.

we now want to express x_0 as a lin. comb. of v_+ and v_-.
possible b/c $\{v_+, v_-\}$ are a basis for $\mathbb{R}^2$ (why?)

so $\exists c_1, c_2 \in \mathbb{R}$ s.t. $x_0 = c_1 v_+ + c_2 v_- = [v_+ \ v_-]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [v_+ \ v_-]^{-1} x_0 = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} .8 \\ .2 \end{bmatrix}$

$\begin{bmatrix} .125 \\ .425 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} 1/8 \\ 17/40 \end{bmatrix}$

we know $Av_+ = v_+$ and $Av_- = .92 v_-$ b/c e.vects. w/ e.vals 1, .92

thus $x_1 = Ax_0 = c_1 Av_+ + c_2 Av_- = c_1 v_+ + c_2 (.92) v_-$

$x_2 = Ax_1 = c_1 Av_+ + c_2 (.92) Av_- = c_1 v_+ + c_2 (.92)^2 v_-$

and in general $x_k = c_1 v_+ + c_2 (.92)^k v_- = .125 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + .425 (.92)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $k = 0, 1, 2, \ldots$

this solves the difference eqn. $x_{k+1} = Ax_k$ w/ $x_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$ and as $k \to \infty$ $(.92)^k \to 0$

long term behavior

thus $x_k \to \begin{bmatrix} .375 \\ .625 \end{bmatrix}$ as $k \to \infty$ (this was the same as the book got w/ $x_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$)

Can you show that for any $x_0 = \begin{bmatrix} p \\ 1-p \end{bmatrix}$ $x_k \to .125\, v_+$ regardless of p.