Sec. 3.1 [redacted]

Intro to Determinants

given $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & a_{11}a_{32}-a_{12}a_{31} & a_{11}a_{33}-a_{13}a_{31} \end{bmatrix}$

$a_{11}R_2$
$a_{11}R_3$

3x3 invertible matrix

$-a_{21}R_1+R_2$
$-a_{31}R_1+R_3$

wlog assume the (2,2) entry is nonzero (If it is, swap $R_2 \leftrightarrow R_3$, relabel entries in 2^{nd} + 3^{rd} rows)

mult. row 3 by (2,2) entry $(a_{11}a_{22}-a_{12}a_{21})$ then take $^{-}(3,2)$ entry times R_2+R_3
det of upper left 2x2 in original A
$-(a_{11}a_{32}-a_{12}a_{31})$

so 2 more ops in order
$(a_{11}a_{22}-a_{12}a_{21})R_3$
$-(a_{11}a_{32}-a_{12}a_{31})R_2+R_3$

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

where $\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

b/c A inv. $\Delta \neq 0$ ($\Delta \neq 0 \Rightarrow A$ inv. also)

Δ is the determinant of a 3x3 matrix A.

for $A=[a_{ij}]$ in 2x2 $\det(A) = a_{11}a_{22}-a_{12}a_{21}$ for a 1x1 $\det(A)=a_{11}$

pair off the 6 terms in Δ as above

$$\Delta = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{bmatrix}$$

or more simply $\Delta = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$

where A_{1j} is obtained from A by deleting the 1^{st} row + j^{th} col.

for any square matrix A, let A_{ij} denote submatrix gotten by removing row i + col j.

$$A = \begin{bmatrix} 1 & 2 & -4 & 6 \\ 2 & -3 & 5 & 7 \\ 3 & 4 & -6 & 8 \\ 4 & -5 & 7 & 9 \end{bmatrix} \qquad A_{24} = \begin{bmatrix} 1 & 2 & -4 \\ 3 & 4 & -6 \\ 4 & -5 & 7 \end{bmatrix}$$

recursion is an iterative process, by defining something for $n=1$ and telling you how to go from n to $n+1$, I define it for all n.

we now give a recursive defn for det of A an $n \times n$ matrix

defn for $n \geq 2$ the determinant of an $n\times n$ matrix $A=[a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}\det(A_{1j})$ w/ +/− alternating where entries $a_{11}, a_{12}, \dots, a_{1n}$ are the 1^{st} row of A.

In symbols

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \dots + \underbrace{(-1)^{j+1}a_{1j}\det(A_{1j})}_{j^{th}\text{ term in the sum}} + \dots + (-1)^{n+1}a_{1n}\det(A_{1n})$$

$$= \sum_{j=1}^{n} (-1)^{j+1} a_{1j}\det(A_{1j})$$

Ex find det of $A = \begin{bmatrix}1&0&5\\2&-5&4\\3&-2&2\end{bmatrix}$

$$\det(A) = 1\cdot\begin{vmatrix}-5&4\\-2&2\end{vmatrix} - 0\cdot\begin{vmatrix}2&4\\3&2\end{vmatrix} + 5\begin{vmatrix}2&-5\\3&-2\end{vmatrix} = 1(-10-(-8))-0(4-12)+5(-4-(-15))$$

$$= -2+55 = 53$$

we also sometimes use the cofactor of the (i,j) entry, called $C_{ij} = (-1)^{j+i}\det(A_{ij})$

so $\det(A) = a_{11}C_{11}+a_{12}C_{12}+\dots+a_{1n}C_{1n} = \sum_{j=1}^{n} a_{1j}C_{1j}$

this is a cofactor expansion across the 1st row of A.

Thm 1 (w/o pf)

det(A) can be found using a cofactor expansion down any col or across any row.

down the jth col

$\det(A) = a_{1j}C_{1j}+a_{2j}C_{2j}+\dots+a_{nj}C_{nj}$

across the ith row

$\det(A) = a_{i1}C_{i1}+a_{i2}C_{i2}+\dots+a_{in}C_{in}$

the $\pm$ of the cofactor, $(-1)^{i+j}$ depends on the pos. in matrix $\rightarrow$ checkerboard pattern of +'s and −'s

$$\begin{bmatrix}+&-&+&\cdots\\-&+&-&\\+&-&+&\\\vdots&&&\ddots\end{bmatrix}$$

if $i+j$ is even $\rightarrow$ "+" sign
if $i+j$ is odd $\rightarrow$ "−" sign

EX2 use cofactor expansion across 3rd row to find det(A)

w/ $A = \begin{bmatrix}1&0&5\\2&-5&4\\3&-2&2\end{bmatrix}$ think $i=3$ in 2nd formula above

$\det(A) = a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33}$

$$\det(A) = 3\begin{vmatrix}0&5\\-5&4\end{vmatrix} - (-2)\begin{vmatrix}1&5\\2&4\end{vmatrix} + 2\begin{vmatrix}1&0\\2&-5\end{vmatrix}$$

$= 3(0+25)+2(4-10)+2(-5-0)$

$= 75-12-10 = 53$ (same as above)

use cofactor expansion down 2nd col $j=2$ in 1st formula

$\det(A) = a_{12}C_{12}+a_{22}C_{22}+a_{32}C_{32}$

$$= -0\begin{vmatrix}2&4\\3&2\end{vmatrix} + (-5)\underbrace{\begin{vmatrix}1&5\\3&2\end{vmatrix}}_{-13} - (-2)\underbrace{\begin{vmatrix}1&5\\2&4\end{vmatrix}}_{-6}$$

$= 0+65-12 = 53$ (also same)

The more zeros a matrix A has the easier it is to compute it's determinant.

Thm1 helps when 1 particular row or col has a lot of zeros.

EX3

Consider

$$A = \begin{bmatrix}9&-6&8&3&-7\\7&3&-5&0&2\\5&0&1&0&0\\4&-1&2&0&0\\-2&0&0&0&0\end{bmatrix}_{5\times5}$$

lets expand along the last row

$$\det(A) = \underset{5+1=\text{even}\,+}{-2}\begin{vmatrix}-6&8&3&-7\\3&-5&0&2\\0&1&0&0\\-1&2&0&0\end{vmatrix} + 0\cdot(\text{other cofactors})$$

all the rest of the a_{5j} terms are 0.

to find this determinant, lets expand down the 3rd col.

$$\det(A) = (-2)\underset{1+3=\text{even}\,+}{(3)}\begin{vmatrix}3&-5&2\\0&1&0\\-1&2&0\end{vmatrix} = (-2)(3)\underset{1+3\text{ even}}{(2)}\underset{(1)}{\begin{vmatrix}0&1\\-1&2\end{vmatrix}} = -12$$

again, 3rd col.

Notes: I swapped cols from ex3 in book. each row/col swap negates the det., I used 2 swaps

Thm 2

If A is a triangular matrix, then $\det(A) = \prod_{i=1}^{n} a_{ii}$ product of diagonal entries.

If a matrix has an entire row or col. of all 0's, then it's determinant is 0.