Thm 5
If A invertible $n \times n$,
then $\forall b \in \mathbb{R}^n$, $Ax = b \Rightarrow \exists! \, x = A^{-1}b$
(has a unique soln. $x = A^{-1}b$)

pf show existence + uniqueness
let $x = A^{-1}b$ $Ax = A(A^{-1}b) = Ib = b$
$\Rightarrow \exists x$
let u solve $Ax = b$
$\Rightarrow Au = b$ mult. by A^{-1} on left $\Rightarrow \cancel{A^{-1}Au} = A^{-1}b$ so $u = A^{-1}b$
$\Rightarrow$ all solns $= A^{-1}b$ so unique

EX 4 using $A + C$ from EX 1
find the soln. to following 10 problems
(E.C. game?) count off 1-10
1st one to board w/ correct answer gets 2pts. bonus on today's matlab quiz!

problems
~~$2i-1)$ $Ax_{2i-1} = b_i$ for $i = 1,\dots,5$ $2i)$ $Cx_{2i} = b_i$ for $i = 1,\dots,4$ for $i=5$ $Cx_{2i} = b_{i+1}$~~
~~$b_1 = \begin{bmatrix}-1\\4\end{bmatrix}$ $b_2 = \begin{bmatrix}0\\1\end{bmatrix}$ $b_3 = \begin{bmatrix}1\\1\end{bmatrix}$ $b_4 = \begin{bmatrix}5\\4\end{bmatrix}$ $b_5 = \begin{bmatrix}-5\\19\end{bmatrix}$ $b_6 = \begin{bmatrix}24\\19\end{bmatrix}$~~

problem #
$2i-1)$ $Ax_{2i-1} = b_i$ for $i \in \{1,\dots,5\}$ $b_1 = \begin{bmatrix}-1\\4\end{bmatrix}$ $b_2 = \begin{bmatrix}0\\1\end{bmatrix}$ $b_3 = \begin{bmatrix}1\\1\end{bmatrix}$

$b_4 = \begin{bmatrix}5\\4\end{bmatrix}$ $b_5 = \begin{bmatrix}-5\\19\end{bmatrix}$

$2i)$ $\begin{cases} Cx_{2i} = b_i & \text{for } i \in \{1,\dots,4\} \\ Cx_{2i} = b_{i+1} & \text{for } i = 5 \end{cases}$ $b_6 = \begin{bmatrix}24\\19\end{bmatrix}$

Thm 6
a) If A invertible $n \times n$, then A^{-1} invertible $n \times n$ and $(A^{-1})^{-1} = A$ (transpose anyone)
the inverse operation, just like the transpose, is order 2!
b) If A, B invertible $n \times n$, then so is AB (and BA)
$(AB)^{-1} = B^{-1}A^{-1}$ $\left[(BA)^{-1} = A^{-1}B^{-1}\right]$
what does $A^{-1}B^{-1}$ equal? $(BA)^{-1}$
c) A invertible $n \times n$, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$ order not important when doing both operations

pf of thm 6 In the spirit of today's quiz...
a) A inv. $n \times n \Rightarrow \exists \, C$ $n \times n$ s.t. $AC = I = CA$
we call it $C = A^{-1}$
so A is inverse of A^{-1} by def.
b) A, B inv. $n \times n \Rightarrow \exists \, A^{-1}$ and B^{-1}
consider $(B^{-1}A^{-1})(AB) = B^{-1}IB = I$
$(AB)(B^{-1}A^{-1}) = AIA^{-1} = I$
swap roles of A and B
$(BA)^{-1} = A^{-1}B^{-1}$
c) $I = I^T = (AA^{-1})^T = (A^{-1})^T A^T$
$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$

for A $n \times n$, A invertible (nonsingular) $\Leftrightarrow$ row equivalent to the identity
i.e. A is singular iff A is not row equivalent to the identity.
$\boxed{A \text{ is not invertible if REF} \neq I_n}$

Thm 7 A $n \times n$ invertible iff A row equiv. to I_n
and any seq. of row ops that reduces A to I_n also turns I_n into A^{-1}
pf Thm 5 $\Rightarrow$ $Ax = b$ has soln $\forall b \in \mathbb{R}^n$, so A has pivot in every row (and col.), by Thm 4 (1.4)
A $n \times n$ inv.
since A is $n \times n$ pivots are on main diag.
i.e. REF$(A) = I_n$ or $A \sim I_n$ (one direction) $\Rightarrow$
$\Leftarrow$ If $A \sim I_n$, each step of row reduction corresponds to left mult. by some elem. mat.
$\Rightarrow \exists \, E_1, \dots, E_p$ s.t. $A \sim E_1A \sim \cdots \sim E_p(E_{p-1} \cdots E_1)A = I_n$
thus if $E = E_p E_{p-1} \cdots E_2 E_1$, then $EA = I$
and so $A^{-1} = E_p \cdots E_1$ and $A = E_1^{-1} \cdots E_p^{-1} = (E_p \cdots E_1)^{-1}$

Algorithm for computing A^{-1} (derived from thm 7)
Augment A w/ I_n $[A \; I]$ row reduce A to REF if A is ~~invertible~~ row equivalent to I then
$[A \; I] \sim [I \; A^{-1}]$
o.w. A has no inverse, A is singular

Alternative view of matrix inversion
let $I_n = [e_1 \cdots e_n]$ row reduction of $[A \; I]$ to $[I \; A^{-1}]$
is simultaneous solns. to n systems: $Ax = e_1, \dots, Ax = e_n$ (2)
augmented cols to each sys. all next to A $[A \; e_1 \cdots e_n] = [A \; I]$
b/c $AA^{-1} = I$ w/ defn. of matrix product, show cols of A^{-1} solve sys. of eqns A (2)
If you only need a few cols. of A^{-1} can solve just corresponding systems from (2)

Note: in practice almost never want to solve a system $Ax = b$ by finding the inverse. Only in rare 2x2 cases is this easier than doing row reduction.

Ex $A = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ find AC and CA. Sol'ns. to Ex's

$AC = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$CA = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ex4
using A and C from previous ex, solve each of the following systems for x.

1) $Ax_1 = b_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ Sheridan $Cx = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2) $Cx_2 = b_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

3) $Ax_3 = b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

4) $Cx_4 = b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Cherilyn $\frac{1}{2}$

5) $Ax_5 = b_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Jeff J.

6) $C\cancel{A}_6 = b_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7) $Ax_7 = b_4 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

8) $Cx_8 = b_4 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ Kangran

9) $Ax_9 = b_5 = \begin{bmatrix} -5 \\ 19 \end{bmatrix}$ William

10) $Cx_{10} = b_5 = \begin{bmatrix} 24 \\ 19 \end{bmatrix}$ Emma

x_3: $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ x_6: $C\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

x_8: $C\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ x_5: $A\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

x_1: $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ x_{10}: $C\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 19 \end{bmatrix}$ x_7

x_4: $C\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ x_9: $A\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 19 \end{bmatrix}$

x_2: $C\begin{bmatrix} -5 \\ 19 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

Ex5
perform 3 types of elem. row ops. on I_3 — add mult. of row to another, swap rows, mult. row by non-0

$2R_2 + R_1$ $\quad R_1 \leftrightarrow R_3$ $\quad -R_2$

let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

- $E_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ • $E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ • $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ find E_1A, E_2A, E_3A (as if same 3 row ops. performed)

- $E_1A = \begin{bmatrix} a+2d & b+2e & c+2f \\ d & e & f \\ g & h & i \end{bmatrix}$ $E_2A = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$ $E_3A = \begin{bmatrix} a & b & c \\ -d & -e & -f \\ g & h & i \end{bmatrix}$

mult. on left by these "elementary matrices", performs the row op they represent elem. row ops. represent on ANY $3\times n$ matrix.

$E_i \cdot I = E_i$ shows this for the identity

Facts: result of elem. row op. performed on A, $m\times n$ is EA, where E is result of the same row op applied to I_m

row ops. "reversible" (can be undone) ⇒ elementary matrices are invertible.
• for every E, $\exists F$ s.t. $EF = I = FE$

If A is invertible repeated use of this fact leads to thm 7.

• Ex6 find $E_1^{-1}, E_2^{-1}, E_3^{-1}$

$F_1 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $-2R_2 + R_1$

$F_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_3$ own inverse order 2 object! (all row swaps + corresponding matrices)

$F_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $-R_2$ again order 2

but in general cR_i has inverse $\frac{1}{c}R_i$

row i $\begin{bmatrix} 1 & \cdots & 0 \\ 0 \cdots & c & \cdots 0 \\ 0 & \cdots & 1 \end{bmatrix} = E$ $\quad F = \begin{bmatrix} 1 & \cdots & 0 \\ 0 \cdots & \frac{1}{c} & \cdots 0 \\ 0 & \cdots & 1 \end{bmatrix}$ row i

• Ex7 find inverse of $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & f \\ 2 & 6 & 0 \end{bmatrix}$ w/ algorithm.

$-2R_1 + R_3,\ 3R_2 + R_1,\ -R_2,\ \frac{1}{2}R_3,\ (1-3f)R_3 + R_1,\ fR_3 + R_2 \Rightarrow A^{-1} = \begin{bmatrix} 3f & 3 & (1-3f)/2 \\ -f & -1 & f/2 \\ -1 & 0 & 1/2 \end{bmatrix}$

verify that $AA^{-1} = A^{-1}A = I$ (check your work!)

exercise for HW
verify that these are in fact inverses by computing the matrix products E_iF_i and F_iE_i for $i = 1,2,3$
practice/challenge HW for weekend!
Can you prove this for arbitrary row operations and arbitrary elementary matrices? Try it!
Hint: split into 3 cases 1/type of row op!

maybe: (1 idea) define mats. in terms of rows/cols formulas for $\mathrm{row}_i(AB) =$ $\quad \mathrm{col}_j(AB) =$

Weekend Challenge (optional)

If you can prove $\exists\, E_1^{-1}, E_2^{-1}, E_3^{-1}$ elem. mats. associated to the elem. row ops.

$cR_i + R_j$, $R_i \leftrightarrow R_j$, and cR_i

I will grade it as an extra quiz for you!

? for the 3 above first express E_1, E_2, E_3 explicitly, and show these are in fact inverses. then give/find there inverse explicitly.