5.1 Eigenvectors and Eigenvalues

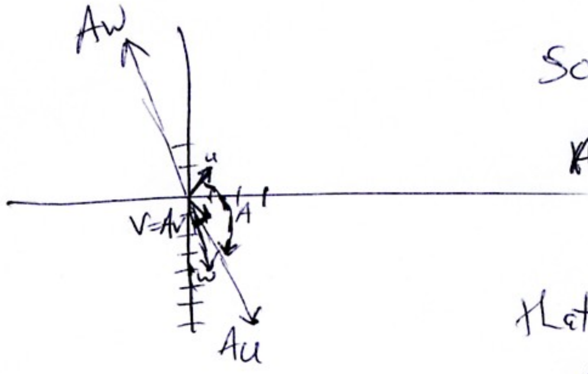
Ex1 Consider

$$A = \begin{bmatrix} 2 & 1 \\ -4 & -3 \end{bmatrix}$$

let $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $w = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

what does A do graphically to u, v and w?

$Au = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$ $Av = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $Aw = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$



So A moves u in the plane off of the line spanned by u.

And A maps v and w onto the lines they span.

In fact v is a fixed pt. of the trans. w/standard mat. A

and w gets mapped to $-2w$.

that is

$Au \neq ku \ \forall k \in \mathbb{R}$

$Av = v$

$Aw = -2w$

We are interested in understanding when the equation $Ax = \lambda x$

has solns. x, w/ A $n \times n$

$\lambda \in \mathbb{R}$ fixed

$x \in \mathbb{R}^n$

for which λ does this consistent?

For each λ that this is consistent what are all solns. to eqn. $Ax = \lambda x$?

<u>Defn</u> An <u>eigenvector</u> of an $n \times n$ mat A is a nonzero vector $x \in \mathbb{R}^n$ s.t. $Ax = \lambda x$ for some scalar $\lambda \in \mathbb{R}$.

We call λ an <u>eigenvalue</u> of A, if there exists a non-trivial soln. to eqn. $Ax = \lambda x$.

a soln. x is the eigenvector associated to λ. or corresponding to λ.

EXS Is a vector an eigenvector?

→ let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ are $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ eigenvectors of A?

soln.

$Au = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2u$ u is. (has evalue 2)

$Av = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \neq k\begin{bmatrix} 1 \\ -1 \end{bmatrix} \ \forall k \in \mathbb{R}$ v is <u>not</u>.

✓ Is a scalar an eigenvalue?

Show that 3 is an eigenvalue of A.

soln. show $Ax = 3x$ has a nonzero soln. equiv. to $Ax - 3x = 0$

$(A - 3I)x = 0$ $A - 3I = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$

lin. dep. cols ⇒ nontriv. homog. solns.

so 3 is an eigenvalue of A.

Can use row ops to find soln. $\xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 - 2x_2 = 0$

general form of solns: $x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

So every vector w/ $x_2 \neq 0$ of the form $x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 3$.

<u>Warning:</u>

row reduction is used on $A - \lambda I$ to find an <u>eigenvector</u> assoc. to λ.

Row reduction <u>cannot</u> be used to find eigen<u>values</u>, λ themselves.

Ex 3.5 $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

How do we find eigenvalues?

need $Ax = \lambda x$ to have a soln., i.e. $(A - \lambda I)x = 0$ has a nontriv. soln.

consider $A - \lambda I = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ So $A - \lambda I$ has nonzero null space, or $A-\lambda I$ is singular.

want $\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}$ to be singular i.e. $\det = 0$

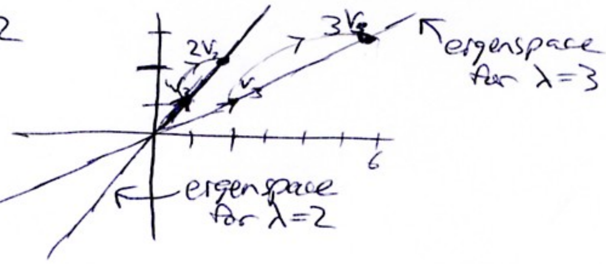
$$(4-\lambda)(1-\lambda) + 2 = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 3)(\lambda - 2) = 0 \Rightarrow \lambda = 2, 3$$

So $\lambda_- = 2$ and $\lambda_+ = 3$ are the e.vals. of A.

Find e.vects. $V_{\lambda_-} = V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ from ex. 2

$V_{\lambda_+} = V_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ from ex. 3.

Fig. 2



Defn
The set of solns. to $(A - \lambda I)x = 0$ for a particular λ is $Nul(A - \lambda I)$ is a subspace of $\mathbb{R}^n$, called the eigenspace of A corresponding to λ.

Ex 4 let $A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 1 & 3 \\ 1 & -2 & 6 \end{bmatrix}$ $\lambda = 3$ is an e.val. of A

Find a basis for the corresponding eigenspace.

Soln. find all solns. to $(A - 3I)x = 0$

$$A - 3I = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 1 & 3 \\ 1 & -2 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 1 & -2 & 3 \end{pmatrix}$$

row reduce aug. mat. $[A - 3I \;\; 0]$

$$\begin{pmatrix} 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 0 \end{pmatrix} \underset{\substack{-R_1+R_2 \\ -R_1+R_3}}{\sim} \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So x_2, x_3 free

$x_1 - 2x_2 + 3x_3 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Thus the eigenspace assoc. to $\lambda = 3$ has a basis $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

see fig. 3 in book!

Note: the presence of free variables in a system $(A - \lambda I)x = 0$ indicates that λ is in fact an eigenvalue of A.

Thm 1 let A be triangular, $n \times n$. (upper or lower or diagonal)
The eigenvalues of A are the diagonal entries, a_{ii} $i = 1, \dots, n$.

pf different from book. λ is an e.val. iff $(A - \lambda I)x = 0$ has non triv. soln. iff $A - \lambda I$ is not invertible by IMT iff $\det(A - \lambda I) = 0$. If A is triangular then $A - \lambda I$ is as well, w/ diagonal entries $a_{ii} - \lambda$. since det of triangular matrices = prod. of diagonal entries $\det(A - \lambda I) = \prod_{i=1}^{n} (a_{ii} - \lambda)$ so this is 0 iff $\lambda = a_{ii}$ for some i, $\therefore$ every diagonal entry of A is an e.val. of A.

When is $\lambda = 0$ an e.val.? when $Ax = 0x$ has nontriv. soln., $Ax = 0$ has nontriv. soln. iff A is not invertible.
So having an e.val. of 0 is equiv. to not being invertible $\Rightarrow$ can add this to IMT.

Thm 2 let A be an $n \times n$ mat. w/ distinct e.vals. $\lambda_1, \dots, \lambda_r$ and corresponding e.vects. $v_1, \dots, v_r$ then the set $\{v_1, \dots, v_r\}$ is linearly independent.

pf suppose $\{v_1, \dots, v_r\}$ is dependent, then thm 7 in 1.7 says $\exists$ v_k which is a ln. comb. of preceeding vectors.
let p be minimal s.t. $v_{p+1} = c_1 v_1 + \dots + c_p v_p$ (5), $\{v_1, \dots, v_p\}$ is ln. indep. b/c p minimal. for some scalars $c_1, \dots, c_p$ not zero. Mult. eqn. (5) by A and by λ_{p+1}

$c_1 A v_1 + \dots + c_p A v_p = A v_{p+1}$

$c_1 \lambda_1 v_1 + \dots + c_p \lambda_p v_p = \lambda_{p+1} v_{p+1}$ (6)

$c_1 \lambda_{p+1} v_1 + \dots + c_p \lambda_{p+1} v_p = \lambda_{p+1} v_{p+1}$ (7)

Sub. (6) (7)

$$c_1(\lambda_1 - \lambda_{p+1})v_1 + \dots + c_p(\lambda_p - \lambda_{p+1})v_p = 0 \quad (8)$$

Note that b/c $\lambda_1, \dots, \lambda_r$ are all distinct $\lambda_i - \lambda_{p+1}$ is nonzero $\forall i = 1, \dots, p$ and not all of the c_i's are 0 b/c of v_{p+1} depending on $v_1, \dots, v_p$ - indep. set.
so (8) is a ln. dep. relation for the set $\{v_1, \dots, v_p\} \Rightarrow$ ln. dep. $\lightning$ $\{v_1, \dots, v_p\}$ was indep.

$\therefore$ $\{v_1, \dots, v_r\}$ is independent.

Eigenvectors and Difference Equations

1st order differential equations

let $x_{k+1} = Ax_k \quad k=0,1,2,\dots$

If A is an $n\times n$ matrix, this defines a recursive sequence $\{x_k\}$ in $\mathbb{R}^n$.

Solns. of this equation are a formula or description of x_k for every k which is not dependent on A or preceeding terms, only on x_0, the initial term.

suppose x_0 was an e.vector of A w/ e.val λ.

then $x_1 = Ax_0 = \lambda x_0$

$x_2 = Ax_1 = \lambda Ax_0 = \lambda \cdot \lambda x_0 = \lambda^2 x_0$

$\vdots$

$x_k = \lambda^k x_0 \quad \forall k = 1,2,\dots$

linear combinations of such solns. are also solns.

see exercise 33 for more like this.