4.5 Dimension of a Vector Space

recall thm 8 says if V has basis w/ n vectors then $V \cong \mathbb{R}^n$
this n is indep. of the basis chosen, and thus intrinsic to V itself.
we call n the <u>dimension of the vectorspace</u>

<u>Thm 9</u> for any vect. sp. V w/ basis $\mathcal{B} = \{b_1, \dots, b_n\}$
any set of more than n vectors in V is linearly dependent.

<u>pf</u> many ways to prove this. a nice way is w/ coordinate vectors, i.e. vectors in $\mathbb{R}^n$ and apply to the thm. 8 from sec. 1.7. see text for formal version.

<u>Thm 10</u> If V, a vect. sp. has a basis w/ n vectors, then every basis has n vectors.

<u>corr</u> any lin. indep. set of vectors in V can have no more than n elements.

<u>pf</u> very straightforward, see text.

<u>Defn</u> If V has a finite spanning set, we say V is <u>finite dimensional</u>
the <u>dimension</u> of V, written $\dim V$, is # of vectors in a basis for V.
The vect. sp. $\{0\}$ is defined to have dimension 0 (since it really has no basis!).
If V is not spanned by a finite set, then V is called <u>infinite dimensional</u>.

<u>Ex1</u> $\dim \mathbb{R}^n = n$ has basis $\mathcal{E} = \{e_1, \dots, e_n\}$ standard basis

$\dim \mathbb{P}_n = n+1$ has basis $\mathcal{B} = \{\underbrace{1, t, t^2, \dots, t^n}_{n+1 \text{ many vectors here}}\}$ standard basis

Note that $\mathbb{P}_n$ has 1 more basis element than $\mathbb{R}^n$.

Note that
$\mathbb{P}_n \cong \mathbb{R}^{n+1}$
$\mathbb{R}^n \cong \mathbb{P}_{n-1}$

$\dim \mathbb{P} = \infty$ b/c $\forall n$, a pos. integer $\mathbb{P}_n \subseteq \mathbb{P}$ subspace
so no matter what size finite set you choose, it cannot span $\mathbb{P}$ b/c $\exists$ a bigger N s.t. $\mathbb{P}_N$ is a subspace of $\mathbb{P}$, and $\mathbb{P}_N$ has more vectors in its basis. see exercise 27?

<u>EX2</u>
$H = \text{Span}\{v_1, v_2\}$
$v_1 \neq k v_2 \Rightarrow$ lin. indep.
so $\{v_1, v_2\}$ is a basis for H
$\therefore \dim H = 2$.

<u>EX3</u> Find dimension of
$$W = \left\{ \begin{bmatrix} a - b + 2c \\ 2a + d \\ b - 2c + 2d \\ -d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$
so
$$W = \text{Span}\left\{ \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \end{array} \right\}$$
$\therefore \dim W \leq 4$

Need to find a basis, put vectors into matrix + find pivot cols...

Note $v_3 = -2v_2$ so spanning set thm. says discard v_3
v_1, v_2, v_4 lin. indep. b/c of where nonzero entries are!

Thus W has basis $\{v_1, v_2, v_4\}$
$\Rightarrow \dim W = 3$

<u>EX4</u> classify all subspaces of $\mathbb{R}^3$, by dimension.

0-dimension subspaces $= \{0\}$ trivial subspace only!

1-dim'l subsps.
Span of single nonzero vector
i.e. a line through the origin.

2-dim'l subsps.
Span of 2 vectors, not scalar mults. of each other
i.e. a plane through the origin.

3 dim'l subsps.
all of $\mathbb{R}^3$, only!
any 3 lin. indep. vectors in $\mathbb{R}^3$, span $\mathbb{R}^3$, by IMT.

Subspaces of finite dimensional spaces

counterpart of the spanning set thm. **Thm 11** Let H be a subspace of a finite-dim'l vect. sp. V.
Any lin. indep. set in H can be expanded (if necessary) to a basis for H. $\dim H \leq \dim V$ and thus H is also finite dim'l.

pf Since any set of lin. indep. vectors in H could be expanded to a basis for V simply ~~do~~ this expansion by choosing vectors in H ~~to~~ add at each step, stop when the set spans H (and not V).
Since V has a basis and $H \subseteq V$ clearly $H \subseteq \text{Span}\{\text{basis } V\}$
$\Rightarrow \dim H \leq \dim V$.

* Thm 12 *
The Basis Thm.

for $p \geq 1$, let V be a p-dim'l vect. sp.
Any lin. indep. set of p elements in V is automatically a basis for V.
Any set of p elements which span V is also automatically a basis for V.

pf let S be a set of p lin. indep. elements. By thm 11 can extend S to a basis for V, but by thm. 10 every basis must have p elems. So S is a basis for V. Suppose S has p elems. + spans V. By spanning set thm. $\exists$ a subset which is a basis, but again by thm. 10 the subset must have p elems., so S is a basis for V.

Dimensions of Nul A and Col A

Let A be an $m \times n$ matrix, and $Ax = 0$ has k free variables (i.e. k lin. indep. sol'ns)
~~Let~~ A ~~have~~ p many pivot columns (i.e. A has p basic variables in $Ax=0$, and p lin. indep. cols)
$\therefore \dim \text{Nul} A = k$ and $\dim \text{Col} A = p$
but since A has n columns and $Ax=0$ has n variables, each either basic or free,
$p + k = n$, always!

While we typically cannot relate Nul A and Col A (b/c when $m \neq n$ these are subspaces of different real vect. sps)
we can relate their dimensions as follows:

(See EX5 in text for a 3×5 matrix)

Look at practice problems!
1. False 2. True

<u>Lemma</u> for any $m \times n$ matrix A,
$\dim \text{Nul} A + \dim \text{Col} A = n$.