Test 2 recap

remaining material 4.2 – 4.6, 5.1 – 5.3, 6.1 – 6.5
Test 3 somewhere in here.

We may end up between 15 + 20 quizzes in which case I will count the best 15 + drop lowest others then rescale by $\frac{20}{15} = \frac{4}{3}$ to make pts. match!

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Subspaces of $\mathbb{R}^n$

type ① set of solns. to homog. eqn. for some matrix $A$
set of $x$ s.t. $Ax=0$

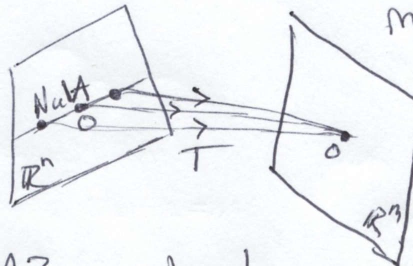
**Def** The <u>null space</u> of an $m\times n$ matrix $A$ is the set of solns. to homog. eqn. $Ax=0$

$\{x \mid Ax=0\}$ or $\{x : Ax=0, x\in\mathbb{R}^n\}$
so $A$ must be $m\times n$

$\mathrm{Nul}A = \{x : x\in\mathbb{R}^n, Ax=0\}$

So $\mathrm{Nul}A \subseteq \mathbb{R}^n$, vectors in $\mathbb{R}^n$ which map to $\vec{0}$ in $\mathbb{R}^m$ by $T$.

Think of a lin. trans. $T$ w/ standard matrix $A$
then $T(x)=Ax$ and $\mathrm{Nul}A$ = set that is mapped to $\mathbb{R}^m \ni 0$ by $T$.




**ex** for $A=\begin{bmatrix}2 & 1\\ -2 & -1\end{bmatrix}$ is $u=\begin{bmatrix}1\\-2\end{bmatrix}$ in $\mathrm{Nul}A$?
check if $Au=\vec{0}$ if yes then yes
if no then no.

why do we call this the null <u>space</u>?

**Thm 2** for $A$ $m\times n$
$\mathrm{Nul}A$ is a subspace of $\mathbb{R}^n$
$\therefore$ set of all solns. to $Ax=0$ of $m$ homog. eqns. in $n$ unknowns is a subspace of $\mathbb{R}^n$!

<u>pf of thm 2</u>
$\mathrm{Nul}A \subseteq \mathbb{R}^n$ b/c $A$ has $n$ cols.

$\therefore$ $x\in\mathrm{Nul}A$
$\Rightarrow x\in\mathbb{R}^n$
so $\mathrm{Nul}A\subseteq\mathbb{R}^n$

show ① $\vec{0}\in\mathrm{Nul}A$
② $\forall x,y\in\mathrm{Nul}A$ $x+y\in\mathrm{Nul}A$
③ $\forall x\in\mathrm{Nul}A, \forall c\in\mathbb{R}$ $cx\in\mathrm{Nul}A$

$\mathrm{Nul}A = \{x : Ax=0, x\in\mathbb{R}^n\}\subseteq\mathbb{R}^n$

$A(\underset{\mathbb{R}^n}{\vec{0}}) = \underset{\mathbb{R}^m}{\vec{0}}$ so $\vec{0}\in\mathrm{Nul}A$

let $x,y\in\mathrm{Nul}A$
$\therefore$ $Ax=\vec{0}$ and $Ay=\vec{0}$
so consider $A(x+y) \underset{\text{by linearity}}{=} Ax+Ay = \vec{0}+\vec{0}=\vec{0}$
and let $c\in\mathbb{R}$
$A(cx) \underset{\text{by linearity}}{=} cAx = c\vec{0}=\vec{0}$
thus $\mathrm{Nul}A$ is a subspace of $\mathbb{R}^n$.

**EX 2** let $H=\left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\in\mathbb{R}^4 : a-b+3c=4d \text{ and } d-b=c\right\}$

Show $H$ is a subspace of $\mathbb{R}^4$.
rearrange eqns.

$a-b+3c-4d=0$
$-b-c+d=0$

$\therefore$ if $A=\begin{bmatrix}1 & -1 & 3 & -4\\ 0 & -1 & -1 & 1\end{bmatrix}$ and $x=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$

the solns. to $Ax=0$ is $H$. $\therefore$ $H=\mathrm{Nul}A$ and by thm 2 $H$ is a subspace of $\mathbb{R}^4$ (b/c $A$ is $2\times 4$)

You must have homog. eqns. or a homog. system, o.w. $\vec{0}$ is not a soln.
If not homog., soln. set could be empty, if sys. was inconsistent!

## Explicit description of Nul A

$\text{Nul}\,A = \{x : Ax=0,\ x \in \mathbb{R}^n\}$ is an implicit defn.
you must check a condition to know if $x \in \text{Nul}\,A$
($\nexists$ a formula for what objects in Nul A look like.)
that would be explicit for ex. how H was defined in last example!

solving $Ax=0$ gives an explicit description of Nul A

**Ex3** from sec. 1.5

let $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$ find a spanning set for the nullspace of A.

1st, solve $Ax=0$ for general soln. w/ free variables (if any?)
row reduce aug. mat. $[A\ 0]$ to REF.

$$A \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - \frac{4}{3}x_3 = 0$
$x_2 = 0$
$0 = 0$

$x_3$ is free

so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$ (p.v.f.)

we decompose general soln. into lin. comb. of vectors w/ wts. = free variables.

**Ex 3.5** suppose $B = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & 4 & 5 & 8 & -4 \end{bmatrix}$ and $[B\ 0] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

recall: pivots are basic, all others are free

thus $x_1 - 2x_2 - x_4 + 3x_5 = 0$
basic $x_3 + \underbrace{2x_4 - 2x_5}_{\text{free}} = 0$

so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}}$

so $\vec{x} = x_2\vec{u} + x_4\vec{v} + x_5\vec{w}$
all lin. combs. of $\vec{u}, \vec{v}, \vec{w}$ form Nul A
so $\text{Nul}\,A = \text{Span}\{\vec{u}, \vec{v}, \vec{w}\}$

**Facts:** ① since these vectors are nonzero and correspond to spanning vectors w/ wts. equal to free variables, the only way a lin. comb. of these is $\vec{0}$ is if all free variables are 0. (wts)
② If Nul A has more than just $\vec{0}$, then # of spanning vectors = # free variables in $Ax=0$.

## Column Space of a Matrix

defined explicitly via lin. combs.

**Defn** The column space of an $m \times n$ matrix A is the set of all lin. combs. of the cols. of A.
let $A = [a_1 \cdots a_n]$, then $\text{Col}\,A = \text{Span}\{a_1, \ldots, a_n\}$

Note: $\text{Col}\,A = \{b : b = Ax,\ x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$
$Ax$ denotes a lin comb. of cols of A w/ wts. x
This note also says that $\text{Col}\,A = \{Ax : x \in \mathbb{R}^n\} = \text{Range}(T)$
where $T: \mathbb{R}^n \to \mathbb{R}^m$
$x \mapsto Ax$

**Thm 3** If A $m \times n$ then Col A is a subspace of $\mathbb{R}^m$.
(Note the subtle diff. w/ Thm 2.)

## Ex 4

let $W = \left\{ \begin{bmatrix} 2a-b \\ -b \\ b-a \end{bmatrix} : a,b \in \mathbb{R} \right\} = \left\{ a\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : a,b \in \mathbb{R} \right\}$

find A s.t. Col A = W

rewrite W as lin. combs.

$= \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

thus let $A = \begin{bmatrix} 2 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$ so $W = \text{Col}A$.

recall (Thm 4 sec 1.4) cols of A span $\mathbb{R}^m$ iff $Ax=b$ has a soln. $\forall b \in \mathbb{R}^m$ $\longrightarrow$ fact: for A $m\times n$, Col A $= \mathbb{R}^m$ iff $Ax=b$ has a soln. $\forall b \in \mathbb{R}^m$

cor: If A $n \times n$, Col A $= \mathbb{R}^n$ iff A invertible (by IMT)

## Contrast between Nul A and Col A

relationship between Nul A and Col A? really they are dissimilar Exs 5-7

relationship given in sec. 4.6 w/more theory!

## Ex 5

let A be $3\times 4$

① Col A $\subseteq \mathbb{R}^k$, find k.

② Nul A $\subseteq \mathbb{R}^l$, find l

soln. ① cols of A $\in \mathbb{R}^3$ so Col A $\subseteq \mathbb{R}^3$

② $x \in \mathbb{R}^4$ so that $Ax$ is defined $\therefore$ Nul A $\subseteq \mathbb{R}^4$.

Aside for non-square matrices $A_{m\times n}$ w/ $m \neq n$ Nul A and Col A live in different universes! No lin comb. of vectors in $\mathbb{R}^m$ can ever be a vector in $\mathbb{R}^n$ unless $m=n$ (think $m=2, n=3$)

what is $\text{Nul}A \cap \text{Col}A = \begin{cases} \emptyset & \text{when } n \neq m \\ \vec{0} & \text{usually when } n=m \\ \text{some set containing } \vec{0} + \text{other vectors} & \text{in rare special cases.} \end{cases}$

See exs 6+7 in book

## Ex 7.5

let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}_{2\times 2}$

find Nul A and Col A.

what is Nul A $\cap$ Col A?

As a subspace of $\mathbb{R}^2$ what is this isomorphic to?

See table on pg. 232

$\text{Nul}A = \{x : Ax=0, x \in \mathbb{R}^2\} = \left\{ x : \begin{bmatrix} x_1 - x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ so $x_1 = x_2$

$\text{Col}A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} y \\ y \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \right\} \subseteq \mathbb{R}^2$

so Nul A = Col A = set of vectors in $\mathbb{R}^2$ w/ 1st + 2nd entries the same.

Graphically it's the line $y=x$ in $\mathbb{R}^2$!

$\therefore$ this space is isomorphic to $\mathbb{R}$.

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kernel + Range of Lin. Trans.

generalization of lin. trans. defn. from 1.8

A linear transformation T from a vect. sp. V to a vect. sp. W is a map that assigns a unique vector $T(x) \in W$ to each vector $x \in V$ s.t.

(i) $T(u+v) = T(u) + T(v)$ $\quad \forall u,v \in V, \forall c \in \mathbb{R}$

(ii) $T(cu) = cT(u)$

The kernel (also called null space) of T is set of $u \in V$ s.t. $T(u) = 0 \in W$.

The range of T is set of $y \in W$ s.t. $\exists x \in V$ s.t. $y = T(x)$

So kernel or $\ker(T) = \{u \in V : T(u) = 0 \in W\}$

and range or $\text{Range}(T) = \{T(x) \in W : x \in V\}$

$\therefore$ if $\exists$ a matrix A s.t. $T(x) = Ax$

then $\ker(T) = \text{Nul}A$ and $\text{Range}(T) = \text{Col}A$

Recall I mentioned that the derivative was a linear operator...

$$\frac{d}{dx}(af(x)+g(x)) = a f'(x) + g'(x)$$

EX8 let $V$ = vect. sp. of all real-valued fcns. $f$ defined on $[a,b]$ that are differentiable and have cts. derivatives $f'$ on $[a,b]$

let $W$ = vect. sp. of all cts. fcns. on $[a,b]$ and

$D: V \to W$, $f \mapsto f'$ so $D(f+g) = D(f) + D(g)$ and $D(cf) = cD(f)$ by props. of the derivative. So $D$ is linear.

What is $\ker(D)$ and $\mathrm{Range}(D)$? $\ker(D)$ = constant fcns. on $[a,b]$ $\cong \mathbb{R}$, $\mathrm{Range}(D) = W$.

EX9 consider $y'' + \omega^2 y = 0$ w/ $\omega$ = constant (models spring vibrations, pendulum, voltage)

So $\exists$ a linear transformation which maps $y = f(t)$ to $f''(t) + \omega^2 f(t)$. The kernel of this lin. trans. is the solutions to the DIFFEQ. for soln. see ex. 19 in sec. 4.1.

practice problems

1. Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + 2b - 3c = 0 \right\} = \left\{ x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : Ax = 0 \right\}$ where $A = [1 \ 2 \ -3]$

Show subspace of $\mathbb{R}^3$ using 2 diff. Thms.

So $W = \mathrm{Nul}\, A$

$W = \left\{ b\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} : b, c \in \mathbb{R} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\} = W = \left\{ \begin{bmatrix} -2b+3c \\ b \\ c \end{bmatrix} : b, c \in \mathbb{R} \right\}$

$\therefore$ if $B = \begin{bmatrix} -2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $W = \mathrm{Col}\, B$.

So $W$ is a subspace of $\mathbb{R}^3$ by thm 2 since $W = \mathrm{Nul}\, A_{1\times 3}$ or by thm 3 since $W = \mathrm{Col}\, B_{3\times 2}$

2. let $A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{3\times 3}$ $v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ $w = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$, Suppose you know $Ax = v$ and $Ax = w$ are both consistent. What can you say about eqn. $Ax = v + w$?

$Ax = v$, $Ax = w$ consistent $\Rightarrow \exists\, y, z \in \mathbb{R}^3$ s.t. $Ay = v$ and $Az = w$

so let $x = y + z$ then $Ax = A(y+z) = Ay + Az = v + w$

$\Rightarrow \exists$ a soln. to sys. $Ax = v + w$ $\therefore$ $Ax = v + w$ is also consistent!

(could also state this in terms of Col A)