props. of Determinants 3.2

Note: from last time
corrollary of <u>thm 2</u>
If a matrix has a row or col. of all 0's, then det = 0.

<u>Thm 3</u> let A be square

a) $A \underset{cR_p+R_q}{\sim} B \Rightarrow \det(B) = \det(A)$

b) $A \underset{R_p \leftrightarrow R_q}{\sim} B \Rightarrow \det(B) = -\det(A)$

c) $A \underset{kR_p}{\sim} B \Rightarrow \det(B) = k \cdot \det(A)$

<u>EX#2</u> use computer software packages

suppose $A_{n\times n} \sim U_{n\times n}$ echelon form by row replacements + interchanges (always possible, algorithm says only need to mult. row by a scalar to get to REF).

w/ r row swaps

$\boxed{\det(A) = (-1)^r \det(U)}$ by thm. 3

Since U is echelon form it is triangular.
so $\det(U) = \prod_{i=1} u_{ii}$

$$\det(A) = \begin{cases} (-1)^r \cdot \left(\begin{matrix}\text{prod. of}\\ \text{pivots of } U\end{matrix}\right) & \text{if } A \text{ inv.} \\ 0 & \text{if } A \text{ not inv. (singular)} \end{cases}$$

<u>Note</u>: even though echelon form of A, U is not unique and pivots, u_{ii} are not unique, their product is, up to a possible negative sign.

<u>Thm 4</u> A square matrix is invertible iff $\det(A) \neq 0$.

can add this to the IMT, so a corollary is lin. dep. cols $\Rightarrow$ det = 0.
also if rows are lin. dep., then det = 0.

<u>EX3</u> find det(A) if $A = \begin{bmatrix} 2 & -2 & 3 & -6 \\ 0 & 5 & -1 & -4 \\ -4 & 9 & -7 & 8 \\ 1 & 0 & 3 & -2 \end{bmatrix}$ $\underset{2R_1+R_3}{\sim} \begin{bmatrix} 2 & -2 & 3 & -6 \\ 0 & 5 & -1 & -4 \\ 0 & 5 & -1 & -4 \\ 1 & 0 & 3 & -2 \end{bmatrix}$

A has lin. dep rows b/c $2R_1 + R_3 = R_2 \Rightarrow \det(A) = 0$.

<u>EX4</u> find det(A)

$A = \begin{bmatrix} 0 & 1 & 3 & -2 \\ 3 & 5 & -7 & 2 \\ 0 & 2 & -4 & 1 \\ -3 & -5 & 4 & -2 \end{bmatrix} \xrightarrow{R_2+R_4} \begin{vmatrix} 0 & 1 & 3 & -2 \\ 3 & 5 & -7 & 2 \\ 0 & 2 & -4 & 1 \\ 0 & 0 & -3 & 0 \end{vmatrix}$

use cofactor expansion on col 1

$= -3 \begin{vmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 0 & -3 & 0 \end{vmatrix}$

use cofactor expansion on row 3

$= (-3)(+3) \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = -9 \cdot 5 = -45$

+ b/c in 3,2 pos. $-(-3) = 3$

Col operations

Thm 5 A $n\times n$, $\det(A^T) = \det(A)$

we now have a thm 3' which subst. the word 'column' for the word 'row' in thm. 3.

pf clear for $n=1$
suppose true for all $k\times k$ dets.
let $n=k+1$, cofactor of a_{ij}, C_{ij} in A
is the same as the cofactor of a_{ji}, C_{ji} in A^T by the assumption of the thm for $k\times k$ dets.
So, the cofactor expansion along the ith row of A (equiv. jth col)
is equal to the cofactor exp. down the ith col of A^T! (equiv. jth row).
This is a proof by induction.

Determinants + Matrix Products

Thm 6 if A, B $n\times n$
then $\det(AB) = \det(A)\cdot\det(B)$

Ex 6 $A^T = \begin{bmatrix} 6 & 3 \\ 1 & 2 \end{bmatrix}$ $B^T = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 14 \\ 20 & 13 \end{bmatrix}$ note: $=(AB)^T$

$\det(B^T A^T) = 25\cdot 13 - 20\cdot 14 = 45$
$\det(A^T) = 12-3 = 9$
$\det(B^T) = 8-3 = 5$

compare w/ book's ex 6 to show thm 5 + 6 in action

Linearity of the determinant fcn.

Warning $\det(A+B) \neq \det(A) + \det(B)$

Ex $A^T+B^T = \begin{bmatrix} 10 & 4 \\ 4 & 4 \end{bmatrix}$ $\det(A^T+B^T) = 40-16 = 24$

view $\det(A)$ as a fcn. of the n cols of $A = [a_1 \cdots a_n]$. replace col j by $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ a vector of variables.
let $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n$ remain fixed!

$A = [a_1 \cdots a_{j-1}\ X\ a_{j+1} \cdots a_n] = A^j_X$

define transformation $T: \mathbb{R}^n \to \mathbb{R}$
$X \longmapsto \det(A^j_X)$

claim $T(X)$ is a linear transformation

pf show (1) $T(cX) = cT(X)$ $\forall$ scalars $c \in \mathbb{R}$ $\forall X \in \mathbb{R}^n$
and (2) $T(u+v) = T(u) + T(v)$ $\forall u, v \in \mathbb{R}^n$
apply thm 3'(c) to cols of A to show (1)
see exercise 43 for cofactor expansion down the jth col (the one w/ X in it) of A.
This is called multi-linearity of the det.
every row + col has this prop.
very useful + well studied in more advanced mathematics

read pfs of thms 3+6 in book for HW.

Quiz 4 in class (10 min)

given
$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$

use props. of the determinant to find the following dets:

3.2

① $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ -g & -h & -i \end{vmatrix}$ ② $\begin{vmatrix} d & e & f \\ a & b & c \\ 2g & 2h & 2i \end{vmatrix}$

③ $\begin{vmatrix} 3a & 3b & 12c \\ d & e & 4f \\ g & h & 4i \end{vmatrix}$

2pts each

3pts. each

3.1 use cofactor expansions

① $\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix}$

② $\begin{vmatrix} 4 & 0 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 & 0 \\ 2 & 6 & 3 & 0 & 0 \\ 5 & -8 & 4 & -3 & 0 \\ 6 & 0 & 9 & -5 & 1 \end{vmatrix}$

③ $\begin{vmatrix} 6 & -6 & 8 & 3 & -7 \\ 9 & 3 & -5 & 0 & 2 \\ 8 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{vmatrix}$