Comments
we are covering
2.1, 2.2, 2.3, 3.1, 3.2, ~~[illegible]~~
~~[illegible]~~ 4.1, 4.2, 4.3

chaps. secs.
Test 2 — { 2.1-3, 3.1-2, 4.1-3 }

2.4-2.9 Not covered in class or on exams

However, 2.4 Partitioned Matrices, 2.5 Matrix Factorizations } valuable skills in future lin. alg. courses, (410)

If still struggling w/ span look at these! { *2.8 Subspaces of $\mathbb{R}^n$, *2.9 Dimension + Rank }

VanderMonde in 2.8 on pg. 184 look @ #11

if of use to you in your major, look these over
2.6 Leontif input/output model (production applications)
2.7 Computer Graphics applications

2.3 (IMT) Invertible Matrix Thm.

Thm 8 TFAE, for $A_{n \times n}$

all are either true or false.

① A is invertible (nonsingular)
② A is row equiv. to I_n (A has REF I_n)
③ A has n pivots (pivot cols)
④ $Ax=0$ has only triv. soln ($x=0$ is unique soln)
⑤ Cols of A are lin. indep. (in $\mathbb{R}^n$)
⑥ The lin. trans. $x \mapsto Ax$ is 1-1.
⑦ The eqn. $Ax=b$ has a soln. $\forall b \in \mathbb{R}^n$
⑧ Cols of A span $\mathbb{R}^n$ (Span{cols of A} $= \mathbb{R}^n$)
⑨ lin. trans. $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
⑩ $\exists$ $n \times n$ matrix C s.t. $CA = I_n$
⑪ $\exists$ $n \times n$ matrix D s.t. $AD = I_n$
⑫ A^T is invertible.

Notation
*
if ⓘ, then ⓙ
ⓘ ⇒ ⓙ
ⓘ iff ⓙ
ⓘ ⇔ ⓙ

*
Notation: If by assuming any of these statements ⓘ we can show another one is true, ⓙ we write ⓘ ⇒ ⓙ (if ⓙ also implies ⓘ we write ⓘ ⇔ ⓙ)

we will prove a circle of implications and tie all the others in as follows

② → ① → ⑩ → ④ → ③ → ②
① → ⑪ → ⑦ → ①
⑦ ⇔ ⑧ ⇔ ⑨
④ ⇔ ⑤ ⇔ ⑥
① ⇔ 12

pf A inv., so $A^{-1} = C$ ∴ ① ⇒ ⑩. ⑩ ⇒ ④ by exercise 23 in 2.1. ④ ⇒ ③ by #23 in 2.2.

If A square, w/ n pivots, then pivots are all on main diag., so REF $= I_n$ by def., ∴ ③ ⇒ ②

② ⇒ ① by Thm 7 in 2.2. Pentagon complete! ① ⇒ ⑪ since $A^{-1} = D$ also works. ⑪ ⇒ ⑦ by #24 in 2.1.

⑦ ⇒ ① by #24 in 2.2. Triangle complete (⑦ and ⑪ are attached to pentagon @ ①) ⑦, ⑧, ⑨ all equiv. by Thm 4 in 1.4 and Thm 12(a) in 1.9

so ⑧ and ⑨ linked via ⑦

④, ⑤, ⑥ all equiv. for any matrix A (not just $n \times n$) by sec. 1.7 and Thm. 12b in 1.9.

so ⑤ and ⑥ linked via ④.

lastly, ① ⇒ ⑫ by Thm. 6c in 2.2
and ⑫ ⇒ ① by same thm. i.s. swap A^T w/ A.
QED, ■

notes/facts

Could say in ⑦, soln to $Ax=b$ is unique $\forall b \in \mathbb{R}^n$.

let A, B be square matrices. If $AB = I$, then A and B are invertible, with $A = B^{-1}$ and $B = A^{-1}$!

You may cite this thm. as a justification in your solns., wherever appropriate, by writing **[by IMT]**
"USE IT! USE IT WELL!"

~~This thm.~~ The IMT partitions or divides all $n \times n$ matrices into 2 disjoint classes (sets): The invertible/non-singular and the non-invertible/singular

① – ⑫ are props. of invertible $n \times n$ matrices
the negation of any of these statements is a prop. of every singular $n \times n$ matrix.

ex any singular, $n \times n$ matrix A, is not row equiv. to I, does not have n pivots, etc. has lin. dep. cols., cols A do not span $\mathbb{R}^n$, etc.

EX1 use this in practice! does sq. matrix have n pivots? If so, then invertible!

This is a very powerful result. See HW probs
But, it only applies to square matrices

Invertible Transformations

matrix mult. corresponds to composition of lin. transformations

consider $A^{-1}Ax = x$ in terms of associated lin. trans.

A lin. trans. $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible ~~$x \mapsto Ax$~~ if $\exists$ $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t. $S(T(x)) = x$ (1) and $T(S(x)) = x$ (2) $\forall x \in \mathbb{R}^n$

Thm 9 let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a lin. trans. w/ standard matrix A.

given $T: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$, $\exists$ A $n \times n$ matrix

T is invertible iff A is invertible.
if so, then $T^{-1} = S: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto A^{-1}x$ i.e. $T^{-1}(x) = A^{-1}x$.

I may sometimes write T_A for the lin. trans. assoc. to A.
then $T_{A^{-1}} = (T_A)^{-1}$ by Thm. 9.

pf T inv., (2) ⇒ T is onto $\mathbb{R}^n$, b/c if $b \in \mathbb{R}^n$ and $x = S(b)$, then $T(x) = T(S(b)) = b$ so $\forall b \in \mathbb{R}^n$, $b \in$ Range(T). ∴ A inv. by IMT ⑨

conversely A inv., let $S(x) = A^{-1}x$. S is linear trans., and clearly satisfies (1) and (2).

For example $S(T(x)) = S(Ax) = A^{-1}Ax = x$.
$T(S(x)) = T(A^{-1}x) = AA^{-1}x = x$. so T is inv. by defn.

see exercise 29 for uniqueness of the inverse.