Defn A **Vector Space** is a non-empty set V of vectors w/ 2 defined operations: addition + scalar mult. (by #'s in $\mathbb{R}$) subject to 10 axioms or rules below.

these axioms must be true $\forall u, v, w \in V$ and $\forall c, d \in \mathbb{R}$

		Name
1)	$u+v \in V$	closed under addition
2)	$u+v = v+u$	addition is commutative
3)	$(u+v)+w = u+(v+w)$	addition is associative
4)	$\exists$ a zero vector, $\vec{0} \in V$ s.t. $\vec{0}+u = u+\vec{0} = u$	existence of additive identity
5)	$\forall u \in V$, $\exists -u \in V$ s.t. $u+(-u)=0$	existence of additive inverse (or closed under add. inverse operation).
6)	scalar mult. of u by c, denote $cu \in V$	closed under scalar mult.
7)	$c(u+v) = cu+cv$	} dist. laws for scalar mult.
8)	$(c+d)u = cu+du$	
9)	$c(du) = (cd)u$	associativity for scalar mult.
10)	$1 \cdot u = u$	Identity for scalar mult.

using these axioms can show $\underline{\underline{0}} \in V$ and $\forall u \in V$, $\underline{\underline{-u}} \in V$ are both unique!

other facts:

$\forall u \in V, c \in \mathbb{R}$

(1) $0u = 0$

(2) $c\vec{0} = \vec{0}$

(3) $-u = (-1)u$

Ex 1 $\mathbb{R}^n$ for $n \geq 1$ are premier examples

use $\mathbb{R}^3$ to understand + visualize concepts.

Ex 2 V is space of all arrows (directed line segs) in 3-dimensions

2 arrows are equal if same length + point is same direction

addition defined by parallelogram rule (Sec. 1.3)

$\forall v \in V$ let cv be the arrow w/ length $|c|$ times length of v pointing in same direction as v if $c>0$, opposite if $c<0$.

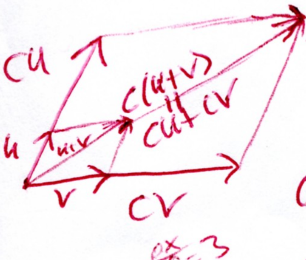


Fig 7

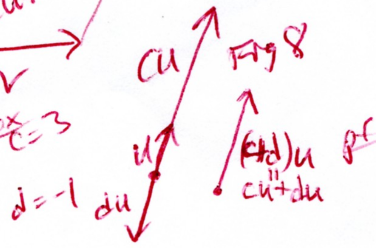


Fig 8

Show V is a vect. sp.

pf 1, 4, 5, 6, 10 clear b/c arrow of zero length is a point representing the $\vec{0} \in V$.

negative of v is $(-1)v$.

rest of 2, 3, 7, 8, 9 done w/ geometry!

EX 3 ~~$\mathbb{S}$~~ $\mathbb{S}$ = space of doubly infinite seqs. of #'s

$$\{y_k\} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$$

if $\{y_k\}, \{z_k\} \in \mathbb{S}$ define sum $\{y_k\} + \{z_k\}$ as $\{y_k + z_k\}$

and scalar mult. by $c \in \mathbb{R}$ as $c\{y_k\} = \{cy_k\}$

Show $\mathbb{S}$ satisfies vect. sp. axioms

EX 4

for $n \geq 0$ $\mathbb{P}_n$ = space of polynomials of deg. at most n

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n \quad (4)$$

where coeffs. $a_0, \ldots, a_n$ are real #'s and so is the variable t.

the degree of p denoted $\deg(p)$ is the highest power of t in (4) w/ nonzero coeff.

if $p(t) = a_0 \neq 0$, $\deg(p) = 0$ constant polynomials

if $a_i = 0 \ \forall i \in \{0, \ldots, n\}$ then $p = 0$ the zero polynomial

this is in $\mathbb{P}_n$ even though it's degree is undefined!

for any $p \in \mathbb{P}_n$ given by (4)

and $q \in \mathbb{P}_n$ given by $q(t) = b_0 + b_1 t + \cdots + b_n t^n$

define the sum $p + q$ by $(p+q)(t) = p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n$

and scalar mult. by $c \in \mathbb{R}$ denoted by cp by $(c \cdot p)(t) = c \cdot p(t) = ca_0 + ca_1 t + \cdots + ca_n t^n$

Show $\mathbb{P}_n$ is a vect. sp. (show axioms hold)

1+6 by defns ↑ $p+q$, cp polys w/ $\deg \leq n$

2, 3, 7-10 follow by props. of real #'s

0 poly. acts like 0 vector $\Rightarrow$ 4

and $(-1)p = -p$, negative of $p \Rightarrow 5$.

EX 5

V = set of real valued fcns. on domain $\mathbb{D}$.

$f + g$ is fcn. w/ value @ $t \in \mathbb{D}$ given by $f(t) + g(t)$

cf has value @ $t \in \mathbb{D}$ $c \cdot f(t)$

EXS if $\mathbb{D} = \mathbb{R}$ $f(t) = 2 - \cos(t)$ $g(t) = t^3 + t^{1/2}$

$$(f+g)(t) = 2 - \cos(t) + t^3 + t^{1/2} \quad \text{and} \quad (4f)(t) = 8 - 4\cos(t)$$

2 fcns. in V are equal iff $f(t) = g(t) \ \forall t \in \mathbb{D}$
f and g

$\vec{0} \in V$ is fcn f s.t. $f(t) = 0 \ \forall t \in \mathbb{D}$

negative of f is $(-1)f$

axioms shown by real # props.

In ex5 the "points" in the vect. sp. are now fcns.

Subspaces

defn a subspace $H \subseteq V$ (⊆ means subset) of a vect. sp. V has 3 props.

1) $\vec{0} \in H$ is the same $\vec{0} \in V$.
2) H is closed under addition $(\forall u, v \in H,\ u+v \in H)$
3) H closed under scalar mult. $(\forall u \in H, \forall c \in \mathbb{R},\ cu \in H)$

these props ensure that H itself is a vect. sp.

other axioms inherited from V b/c of these props.

subspace of V denotes a space inside V

every vect. sp. V is a subspace of itself

Ex6 $\vec{0} \in V$ is a proper subspace of any vect. space w/ nonzero elems. called the zero subspace, denoted by $\{0\}$

Ex7 $\mathbb{P}$ set of all polynomials w/ real coeffs. operations defined as for fcns. (see ex5)

then $\mathbb{P}$ is a subspace of real-valued fcns. defined on $\mathbb{R}$.

and $\forall n \geq 0$ $\mathbb{P}_n$ is a subspace of $\mathbb{P}$

Lets call vect. sp. of all real valued fcns. $\mathbb{F}$ ex5 w/ $\mathbb{D} = \mathbb{R}$

then $\mathbb{P}_n \subseteq \mathbb{P} \subseteq \mathbb{F}$ $\forall n \geq 0$

Ex8 $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$

So $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$

however

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$ which "looks" + "acts" ~~just~~ like $\mathbb{R}^2$ but is in fact ~~different~~.

Note: We say H here is isomorphic (eye-so-more-fick) to $\mathbb{R}^2$. (i.e. $\exists$ an isomorphism between them).

a morphism or map is an isomorphism iff it is a bijection which preserves or respects everything. {all notions of length + distance as well as all operations (addition + scalar mult.)

Soln: $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$ (let $s = t = 0 \in \mathbb{R}$)

$\forall h_1, h_2 \in H$ $h_1 = \begin{bmatrix} s_1 \\ t_1 \\ 0 \end{bmatrix}$ $h_2 = \begin{bmatrix} s_2 \\ t_2 \\ 0 \end{bmatrix}$

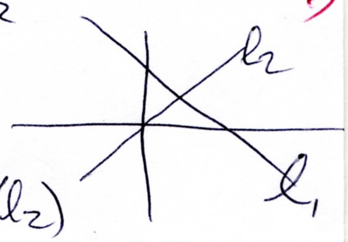
So $h_1 + h_2 = \begin{bmatrix} s_1 + s_2 \\ t_1 + t_2 \\ 0 \end{bmatrix} \in H$

and $ch_1 = \begin{bmatrix} cs_1 \\ ct_1 \\ 0 \end{bmatrix} \in H$

So all 3 props hold for H $\therefore$ H is a subspace of $\mathbb{R}^3$.

Ex9 A plane in $\mathbb{R}^3$ not through the origin, is not a subspace of $\mathbb{R}^2$ b/c $\vec{0}$ is not in the plane.

Similarly a line in $\mathbb{R}^2$ not through the origin (l_1) is also not a subspace, but a line through the origin is (l_2)



A subspace spanned by a set

Thm 1 given vectors $V_1, \dots, V_p$ in a vect. sp. V,
$H = \text{Span}\{V_1, \dots, V_p\}$ is a subspace of V.

EX10 let $p=2$

$H = \text{Span}\{V_1, V_2\} = \{cV_1 + dV_2 : c, d \in \mathbb{R}\}$

so $\vec{0} \in H$ b/c $c=d=0 \in \mathbb{R}$.

further let $h_1 = c_1V_1 + d_1V_2$, $h_2 = c_2V_1 + d_2V_2$ be any 2 elems. in H

so $h_1 + h_2 = (c_1+c_2)V_1 + (d_1+d_2)V_2 \in H \Rightarrow$ closed under addition

and $rh_1 = rc_1V_1 + rd_1V_2 \in H \quad \forall r \in \mathbb{R} \Rightarrow$ closed under scalar mult.

arbitrary scalar multiple

So H is a subspace. //

EX11

$H = \{(a+2b, b-3a, b, a) : a, b \in \mathbb{R}\}$ show H is a subspace of $\mathbb{R}^4$.

soln let $h_1, h_2 \in H$ then $h_1 = a_1\begin{bmatrix}1\\-3\\0\\1\end{bmatrix} + b_1\begin{bmatrix}2\\1\\1\\0\end{bmatrix}$ and $h_2 = a_2\begin{bmatrix}1\\-3\\0\\1\end{bmatrix} + b_2\begin{bmatrix}2\\1\\1\\0\end{bmatrix}$

so $h_1 + h_2 = (a_1+a_2)\begin{bmatrix}1\\-3\\0\\1\end{bmatrix} + (b_1+b_2)\begin{bmatrix}2\\1\\1\\0\end{bmatrix}$ and $ch_1 = ca_1\begin{bmatrix}1\\-3\\0\\1\end{bmatrix} + cb_1\begin{bmatrix}2\\1\\1\\0\end{bmatrix}$. //

The idea here is to represent H as linear combinations of $v_1 = \begin{bmatrix}1\\-3\\0\\1\end{bmatrix}$ and $v_2 = \begin{bmatrix}2\\1\\1\\0\end{bmatrix}$, i.e. $H = \text{Span}\{v_1, v_2\}$ for these $v_1, v_2 \in \mathbb{R}^4$.

EX12

For what values of h will y be in subsp. of $\mathbb{R}^3$ spanned by V_1, V_2, V_3.

$V_1 = \begin{bmatrix}1\\-1\\-2\end{bmatrix}$ $V_2 = \begin{bmatrix}5\\-4\\-7\end{bmatrix}$ $V_3 = \begin{bmatrix}-3\\1\\0\end{bmatrix}$ and $\vec{Y} = \begin{bmatrix}-4\\3\\h\end{bmatrix}$.

Note same problem as practice problem #2 in sec 1.3

$H = \text{Span}\{V_1, V_2, V_3\} = \left\{x_1\begin{bmatrix}1\\-1\\-2\end{bmatrix} + x_2\begin{bmatrix}5\\-4\\-7\end{bmatrix} + x_3\begin{bmatrix}-3\\1\\0\end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R}\right\}$

$\therefore \vec{a} \in H \Rightarrow \vec{a} = \begin{bmatrix}1 & 5 & -3\\-1 & -4 & 1\\-2 & -7 & 0\end{bmatrix}\vec{x}$ for $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3$

so for $\vec{y}$ to be in H there must exist a soln. to the system $\begin{bmatrix}1 & 5 & -3\\-1 & -4 & 1\\-2 & -7 & 0\end{bmatrix}\vec{x} = \begin{bmatrix}-4\\3\\h\end{bmatrix}$ $\begin{bmatrix}1 & 5 & -3 & -4\\-1 & -4 & 1 & 3\\-2 & -7 & 0 & h\end{bmatrix} \sim \begin{bmatrix}1 & 5 & -3 & -4\\0 & 1 & -2 & -1\\0 & 0 & 0 & h-5\end{bmatrix}$ $\therefore h=5$ is necessary.