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The Crystals Associated to Barsotti-Tate Groups: with Applications to Abelian Schemes

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INTRODUCTION

The concept of Barsotti-Tate group was introduced in [1] where the name equidimensional hyperdomain was used (actually for an equivalent concept) and in [30] where the name p-divisible group was used. Following Grothendieck, we prefer the term Barsotti-Tate group because the concept of "p-divisible group" has a meaning for any abelian group object in an arbitrary category and does not indicate any relation with algebraic geometry.

Barsotti-Tate groups arise in "nature" when one considers the sequence of kernels of multiplication by successive powers of p on an abelian variety. Also, as Grothendieck has observed, there are Barsotti-Tate groups which are naturally associated with the crystalline cohomology of a proper smooth scheme which is defined over a perfect field of characteristic p. Since we do not discuss crystalline cohomology no further mention is made of this example.

Returning to the situation where A is an abelian variety over a (perfect) field of characteristic p, let $A(n)$ denote the kernel of multiplication by p^n on A. The system $(A(n))_{n \geq 1}$ constitute a Barsotti-Tate group. As opposed to the situation where one looks at the kernel of multiplication by ℓ, $(\ell, p) = 1$, the groups $A(n)$ are never étale unless $A = (0)$. Each $A(n)$ can be written uniquely as a product of a connected group with an étale group. The sequence of these connected components yield, by passage to the limit, the formal group obtained by completing A along the identity. Thus the Barsotti-Tate group contains more information than just this formal group. Another advantage which Barsotti-Tate groups possess over formal groups is that there is a duality (generalizing Cartier duality) defined for them.

In this paper Barsotti-Tate groups over an arbitrary base scheme S (on which p is locally nilpotent) are studied. Intuitively we can think of these as nicely varying families of Barsotti-Tate groups parametrized by S. The basic theorems are those which allow us to associate various crystals to such groups. Actually in order to construct these crystals we assume that the Barsotti-Tate groups in question are, locally, liftable to infinitesimal neighborhoods. This restriction is surely unnecessary. In fact, recent work of L. Illusie (modulo several verifications of compatabilities he has yet to make) shows that the obstruction to the existence of such liftings lie in a particular Ext group.* Combining this with a result of Grothendieck showing that the obstruction question is zero, we see it has "essentially" been proven that all Barsotti-Tate groups satisfy the above lifting hypothesis. Thus, for the rest of the introduction, assume all Barsotti-Tate groups are locally liftable in the above sense.

Let us now give a more detailed summary of the various chapters. For simplicity we restrict to the affine case: Let A be a ring, I a nilpotent ideal of A, $A_o = A/I$. Assume p is nilpotent in A. Set $S = \mathrm{Spec}(A)$, $S_o = \mathrm{Spec}(A_o)$ and generally denote with the subscript "o" the result of restricting an object defined over S to S_o.

In Chapter I the concept of a Barsotti-Tate group over S is defined, several sorites and several examples are given. This chapter consists

* These verifications have been made [17 bis].

primarily of definitions.

In Chapter II the relation between formal Lie groups and Barsotti-Tate groups is studied. First assuming A_o is of characteristic p, we show how to associate a formal Lie group to any Barsotti-Tate group G defined on S_o. Via the use of the relative cotangent complex this result is extended to a Barsotti-Tate group G over S (in the case where $I = pA$). The procedure employed is to actually reduce to the case already treated when the base is of characteristic p. Next we treat the question of when is a formal Lie group a Barsotti-Tate group. The results here are certainly partial and only the case when S is artin is explicitly dealt with. Finally necessary and sufficient conditions for a Barsotti-Tate group to be expressible as an extension of an ind-étale Barsotti-Tate group by a formal Lie group are given.

Chapter III is largely preliminary. The first and third sections recall the relevant definitions and properties of divided powers and crystals respectively. The second section treats the "exponential" which is used extensively later. In order to illustrate this concept assume the ideal I has nilpotent divided powers. Let $H = \mathrm{Spec}(B)$ be an affine commutative S-group such that $\underline{\mathrm{Lie}}(H)$ is locally-free of finite type. Denote by $\eta: B \to A$ the augmentation corresponding to the unit section of H. The exponential will be a homomorphism:

$$\exp: \mathrm{Hom}_{\mathcal{O}_S}(\mathbb{G}_a, I \otimes \underline{\mathrm{Lie}}(H)) \hookrightarrow \mathrm{Ker}\,[\mathrm{Hom}_{S\text{-gr.}}(\mathbb{G}_{a_S}, H) \longrightarrow \mathrm{Hom}_{S_o\text{-gr.}}(\mathbb{G}_{a_{S_o}}, H_o)].$$

For $\theta: \mathbb{G}_a \to I \otimes \underline{\mathrm{Lie}}(H)$, let $\theta(1) = \sum i_j \otimes D_j$ where D_j is an A-linear invariant derivation of B to itself. $\exp(\theta)$ is to be a homomorphism $\mathbb{G}_a \longrightarrow H$ and hence to describe it we give the corresponding algebra map $B \longrightarrow A[T]$. This algebra map is given explicitly via:

$$b \longmapsto \sum_{n \geq 0} \eta((\Sigma\, i_j D_j)^{(n)}(b))\, T^n .$$

This is an algebra map by Leibniz's rule, and is a bi-algebra map because the D_j's are invariant. Additivity in θ is checked directly and finally it is clear that $\exp(\theta)\colon \mathbb{G}_a \longrightarrow \mathrm{Ker}(H \longrightarrow H_o)$.

The exponential is studied in various contexts more general than the above. Unfortunately we have to resort to an ad hoc construction at one point and hence the discussion can not be regarded as truly satisfactory. Finally we utilize the exponential in order to define a notion of linear equivalence between certain vector subgroups of a given smooth group. This allows us to "linearize" the deformation theory of Chapter V.

Chapter IV is technically the heart of the work. It is here that we construct the crystals alluded to above. The first section constructs for any Barsotti-Tate group G on S a "universal" extension of G by a vector group:

$$(*) \qquad 0 \longrightarrow \underline{V}(G) \longrightarrow E(G) \longrightarrow G \longrightarrow 0$$

Here $\underline{V}(G) = \underline{\omega}_{G^*}$, the invariant differentials on the Cartier dual of G. This extension is universal in the sense that given any extension $0 \longrightarrow W \longrightarrow E \longrightarrow G \longrightarrow 0$ of G by a vector group W, there is a unique linear homomorphism $\underline{V}(G) \longrightarrow W$ which induces it from the extension (*).

The construction of the universal extension is functorial in G and commutes with all base changes.

Let us endeavor to explain how the crystals are obtained. For each G_o, a Barsotti-Tate group on S_o, crystals $\mathbb{E}(G_o)$, $\overline{\mathbb{E}}(G_o)$, $\mathbb{D}(G_o)$ are defined. The latter two are obtained from $\mathbb{E}(G_o)$ by "completing along the identity" or by taking the Lie algebra in the respective cases. Thus $\mathbb{E}(G_o)$ is the fundamental crystal to be constructed. The idea is to "crystallize" $E(G_o)$, the universal extension mentioned above. This means we want to show that if G_1 and G_2 are two liftings of G_o to S, then $E(G_1)$ and $E(G_2)$ are canonically isomorphic. The main theorem of Chapter IV proves this. The proof relies heavily on the exponential. Why is this? Well, we want an isomorphism $v\colon E(G_1) \xrightarrow{\sim} E(G_2)$ which reduces to $\mathrm{id}_{E(G_o)}$. Let us consider the two extensions:

$$0 \longrightarrow \underline{V}(G_1) \xrightarrow{i_1} E(G_1) \longrightarrow G_1 \longrightarrow 0$$

$$0 \longrightarrow \underline{V}(G_2) \xrightarrow{i_2} E(G_2) \longrightarrow G_2 \longrightarrow 0$$

Let w be any lifting of $\mathrm{id}_{V(G_o)}$ to a linear map $\underline{V}(G_1) \longrightarrow \underline{V}(G_2)$. If we could construct v and w such that $v \circ i_1 = i_2 \circ w$, then v would induce a homomorphism $G_1 \longrightarrow G_2$ reducing to id_{G_o}. It is easy to see that this would imply $G_1 \cong G_2$. But it is definitely false that G_1 is necessarily isomorphic to G_2. In fact if $A_o = \mathbb{Z}/p\mathbb{Z}$, $A = \mathbb{Z}/p^2\mathbb{Z}$, $G_o = \hat{\mathbb{G}}_m \times \mathbb{Q}_p/\mathbb{Z}_p$, then there are (up to canonical isomorphism) precisely p distinct liftings of G_o to S. Thus the commutativity condition above is too strong. Surely though we want $d = v \circ i_1 - i_2 \circ w$ to induce zero on S_o (since v is to lift $id_{E(G_o)}$ and w is to lift $id_{\underline{V}(G_o)}$). Thus d need not be zero but d_o should be zero; i.e., $d \in \operatorname{Ker}[\operatorname{Hom}(V(G_1), E(G_2)) \longrightarrow \operatorname{Hom}(\underline{V}(G_o), E(G_o))]$. It is precisely maps of this type which will be studied in Chapter III: the exponentials. Thus it is natural to ask if v can be chosen such that d is an exponential (it is easy to see the answer is independent of the w chosen to lift $id_{\underline{V}(G_o)}$). The answer is yes: there is a unique such v.

This enables us to construct the crystal $\mathbb{E}(G_o)$. Finally let us note that the developments of Chapter IV go through mutatis mutandis to the situation when the Barsotti-Tate groups are replaced by abelian schemes. This will be of crucial importance in Chapter V.

Chapter V treats the deformation theory of Barsotti-Tate groups and abelian schemes. We continue to assume the ideal I has nilpotent divided powers. We prove that to lift a Barsotti-Tate group G_o to S is equivalent to lifting the natural vector subgroup $\underline{V}(G_o) \hookrightarrow \underline{\text{Lie}}(E(G_o)) = \mathbb{D}(G_o)_{S_o}$ to an "admissible" filtration $\text{Fil}^1 \hookrightarrow \mathbb{D}(G_o)_S$. Actually a more precise result giving us information about lifting homomorphisms also, is proven. There is a completely analogous theorem concerning abelian schemes. In order to relate the deformation theory for abelian schemes with that for Barsotti-Tate groups we prove the following result: If A_o is an abelian scheme on S_o and $\bar{A}_o$ the corresponding Barsotti-Tate group, then there is a functorial isomorphism $\mathbb{D}(\bar{A}_o) \xrightarrow{\sim} \mathbb{D}(A_o)$.

Combining this result with the already stated results on deformations of abelian schemes and of Barsotti-Tate groups, it is quite easy to obtain the theorem of Serre-Tate, which says essentially: To lift A_o to S is equivalent to lifting $\bar{A}_o$ to S (and similarly for morphisms). (There are of course no assumptions concerning divided powers in this theorem.)

In the final section of the chapter we apply the above results to obtain the Serre-Tate canonical lifting of an ordinary abelian variety defined over a perfect field.

Results of the type obtained here have been previously announced in Cartier's Bourbaki talk [7]. He assumes the ring A_o is a perfect field of characteristic p and that the Barsotti-Tate groups in question are p-divisible formal Lie groups. Apparently he uses a more "Dieudonné module theoretic" approach. He obtains finer results about the structure of the crystal $\mathbb{D}(G)$ than those obtained here, in particular the result labeled 1) below.

In the appendix a characterization of the canonical lifting of an ordinary abelian variety is given. In the case of ordinary elliptic curves * the theorem of Serre-Tate is illustrated by giving Tate's elegant formulation of the result in terms of "q". This portion of the appendix is taken entirely from a letter of Tate to Dwork, dated November 1968.

Professor Serre has kindly informed me that for elliptic curves, the canonical lifting has a rather long history. Hasse used it in his first demonstration of the Riemann hypothesis for elliptic curves [16bis, 16ter]. Deuring in his fundamental paper [9bis] proved that for any elliptic curve E_o over a finite field and any endomorphism φ_o of E_o, there exists a "lifting" of the pair (E_o, φ_o) to characteristic zero. In our context the analogue is proposition (1.2) of the appendix. For the analytical aspects of the theory one should consult [9ter].

There are several results and problems which are not mentioned in the text. I shall limit myself to mentioning three such. The first two have been announced by Grothendieck while the third is, as far as I know, completely open.

1) If S_o is a perfect field of characteristic p, then $\mathbb{D}^*(G) = \mathbb{D}(G^*)$ is canonically isomorphic to the ordinary Dieudonné module of G.

2) If $\pi: A_o \longrightarrow S_o$ is an abelian scheme, then $\mathbb{D}(A_o^*)$ is canonically isomorphic to $R^1\pi_{*,\text{crys}}(\mathcal{O}_{A_o,\text{crys}})$.

3) Assuming S_o is of characteristic p and is perfect, to study more closely the functor $\mathbb{D}$. In particular to attempt to describe its essential image. Also to decide if this functor is fully-faithful.

As Grothendieck has observed his result 1) above renders "plausible" the belief that $\mathbb{D}$ is fully-faithful since $\mathbb{D}$ commutes with base change and is an equivalence of categories when the base is a perfect field.

During the summer of 1970 Grothendieck lectured in Montreal on "Barsotti-Tate groups and Dieudonné crystals". It was my good fortune to not only be able to attend these lectures, but also to have extensive personal contact with Grothendieck through which I was able to profit greatly.

Therefore, it is indeed my pleasure to thank A. Grothendieck for the time and effort he was so generously willing to devote to me as well as for his permission to use many of his unpublished results and ideas.

During the fall semester, 1970, N. Katz and I conducted a seminar at Princeton on Dieudonné theory and Barsotti-Tate groups. In the course of the seminar I presented most of the results which appear here. Throughout the entire seminar and even after it officially terminated Katz showed great patience and interest in discussing this work. This is especially true of the appendix where his assistance both in its initial conception and in the execution of several technical results was invaluable.