4.6 Rank

Row Space for A $m\times n$ each row has $\underline{n}$ entries, so could be thought of as a vector in $\mathbb{R}^n$

All lin. combs. of the row vectors is the <u>row space</u> of A, denoted Row A.

Row A is a subspace of $\mathbb{R}^n$

$\text{Row}\,A = \text{Col}\,A^T$

<u>Thm 13</u> If 2 matrices A and B are row equiv., then Row A = Row B

If B is in E.F., nonzero rows of B form a basis for row A and row B.

Pf $A \sim B \Rightarrow$ rows of B are lin. combs. of rows of A, So any lin. comb. of rows of B is automatically a lin. comb of rows of A, Thus row $B \subseteq$ row A.

row ops reversible $\Rightarrow$ row $A \subseteq$ row B $\Rightarrow$ row A = row B.

If B in E.F., then rows are lin dep. b/c no nonzero row is lin. comb. of nonzero rows below it). Apply thm. 4 to nonzero rows of B in reverse order. 1st row last.

So nonzero rows of B form a basis.

Consider row A, col A, Nul A

<u>ex 2</u>

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

find bases for above 3 ~~vectspaces~~ spaces

for row A and col A row reduce to E.F.

$$A \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

pivot cols of B

Thm 13 says $(1,3,-5,1,5)$, $(0,1,-2,2,-7)$ and $(0,0,0,-4,20)$ form a basis for row B and row A.

and from pivots get $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$ form a basis for Col A.

remember to go back to original matrix A, to find these once pivot cols are identified... (the pivot cols in B do not form a basis for Col A!).

Need REF for Nul A

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $Ax = 0 \Leftrightarrow Cx = 0$

so
$$x_1 + x_3 + x_5 = 0$$
$$x_2 - 2x_3 + 3x_5 = 0$$
$$x_4 - 5x_5 = 0$$

x_3, x_5 - free

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$ is P.V.F. of all solns to $Ax=0$

$$\therefore \text{Nul}\,A = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

basis for Col A are columns of A itself while bases for row A and Nul A have no <u>direct</u> connection to A itself.

<u>Warning</u>: lin dep. relations among <u>ROWS</u> are not preserved by row ops.

1st 3 rows of B are lin. indep. but 1st 3 rows of A are not! $r_3 = 2r_1 + 7r_2$ in A!

Rank Theorem

<u>Defn</u> The <u>rank</u> of A is the dimension of the column space of A.
since $\text{row} A = \text{col} A^T$, $\dim \text{Row} A = \text{rank} A^T$, $\dim \text{Nul} A$ called <u>nullity</u> of A.

<u>Rank Thm.</u> (Thm 14) For A an $m \times n$ matrix,
$\dim \text{Row} A = \dim \text{Col} A = \text{rank of } A = \#\text{pivot positions in } A$ and satisfies eqn.
$\text{rank } A + \dim \text{Nul} A = n$

pf Thm 6 in 4.3 says $\text{rank } A = \#\text{pivot positions in } A$
$= \#\text{pivot positions in } B$, an E.F. of A.
B has nonzero row/pivot, these form a basis for $\text{row} A \Rightarrow \text{rank } A = \dim \text{row} A$.
from 4.5 $\dim \text{Nul} A = \#$ free variables in $Ax=0$, i.e. $\dim \text{Nul} A = \#$ cols of A that are <u>NOT</u> pivot cols.
Clearly $\underbrace{\left\{\begin{matrix}\#\text{pivot}\\ \text{cols}\end{matrix}\right\}}_{\text{rank} A} + \underbrace{\left\{\begin{matrix}\#\text{nonpivot}\\ \text{cols}\end{matrix}\right\}}_{\dim \text{Nul} A} = \underbrace{\left\{\begin{matrix}\#\text{of}\\ \text{cols}\end{matrix}\right\}}_{n}$, thus thm. is proved. (only the # of those is related to Nul A, not these cols themselves)

EX 3

a) let A be 5×8 matrix w/ a 3-dim'l null space, what is rank A? so

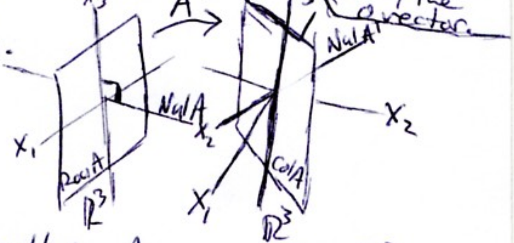
8 cols $\Rightarrow n = 8$
$\text{rank} A + 3 = 8$
$\Rightarrow \text{rank} A = 5$
(called full rank (per row!))

b) Can an $m \times (m+k)$ matrix have a $k-1$ dim'l null space?
Suppose a matrix B has this $k-1$ dim'l null space and is size $m \times (m+k)$
by rank thm., $n = m+k = \text{rank} B + k - 1 \Rightarrow \text{rank} = m+1$ but only has m rows
so the vectors in $\text{Col} B$ are in $\mathbb{R}^m$ $\therefore$ largest basis $\text{Col} B$ can have is $m \Rightarrow$
$\dim \text{Col} A \le m$ ↯ since $\text{rank} B = m$.
Could also argue that B only has m rows,
so those m rows span $\text{row} B$, by spanning set thm. basis for $\text{row} B$ has $\le m$ elements
$\Rightarrow \dim \text{row} B \underset{\text{by rank thm.}}{=} \text{rank} B \le m$ ↯ $\therefore$ an $m \times (m+k)$ matrix cannot have a $k-1$ dim'l null space (it is too small) must be at least k-dim'l!

In Ch. 6 we will see that in some way $\text{Row} A \perp \text{Nul} A$
that is $\text{row} A$ is perpendicular to $\text{Nul} A$ (in the sense of a dot product in $\mathbb{R}^n$). $\text{Row} A \cap \text{Nul} A = \{0\}$ only the 0 vector.

EX 4

$A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$ $\text{Nul} A$ is x_2-axis, $\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$ $\forall x_2 \in \mathbb{R}$.
(since cols 1 and 3 are lin. indep.)



and thus $\text{row} A$ is the $x_1 x_3$-plane
$\text{Col} A$ is plane w/ eqn. $x_1 - x_2 = 0$

$\text{Nul} A^T$ is $k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

consider lin. trans.
$x \mapsto Ax$
from $\mathbb{R}^3$ to $\mathbb{R}^3$

What is the image of $\text{row} A$ and $\text{Nul} A$, under left mult. by A, in $\mathbb{R}^3$?

left mult. by A maps $\mathbb{R}^3$ to $\mathbb{R}^3$
$\text{row} A$ is 2-dim'l subspace in domain
and $\text{Nul} A$ is 1-dim'l subspace in domain.

$r_i = (\text{row } i \text{ of } A)^T$ so is an $n \times 1$ col. vector which is the ith row of A.

for $x \in \text{row} A \Rightarrow x$ is lin. comb. of rows of A.
$x = c_1 r_1 + \cdots + c_m r_m \in \mathbb{R}^n$ write A in column form
$Ax = c_1 A r_1 + \cdots + c_m A r_m$ $A = [a_1 \; a_2 \cdots a_n]$

$Ax = c_1 [a_1 \cdots a_n] r_1 + \cdots + c_m [a_1 \cdots a_n] r_m$
$= c_1 (r_{11} a_1 + \cdots + r_{1n} a_n) + \cdots + c_m (r_{m1} a_1 + \cdots + r_{mn} a_n)$
row-col rule for mult.
$\therefore Ax$ is a lin. comb. of cols of A.
so $\text{Row} A \mapsto \text{Col} A$ by fcn. left mult. by A.

Applications to Systems of Equations.

EX5 Scientist found 2 solns. to homog. system w/ 40 eqns. in 42 variables.
2 solns. are lin. indep. (not multiples) and all other solns. are lin. combs. of these 2.
Does an associated non-homog. sys. (w/ same coeffs) have a soln.?

soln Yes, let A be coeff. matrix of homog. sys., so A is 40×42.
2 solns. are lin. indep. and span $\text{Nul}A$ (basis for $\text{Nul}A$) $\Rightarrow \dim \text{Nul}A = 2$
Rank thm. $\Rightarrow \dim \text{Col}A = 42 - 2 = 40$, but cols of A are in $\mathbb{R}^{40}$
$\Rightarrow$ if $\dim \text{Col}A = 40$ then $\text{Col}A = \mathbb{R}^{40} \Rightarrow Ax = b$ has a soln. $\forall b \in \mathbb{R}^{40}$
(full rank)
rank is its max value $= \min(m,n)$.

Rank + IMT following that of sec. 2.3

let A be $n \times n$ square matrix
TFAE to A is an invertible matrix.
m. Cols of A form a basis for $\mathbb{R}^n$
n. $\text{Col}A = \mathbb{R}^n$
o. $\dim \text{Col}A = n$
p. rank $A = n$
q. $\text{Nul}A = \{0\}$
r. $\dim \text{Nul}A = 0$

pf cols of A span $\mathbb{R}^n$ (h) } since A has n cols. each a vector in $\mathbb{R}^n$
cols of A are lin. indep. (e) } the basis thm. says $m \Leftrightarrow$ (e) and (h)
(g) $Ax = b$ has at least 1 soln. $\forall b \in \mathbb{R}^n \Rightarrow$ (n) b/c $\text{Col}A = \{b : Ax=b \text{ has a soln}\}$.
(n) $\Rightarrow$ (o) $\Rightarrow$ (p) by def of dimension + rank.
If rank $= n = \#$ of cols. of A, then $\text{Nul}A = \{0\}$ by rank thm.
so (p) $\Rightarrow$ (r) $\Rightarrow$ (q). (d) $Ax = 0$ has only triv. soln. follows naturally from (q)
So we've added these new