# 4.4 Coordinate Systems

Specifying a basis for a Vector space $V$ defines a coord. sys. on $V$.
if $\mathcal{B}$ has $n$ vectors, then $V$ will behave like $\mathbb{R}^n$.
if $V$ is $\mathbb{R}^n$, then $\mathcal{B}$ gives a "new view" of $V = \mathbb{R}^n$

**Thm 7** unique Representation Thm.
Let $\mathcal{B} = \{b_1, \dots, b_n\}$ a basis for $V$, a vect. sp.
Then $\forall x \in V$ $\exists!$ set of scalars (wts) $c_1, \dots, c_n$
s.t. (1) $x = c_1 b_1 + \dots + c_n b_n$ ($x$ is a lin. comb. of basis w/ these wts.)

pf
$\mathcal{B}$ spans $V \Rightarrow \exists c_1, \dots, c_n$
suppose $\exists$ another set of wts. $d_1, \dots, d_n$
s.t. $x = d_1 b_1 + \dots + d_n b_n$
subtracting
$x - x = \vec{0} = (c_1 - d_1) b_1 + \dots + (c_n - d_n) b_n$
since basis elems. are lin. indep.
$\Rightarrow c_i - d_i = 0 \ \forall i \in \{1, \dots, n\}$
so $c_j = d_j \ \forall j = 1, \dots, n$
$\therefore$ wts. are unique.

**Defn** let $\mathcal{B}$ be a basis for $V$ (as in thm 7)
the coordinates for $x \in V$ relative to the basis $\mathcal{B}$ (or $\mathcal{B}$-coordinates of $x$) are the unique wts. $(c_1, \dots, c_n)$ s.t. $x = c_1 b_1 + \dots + c_n b_n$

since $(c_1, \dots, c_n)$ is a unique $n$-tuple associated to $x$ relative to $\mathcal{B}$
define $[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$
as the coordinate vector of $x$ relative to $\mathcal{B}$
or $\mathcal{B}$-coordinate vector of $x$

can think of a mapping from $V \to \mathbb{R}^n$, $x \mapsto [x]_{\mathcal{B}}$
as a coordinate mapping (determined by $\mathcal{B}$)

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EXS (1)
consider the basis $b_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $\mathbb{R}^2$
let $[x]_{\mathcal{B}} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$ be the coordinate vector for some $x \in \mathbb{R}^2$
find $x$. $x = -6 b_1 + 5 b_2 = -6 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$

(1b) Consider the vector $x = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$ in $\mathbb{R}^2$
what are the $\mathcal{B}$ coords. for $x$ w/ same $\mathcal{B}$ as in (1)
$\begin{bmatrix} -6 \\ 5 \end{bmatrix} = x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
$\Rightarrow x_2 = -3, \ x_1 = 8$
so $[x]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$ for $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^2$.

(2) Consider the standard basis for $\mathbb{R}^n$
$\mathcal{E} = \{e_1, \dots, e_n\}$ what are the $\mathcal{E}$-coords. for any $x \in \mathbb{R}^n$
$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + \dots + x_n e_n$ so $[x]_{\mathcal{E}} = x$ for $\mathcal{E} = \{e_1, \dots, e_n\}$ a basis for $\mathbb{R}^n$.

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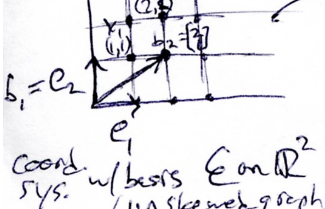
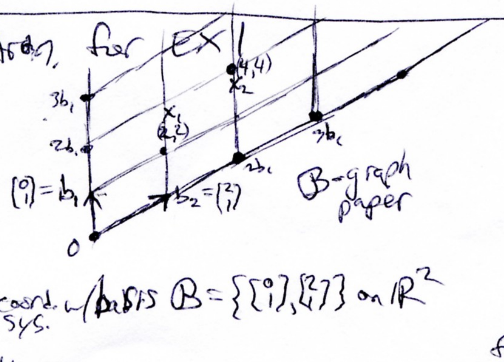
Graphical Interpretation for EX (1)



coord. sys. w/ basis $\mathcal{E}$ on $\mathbb{R}^2$ (unskewed graph paper!)



coord. sys. w/ basis $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ on $\mathbb{R}^2$

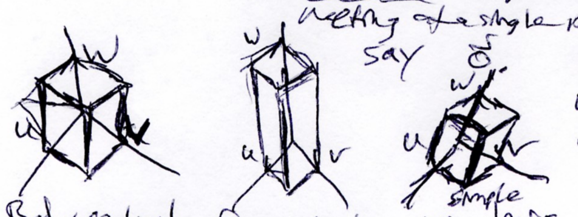
EX (1) w/ $[x_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
$[x_2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

think about $x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
$\left[ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ says $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1 \cdot b_1 + 1 \cdot b_2$
so the pt. $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ in standard coords. is gotten by adding $b_1 + b_2$ (the basis elems. in $\mathcal{B}$-coords)

also see a mapping from $\mathcal{E}$ coords on $\mathbb{R}^2 \to \mathcal{B}$ coords on $\mathbb{R}^2$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

EX3 crystallography – choose basis $\{u, v, w\}$ for $\mathbb{R}^3$ (recall any 3 lin. indep. vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$)

3 adjacent edges of a "unit cell" meeting at a single pt. say $\vec{0}$

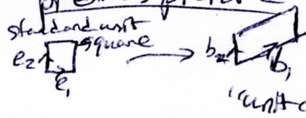



Body-centered cubic — face-centered orthorhombic $(\frac{1}{2}, \frac{1}{2}, 1)$ — monoclinic

EXS of unit cells each basis defines a different unit cell in $\mathbb{R}^3$

Aside: This notion of a basis defining a "unit cell" applies to any basis of any length n. The unit cell is then the n-dim'l cube in $\mathbb{R}^n$ and all of it's linear transformations

previous picture




lin. trans. of unit square gives a new "unit cell" in $\mathbb{R}^2$ associated to the basis $\mathcal{B} = \left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$

matrix $B = \begin{bmatrix}0 & 1\\1 & 2\end{bmatrix}$ is the standard matrix of lin. trans. that does this!

## Coords. in $\mathbb{R}^n$

Specify a basis $\mathcal{B}$ and a vector $x \in V$ the $\mathcal{B}$-coords. of $x$, $[x]_{\mathcal{B}}$ are easily found

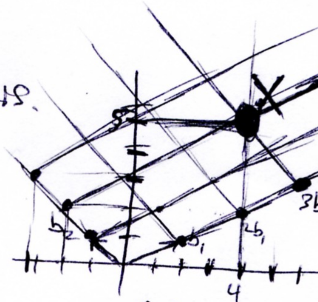
EX4 let $\mathcal{B} = \{b_1, b_2\} = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ $x = \begin{bmatrix}4\\5\end{bmatrix}$

find $[x]_{\mathcal{B}}$. Thus $\underbrace{\begin{bmatrix}4\\5\end{bmatrix}}_{x} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$

can solve w/ row ops. inverses, inspection, etc.

$c_1 = 3$ $c_2 = 2$ solves this.

so $[x]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$




$x = \begin{bmatrix}4\\5\end{bmatrix} = 4e_1 + 5e_2$
$= 3b_1 + 2b_2$

the matrix $\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}$ changes $\mathcal{B}$-coords. into standard coords.

analogous matrix for any $\mathcal{B}$ in $\mathbb{R}^n$

Lemma let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for $\mathbb{R}^n$ then $P_{\mathcal{B}} = [b_1\ b_2 \cdots b_n]$ is an $n \times n$ change of coords. matrix

s.t. $\forall x \in \mathbb{R}^n$ $x = c_1 b_1 + \cdots + c_n b_n = P_{\mathcal{B}}[x]_{\mathcal{B}}$

so we have a trans. $T_{\mathcal{B}}: \mathbb{R}^n \to \mathbb{R}^n$

$[x]_{\mathcal{B}} \to x = P_{\mathcal{B}}[x]_{\mathcal{B}}$

pts. in $\mathbb{R}^n$ w/ $\mathcal{B}$-coords. — pts. in $\mathbb{R}^n$ in standard coords.

Notes:

The cols (and rows) of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, so $P_{\mathcal{B}}$ is invertible by IMT

left mult. by $P_{\mathcal{B}}^{-1}$ converts stand. coords to $\mathcal{B}$-coords.

$P_{\mathcal{B}}^{-1} x = [x]_{\mathcal{B}}$

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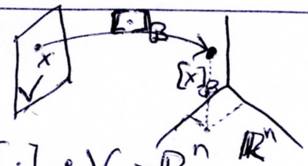
Thm 8 let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for $V$

the coord. mapping $x \mapsto [x]_{\mathcal{B}}$ is a 1-1 lin. trans. from $V$ onto $\mathbb{R}^n$




$[\cdot]_{\mathcal{B}}: V \to \mathbb{R}^n$
$x \mapsto [x]_{\mathcal{B}}$

pf take $u, v \in V$ $u = c_1 b_1 + \cdots + c_n b_n$
$v = d_1 b_1 + \cdots + d_n b_n$

$u + v = (c_1 + d_1) b_1 + \cdots + (c_n + d_n) b_n$

so $[u+v]_{\mathcal{B}} = \begin{bmatrix}c_1 + d_1\\ \vdots \\ c_n + d_n\end{bmatrix} = \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix} + \begin{bmatrix}d_1\\ \vdots \\ d_n\end{bmatrix} = [u]_{\mathcal{B}} + [v]_{\mathcal{B}}$ so addition is preserved.

$\forall r \in \mathbb{R}$ $ru = rc_1 b_1 + \cdots + rc_n b_n$

$[ru]_{\mathcal{B}} = \begin{bmatrix}rc_1\\ \vdots \\ rc_n\end{bmatrix} = r\begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix} = r[u]_{\mathcal{B}}$ so scalar mult. preserved

so $[\cdot]_{\mathcal{B}}$ is linear.

1-1 and onto are verified in exercises 23 + 24.

The linearity of this coordinate map extends to lin. combinations

let $u_1, \dots, u_p$ be vectors in the vect. sp. $V$
and $c_1, \dots, c_p$ be scalars then

$$[c_1u_1 + \dots + c_pu_p]_{\mathcal{B}} = c_1[u_1]_{\mathcal{B}} + \dots + c_p[u_p]_{\mathcal{B}} \quad (5)$$

$\mathcal{B}$ coords. of a lin. comb. of $u_1, \dots, u_p$ is the same lin. comb. of ~~their~~ $\mathcal{B}$-coords.

The coordinate mapping $[\cdot]_{\mathcal{B}}$ from thm. 8 is an isomorphism from $V$ onto $\mathbb{R}^n$.

1-1 lin. trans. from vect. sp. $V$ onto vect. sp. $W$ is called an isomorphism between $V$ and $W$

any vect. sp. calcs. in $V$ are accurately reproduced in $W$ and vice-versa

EXS 5-7 + practice probs.

EX5 $\mathcal{B}$ standard basis for $\mathbb{P}_3$ = polynomials of deg $\leq 3$

$\mathcal{B} = \{1, t, t^2, t^3\}$ $\forall p \in \mathbb{P}_3$ $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3$ for some $a_0, a_1, a_2, a_3 \in \mathbb{R}$

so in this basis, standard basis-coords for $p$ are $[p]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$[\cdot]_{\mathcal{B}} : \mathbb{P}_3 \to \mathbb{R}^4$

$p \mapsto [p]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$ when $p(t) = a_0 + a_1t + a_2t^2 + a_3t^2$

is an isomorphism from $\mathbb{P}_3$ onto $\mathbb{R}^4$! prove it is an isomorphism (Thm 8) (a 1-1, onto lin. trans between vect. spaces)

w/o thm. 8, show this as follows

linearity let $p, q \in \mathbb{P}_3$ be arbitrary elems. $r \in \mathbb{R}$ be arbitrary scalar.

$p = a_0 + a_1t + a_2t^2 + a_3t^3$
$q = b_0 + b_1t + b_2t^2 + b_3t^3$

so $[p]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $[q]_{\mathcal{B}} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$p + q = (b_0 + a_0) + (b_1 + a_1)t + (b_2 + a_2)t^2 + (b_3 + a_3)t^3$
$rp = ra_0 + ra_1t + ra_2t^2 + ra_3t^3$

$[p+q]_{\mathcal{B}} = \begin{bmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} = [p]_{\mathcal{B}} + [q]_{\mathcal{B}}$

$[rp]_{\mathcal{B}} = \begin{bmatrix} ra_0 \\ ra_1 \\ ra_2 \\ ra_3 \end{bmatrix} = r[p]_{\mathcal{B}}$

$\therefore$ $[\cdot]_{\mathcal{B}}$ is a linear transformation!
show 1-1 and onto. (bijection).

1-1 2 ways, by defn. (or $\ker = \{0\}$)

$\forall b \in \text{range}(T)$, $\exists!\, x \in \text{domain}(T)$ s.t. $T(x) = b$

or $\nexists\, x, y \in \text{dom}(T)$ s.t. $T(x) \neq T(y)$, $x \neq y$

or $\forall x, y \in \text{dom}(T)$ if $T(x) = T(y)$ then $x = y$

3rd defn.
let $x, y \in \mathbb{P}_3 \Rightarrow x(t) = x_0 + x_1t + x_2t^2 + x_3t^3 \Rightarrow [x]_{\mathcal{B}} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$y(t) = y_0 + y_1t + y_2t^2 + y_3t^3$ $[y]_{\mathcal{B}} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$

if $[x]_{\mathcal{B}} = [y]_{\mathcal{B}}$, then $x_0 = y_0, x_1 = y_1, x_2 = y_2, x_3 = y_3$ and thus $x(t) = y(t)$ $\forall t \Rightarrow x = y$, as polynomials in $\mathbb{P}_3$, so $[\cdot]_{\mathcal{B}}$ is 1-1.

Equivalently w/ 1st defn.

~~[illegible]~~ $\forall\, b \in Range([\cdot]_{\mathcal{B}})$, $b = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \exists\, x \in \mathbb{P}_3$ s.t. $b_0 + b_1 t + b_2 t^2 + b_3 t^3 = x(t)$

b/c $[x]_{\mathcal{B}} = b \in \mathbb{R}^4$

show x is unique.

suppose $\exists\, y \in \mathbb{P}_3$ s.t. $[y]_{\mathcal{B}} = b$ ; thus $y(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$

(for proof by 2nd defn see 1st or 3rd defs method + alter to do a pf. by contradiction.) $\Rightarrow x = y$ in $\mathbb{P}_3$ so x unique

## ker $= \{\vec{0}\}$

show that kernel or null space of $[\cdot]_{\mathcal{B}} = \{\vec{0}\}$

$$\ker([\cdot]_{\mathcal{B}}) = \{p \in \mathbb{P}_3 : [p]_{\mathcal{B}} = \vec{0} \in \mathbb{R}^4\} = \{p \in \mathbb{P}_3 : p(t) = 0 + 0t + 0t^2 + 0t^3 = 0\}$$

$\therefore \ker([\cdot]_{\mathcal{B}}) = \{0\}$, the zero polynomial in $\mathbb{P}_3$ is the only poly. in the kernel.

show kernel of lin. trans $= \{0\} \Rightarrow$ lin. trans. is 1-1 is given as an iff statement as #8. in table at top of pg. 232

given $\ker(T) = \{0\}$ for a lin. trans. T

suppose T is not 1-1, then $\exists\, x, y \in dom(T)$, $x \neq y$ s.t. $T(x) = T(y)$

then $T(x) - T(y) = 0 \in Range(T)$

but $T(x) - T(y) = T(x - y)$ by linearity so $x - y \in Dom(T)$ maps to $\vec{0} \in Range(T)$

since $x \neq y$, $x - y \neq 0 \Rightarrow x - y \in \ker(T)$ ↯

this contradicts the given information, so our supposition is false $\therefore$ T is 1-1.

> **Lemma** for a linear transformation T between vector spaces, $\ker(T) = \{0\}$ implies that T is 1-1.

Use This Lemma to show things are 1-1, whenever appropriate!

**onto**

$\forall\, a \in \mathbb{R}^4$ w/ $a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\exists\, p \in \mathbb{P}_3$ s.t. $[p]_{\mathcal{B}} = a$

namely $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$.

So $[\cdot]_{\mathcal{B}}$ is a lin. trans, 1-1, and onto, $\therefore$ it is an isomorphism

**EX6** use coord. vectors to show polynomials $1+2t^2$, $4+t+5t^2$, $3+2t$ are linearly dependent in $\mathbb{P}_2$. (note: $\mathbb{P}_2$ not $\mathbb{P}_3$ as in ex5)

get $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ as cols of a 3x3 matrix A, we row reduce

$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ cols dep. w/ cols 1+2 pivot cols $\Rightarrow \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} - 5\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow 3 + 2t = 2(4 + t + 5t^2) - 5(1 + 2t^2)$ so polys. are dep. as well.

**EX7** $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, $\mathcal{B} = \{v_1, v_2\}$, so $\mathcal{B}$ is a basis for $H = Span\{v_1, v_2\}$. Is $x \in H$?

if $x \in H$, then $\exists\, c_1, c_2 \in \mathbb{R}$ s.t. $x = c_1 v_1 + c_2 v_2 \Rightarrow \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

could see if system $A\vec{c} = \vec{x}$ is consistent, o.w. just solve by inspection or by hand

$\Rightarrow c_1 = 2$ from 2nd entry


$H = Span\{v_1, v_2\}$ is a plane through the origin in $\mathbb{R}^3$

So $\begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} - \begin{bmatrix} 6 \\ 12 \\ 4 \end{bmatrix} = c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} = c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ w/ $c_2 = 3$ so Yes $x \in H$

$x = 2v_1 + 3v_2$