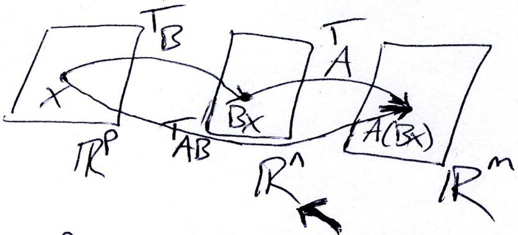
Matrix Mult.



let A be $m\times n$
B be $n\times p$

So $\exists\, T_B : \mathbb{R}^p \to \mathbb{R}^n,\ T_A : \mathbb{R}^n \to \mathbb{R}^m$

composition of fcns.

$\forall$ fcns. f, g s.t. $\mathrm{Dom}(f) \supseteq$ ~~Range~~(g)
why is this necessary? think middle space!

$f\circ g(x) = f(g(x))$

$T_A \circ T_B$ = composition of maps

$T_A \circ T_B(x) = T_A(T_B(x)) = T_A(Bx) = A(Bx)$
where $x \in \mathbb{R}^p$, a $p\times 1$ colvector
so Bx is an $n\times 1$ colvector, i.e. $Bx \in \mathbb{R}^n$
thus in $\mathrm{Dom}(T_A) \Rightarrow A(Bx) \in \mathbb{R}^m$ an $m\times 1$ colvect.

since B is $n\times p$ let A's cols be
$B = [\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_p]$, so $Bx = x_1\vec{b}_1 + \cdots + x_p\vec{b}_p$ since matrix vector mult. is linear

$A(Bx) = Ax_1\vec{b}_1 + \cdots + Ax_p\vec{b}_p = x_1 A\vec{b}_1 + \cdots + x_p A\vec{b}_p \to$ lin comb. of vectors $\{A\vec{b}_1, \ldots, A\vec{b}_p\}$ w/ wts. $x_1, \ldots, x_p$
def. of matrix vect. mult.
$\Rightarrow A(Bx) = [A\vec{b}_1 \cdots A\vec{b}_p]x = (AB)x$

So $A\cdot B$ is defined as $[A\vec{b}_1 \cdots A\vec{b}_p]$
the matrix product of A and B
when A is $m\times n$ and B is $n\times p$.
Since $A\vec{b}_i$ is $m\times 1$, AB is $m\times p$.
i is an index
$\forall i = 1, \ldots, p$

EXs
let $A = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & -6 \\ -3 & 1 & 2 \end{bmatrix}$

~~AB=~~ $Ab_1 = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$ $Ab_2 = \begin{bmatrix} 1 \\ 12 \end{bmatrix}$ $Ab_3 = \begin{bmatrix} -10 \\ -24 \end{bmatrix}$

So $AB = \begin{bmatrix} 8 & 1 & -10 \\ 8 & 12 & -24 \end{bmatrix}$

EX if A is 3×2
and B is 2×3
what are sizes of AB and BA?
(3x3) (2x2)

$$\underset{A}{\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}} \underset{B}{\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

$$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

Fact:
each col. of AB is lin. comb. of cols of A w/ wts. from corresponding col of B.

EX 2nd col. above

$\begin{bmatrix} 1 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 0 \end{bmatrix}$

2nd col of AB; 1st entry in 2nd col of B; 1st col. of A; 2nd entry in 2nd col of B; 2nd col. of A.

true for any col of AB
the ith col of AB = lin. comb. of cols. of A w/ wts. = ith col. of B.

clearly cols of A = rows of B to make lin. combs. here.
Also see that AB has same # of rows as A & same # of cols as B.

Row-Col rule for computing AB

$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

for $A\ m\times n$ and $B\ n\times p$.

This is called a dot product or inner product of the ith row of A and the jth col of B.

if $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ as row representation and $B = [b_1\ b_2 \cdots b_p]$ as col. representation

then $AB = \begin{bmatrix} a_1\cdot b_1 & \cdots & a_1\cdot b_p \\ \vdots & a_i\cdot b_j & \\ a_m\cdot b_1 & \cdots & a_m\cdot b_p \end{bmatrix}$ ith row, jth col.

$a_i\cdot b_j = \langle a_i, b_j\rangle := a_i^T b_j$
dot prod. (on vectors); inner prod. (more general); as a matrix prod.

If a_i and b_j are both thought to be $n\times 1$ vectors then the "T" is needed. O.w. if a_i is $1\times n$ and b_j $n\times 1$ just mult. matrices

$AB = \{a_i\cdot b_j\}_{i=1\, j=1}^{m\quad p}$ where $a_i, b_j \in \mathbb{R}^n$ $\forall i \in \{1, \ldots, m\}\ j \in \{1, \ldots, p\}$
b/c A has n cols. and B has n rows.
i.e. the length of the rows of A = length of cols of B

EX5 use row-col. method to find 2^{nd} row of previous ex.

only need 2^{nd} row of A

$A = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & -6 \\ -3 & 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} \square & \square \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 & -6 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square \\ 8+0 & 12+0 & -24+0 \end{bmatrix} = \begin{bmatrix} \square & \square & \square \\ 8 & 12 & -24 \end{bmatrix}$$

EX6

find 2^{nd} col of CD

for $C = \begin{bmatrix} 4 & 7 & 3 \\ -6 & 1 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 2 & -1 & 6 & -3 \\ -5 & 3 & -8 & 0 \\ 0 & -4 & -7 & 9 \end{bmatrix}$

only need 2^{nd} col. of D.

$$\begin{bmatrix} 4 & 7 & 3 \\ -6 & 1 & 2 \end{bmatrix}\begin{bmatrix} (& -1 & (& (\\ & 3 & & \\ ) & -4 &) &) \end{bmatrix} = \begin{bmatrix} \square & 5 & \square & \square \\ \square & 1 & \square & \square \end{bmatrix}$$

Note: $C = B^T$ $D = A^T$ from EX 6 in book

They look for 2^{nd} row of AB.

recall: $(AB)^T = B^T A^T = CD$

so 2^{nd} col of $(AB)^T$ = 2^{nd} row of AB "flipped on its side"

compare w/ EX6 in book!

Facts: $row_i(AB) = row_i(A) \cdot B$

$col_j(AB) = A \cdot col_j(B)$

I will always try to use "i" as a row index and "j" as a col. index whenever I can.

props. of Matrix Mult.

Exercise: What sizes can B, C be? ~~...~~ what sizes result in Thm.

Thm 4 let A be $m \times n$, B and C are sized so sums + prods are defined!

B here can be $n \times p$ and then C is $p \times \blacksquare$ (any non-zero #) — $m \times \blacksquare \leftarrow$ a) $A(BC) = (AB)C$ (associative law of mult.)

B, C same size, both $n \times \blacksquare$ — $m \times \blacksquare \leftarrow$ b) $A(B+C) = AB + AC$ (left distributive law)

B, C same size, both $\blacksquare \times m$ — $\blacksquare \times n \leftarrow$ c) $(B+C)A = BA + CA$ (right distributive law)

B must be $n \times \blacksquare$ — $m \times \blacksquare \leftarrow$ d) $r(AB) = (rA)B = A(rB)$ $\forall$ scalars $r \in \mathbb{R}$ (scalar factor)

I_m is $m \times m$, I_n is $n \times n$ — $m \times n \leftarrow$ e) $I_m A = A = A I_n$ (identity property) see book for pf

warnings ① in general $AB \neq BA$

② cancellation laws do not hold, if $AB = AC$ does not mean that $B = C$ (HW #10)

③ if $AB = 0_{matrix}$ does not imply that $A = 0$ or $B = 0$ (HW 12)

EX7 + 7.5

$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix}$ find AB and BA

$AB = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 12 & 6 \end{bmatrix}$ $BA = \begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -8 \\ 0 & 6 \end{bmatrix}$

definitely have similar entries, but $AB \neq BA$!

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$ find AB

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

let $C = \begin{bmatrix} 5 & 4 \\ 0 & 0 \end{bmatrix}$ $AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

so no cancellation, $B \neq C$

and $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

powers of a matrix

if A is square, $n \times n$

for a pos. integer k, A^k denotes prod. of k copies of A.

$A^k = \underbrace{A \cdot A \cdots A}_{k\ many}$

for A nonzero $x \in \mathbb{R}^n$ then $A^k x$ is well defined as left mult. x by A k times.

and so $A^0 x := x \Rightarrow A^0 := I_n$

Sec. 2.2 Inverse of a Matrix

A ($n \times n$) invertible if $\exists C_{n \times n}$ s.t.

$AC = CA = I_n$ ✓ (from now on n will be omitted b/c clear from context all matrices are $n \times n$)

we call C the inverse of A, C is uniquely determined by A.

suppose B is another inverse of A,

then $B = BI = B(AC) = (BA)C = IC = C$

so $B = C$ all inverses of A are equal! $\Rightarrow$ uniqueness

we denote the unique inverse of A (when it exists) by A^{-1}, so that $A^{-1}A = AA^{-1} = I$

when A is not invertible we say it is singular and when it is invertible we call it non-singular

2x2 inverses ⊛ Thm 4 (pf EX 25+26)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. if $ad - bc \neq 0$ then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

when $ad - bc = 0$ A is singular!

$ad - bc = \det(A)$ called the determinant of A

⊛ Ex $A = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ find AC and CA.