Example (3.9) Assume S to be artin and let G be a Barsotti-Tate group on S. Then for each n, we have an extension: $0 \to G(n)^{o} \to G(n) \to G(n)^{ét} \to 0$ where $G(n)^{o}$ is the disjoint union of the connected components of the various fibers of $G(n)$. By reducing to the case when we are over a field we easily see that the various sequences:

$$0 \to G(n-i)^{o} \to G(n)^{o} \xrightarrow{p^{n-i}} G(i)^{o} \to 0$$

$$0 \to G(n-i)^{ét} \to G(n)^{ét} \xrightarrow{p^{n-i}} G(i)^{ét} \to 0$$

are exact [see S.G.A. 3 VIA 5.5]. Thus in passing to the limit we have an extension $0 \to G^{o} \to G \to G^{ét} \to 0$ with G^{o} having connected fibers (and hence, if p is nilpotent on S, a formal Lie group) and $G^{ét}$ ind-étale. Applying Cartier duality to G^{o}, taking the étale quotient of G^{o*} and applying Cartier duality again we have a filtration:

$$\{e\} \subseteq G^{t} \subseteq G^{o} \subseteq G$$

with G^{t}, toroidal. As shall be seen later such representations of G as an extension of an ind-étale by a group with connected fibers do not in general exist.

Chapter II. The Relation Between Barsotti-Tate and Formal Lie Groups

§1. (1.0). Let S be a scheme, and X and Y with $Y \hookrightarrow X$ two sheaves on S for the f.p.p.f. topology. We make the following definition.

Definition (1.01): $\mathrm{Inf}^k_Y(X)$ is the subsheaf of X whose sections over an S-scheme T are given as follows: $\Gamma(T, \mathrm{Inf}^k_Y(X)) = \{t \in \Gamma(T,X) \mid$ there is a covering $\{T_i \to T\}$ and for each T_i a closed subscheme T_i' defined by an ideal whose $k+1^{st}$ power is (0) with the property that $t_{T_i'} \in \Gamma(T_i', X)$ is actually an element of $\Gamma(T_i', Y)\}$. $\Gamma(T_i', Y) \hookrightarrow \Gamma(T_i', X)$

Thus for each integer $k \geq 0$ we have defined the k^{th} infinitesimal neighborhood of Y in X. We first observe the following simple lemma.

Lemma (1.02) If X and Y are schemes and $Y \hookrightarrow X$ is an immersion, then the above definition coincides with the usual one of [E.G.A. IV §16].

Proof: Let us denote by $Y^{(k)}$ the k^{th} infinitesimal neighborhood of Y in X in the sense of E.G.A. We can obviously assume that Y is a closed subscheme of X defined by a quasi-coherent ideal $I \subseteq \mathcal{O}_X$. Thus $\mathcal{O}_{Y^{(k)}} = \mathcal{O}_X / I^{k+1}$ and we certainly have a monomorphism $Y^{(k)} \to \mathrm{Inf}^k_Y(X)$. To show this is an isomorphism we must show that for any affine scheme T (over S) the map $\Gamma(T, Y^{(k)}) \to \Gamma(T, \mathrm{Inf}^k_Y(X))$ is surjective. Thus we are reduced to showing that if $T = \mathrm{Spec}(A)$, $T' = \mathrm{Spec}(A')$ with $A \to A'$ faithfully flat, $J \subseteq A'$ is an ideal with $J^{k+1} = (0)$, and $\varphi: T \to X$ is such that the composite $\mathrm{Spec}(A'/J) \hookrightarrow T' \to T \xrightarrow{\varphi} X$ factors through Y, then φ factors through $Y^{(k)}$. Let ψ denote the morphism $A \longrightarrow A'$ corresponding to $T' \longrightarrow T$. Then the hypothesis tells us that $I \cdot A \subseteq \psi^{-1}(J)$. But $A \longrightarrow A'$ being faithfully flat implies that it is injective, and hence $\psi^{-1}(J)$ has its $k+1^{st}$ power equal to (0). But this certainly implies that $\varphi: T \longrightarrow X$ factors through $Y^{(k)}$.

Lemma (1.03) (Compatability of the formation of infinitesimal neighborhoods with base change.) If $S' \longrightarrow S$ is a morphism then $\mathrm{Inf}^k_{Y_{S'}}(X_{S'}) = (\mathrm{Inf}^k_Y(X))_{S'}$.

Proof: Let $T' \longrightarrow S'$ be given and let $t' \in \Gamma(T', \mathrm{Inf}^k_{Y_{S'}}(X_{S'}))$. Then by definition there is a covering family $\{T'_i \longrightarrow T'\}$ and nilpotent immersions of order k $T'_{i_o} \hookrightarrow T'_i$ such that $t'_{T'_{i_o}} \in \Gamma(T'_{i_o}, Y_{S'})$. But considering T', T'_i, T'_{i_o} as S-schemes via $S' \longrightarrow S$ we see that this this t' belongs to $\Gamma(T', (\mathrm{Inf}^k_Y(X))_{S'})$. Conversely, if we begin with a t' in this last set and hence have T'_i, T'_{i_o} S-schemes as in the definition then T''_i can obviously be viewed as an S'-scheme in such a way that $\Gamma(T'_i, X_{S'}) = \Gamma(T'_i, X)$. Thus such a t' must belong to $\Gamma(T', (\mathrm{Inf}^k_{Y_{S'}}(X_{S'}))$ which completes the proof.

(1.1) Let X be a sheaf on S and $e_X: S \longrightarrow X$ be a section. If it is understood that X is given together with a section, then we will write $\mathrm{Inf}^k(X)$ rather than $\mathrm{Inf}^k_S(X)$, the latter having been defined in (1.01) via our section $e_X: S \hookrightarrow X$.

Definition (1.1.1) A pointed sheaf (X, e_X) as above is ind-infinitesimal if $X = \varinjlim \mathrm{Inf}^k(X)$.

Remark (1.1.2) It follows immediately from the definition that $\mathrm{Inf}^k(X) = \mathrm{Inf}^k(\mathrm{Inf}^{k+i}(X))$ for any $i \geq 0$. check this!

Recall (1.1.3) Let X be an S-scheme and $e_X: S \longrightarrow X$ a section, then if $\underline{\omega}_{e_X}$ denotes the conormal sheaf of the immersion $e_X: S \hookrightarrow X$, we have $\underline{\omega}_{e_X} \cong e_X^*(\Omega^1_{X/S})$. This is seen by checking that both $\mathcal{O}_S$-modules represent the same functor: namely $\mathcal{F} \mapsto \mathrm{Der}_S(\mathcal{O}_X, e_{X*}(\mathcal{F}))$ or else as in [G.A.I §4,2.2]. Also recall that associated to such an immersion we have a graded quasi-coherent sheaf of $\mathcal{O}_S$-algebras [E.G.A.IV 16.1.5].

The following basic definition can now be made.

Definition (1.1.4) A pointed sheaf (X, e_X) on S is said to be a formal Lie variety if the following conditions are satisfied:

1) X is ind-infinitesimal and $\mathrm{Inf}^k(X)$ is representable for all $k \geq 0$.

2) $\underline{\omega}_X = e_X^*(\Omega^1_{\mathrm{Inf}'(X)/S}) = e_X^*(\Omega^1_{\mathrm{Inf}^k(X)/S})$

is locally free of finite type.

3) Denoting by $\mathrm{gr}^{\mathrm{inf}}(X)$ the unique graded $\mathcal{O}_S$-algebra, such that $\mathrm{gr}^{\mathrm{inf}}_i(X) = \mathrm{gr}_i(\mathrm{Inf}^i(X))$ holds for all $i \geq 0$ (see remark 1.1.3), we have an isomorphism $\mathrm{Sym}(\underline{\omega}_X) \xrightarrow{\sim} \mathrm{gr}^{\mathrm{inf}}(X)$ induced by the canonical mapping $\underline{\omega}_X \xrightarrow{\sim} \mathrm{gr}^{\mathrm{inf}}_1(X)$.

We proceed to translate this definition into more down to earth terms. First condition 1) and remark (1.1.2) tell us that for each k $\mathrm{Inf}^k(X)$ is an affine S-scheme and that both S and $\mathrm{Inf}^k(X)$ have the same underlying topological space. Thus $\mathrm{Inf}^k(X)$ is given by a quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{a}_k$ which is augmented:

$$0 \longrightarrow I_k \longrightarrow \mathcal{a}_k \longrightarrow \mathcal{O}_S \longrightarrow 0$$

and furthermore $I_k^{k+1} = (0)$. We are also told that $\mathcal{a}_{k+1}/I_{k+1}^{k+1} \cong \mathcal{a}_k$, which makes obvious how we define the algebra $\mathrm{gr}^{\mathrm{inf}}(X)$. Conditions 2) and 3) imply that locally on S we have

$$\mathcal{a}_k \xleftarrow{\sim} \mathcal{O}_S\,[T_1, \ldots, T_N]/(T_1, \ldots, T_N)^{k+1},$$

these isomorphisms being compatible [4; Chap. III, §2, #8 Cor. 3]. Hence locally on S, X is given by a power series ring $\mathcal{O}_S[[T_1, \ldots, T_N]]$ in the sense that for an S-scheme S', $\Gamma(S', X) \cong \underbrace{\mathrm{Nil}(\mathcal{O}_{S'}) \times \cdots \times \mathrm{Nil}(\mathcal{O}_{S'})}_{N \text{ factors}}$

where $\mathrm{Nil}(\mathcal{O}_{S'})$ denotes the locally nilpotent sections of $\mathcal{O}_{S'}$. These correspond exactly to the continuous homomorphisms

$$\underline{\mathcal{O}_S[[T_1, \ldots, T_N]] \longrightarrow \Gamma(S', \mathcal{O}_{S'})}$$

when the latter is given the discrete topology.

<u>Definition</u> (1.1.5) A formal Lie group over S, (e_G, G) is a group in the category of formal Lie varieties.

As usual we consider only commutative formal Lie groups and this will be always implicitly assumed in the following without further mention.

<u>Lemma</u> (1.1.6) Let G be a group on a scheme S. Then $\overline{G} = \varinjlim \mathrm{Inf}^k(G)$ is also a group.

<u>Proof:</u> Since $\overline{G}$ is closed under taking inverses it suffices to show it is closed under addition. Since $\overline{G}$ is a sheaf it suffices to check that if $f, g \in \Gamma(T, \mathrm{Inf}^k(G))$ then there is a $k' \geq k$ such that $f + g \in \Gamma(T, \mathrm{Inf}^{k'}(G))$. But since f belongs to $\Gamma(T, \mathrm{Inf}^k(G))$ there is a covering family $\{T_i \longrightarrow T\}$ and nilpotent immersions of order k, $\overline{T}_i \hookrightarrow T_i$ such that $\underline{f|\overline{T}_i = 0}$. Similarly there is a covering family $\{T'_j \longrightarrow T\}$ and nilpotent immersions of order k $\overline{T}'_j \hookrightarrow T'_j$ corresponding to g. But $\{T_i \times_T T'_j \longrightarrow T\}$ is a covering family, $\overline{T}_i \times_T \overline{T}'_j \longrightarrow T_i \times_T T'_j$ is a nilpotent immersion of order $2k$ and obviously $\underline{f + g|\overline{T}_i \times_T \overline{T}'_j = 0}$. Thus $\overline{G}$ is indeed a group.

<u>§2. (2.0)</u> In this paragraph we assume that S is of characteristic p and then associate to any G in B.T.(S) a subgroup $\overline{G}$ which is a formal Lie group. Later we will, by reducing to the case when S is of characteristic p, be able to extend this result to the case where p is locally nilpotent on S.

(2.1) Recall that if G is a flat S-group scheme there are defined two homomorphisms:

$$f_G : G \longrightarrow G^{(p)}$$

$$v_G : G^{(p)} \longrightarrow G$$

satisfying $v_G \circ f_G = p\,\mathrm{id}_G$ and $f_G \circ v_G = p\,\mathrm{id}_{G^{(p)}}$.

Here $G^{(p)} = G \times_S S$ when S is viewed as an S-scheme via the absolute Frobenius $F: S \longrightarrow S$. For the <u>definition of v_G</u> see [G.A. IV §3, 4] or [S.G.A. 3 VII A 4.2]. <u>Note that the definition of f_G</u> has nothing to do with G being either representable or a group but applies to any contravariant functor from Sch/S to Sets. check it!

Notation (2.1.1) $G[n]$ will denote the kernel of the n^{th} iterate of the Frobenius homomorphism:

$$G \xrightarrow{f_G} G^{(p)} \xrightarrow{f_{G^{(p)}}} G^{(p^n)} \cdots \longrightarrow G^{(p^n)}$$

Proposition (2.1.2) Assume G is a finite, locally-free S-group such that $G = G[n]$. Then the following conditions are equivalent:

1) For all $i=0,\dots,n$ $f_G^i : G \longrightarrow G[n-i]^{(p^i)}$ is flat.

2) For all $i=0,\dots,n$ $f_G^i : G \longrightarrow G[n-i]^{(p^i)}$ is an epimorphism of f.p.p.f. sheaves of groups.

3) Locally on S we have an isomorphism of pointed S-schemes $G \cong \mathrm{Spec}\,(\mathcal{O}_S[T_1,\dots,T_N]/(T_1^{p^n},\dots,T_N^{p^n}))$.

Proof: Since we have the following diagram

$$\begin{tikzcd} G \arrow[r, "f_G"] \arrow[rr, bend left, "F_G"] & G^{(p)} \arrow[r] \arrow[d] & G \arrow[d] \\ & S \arrow[r, "F_S"] & S \end{tikzcd}$$

in which the square is cartesian it follows that f_G is certainly bijective on the topological space level. Thus 1) implies that each f_G^i is in fact faithfully flat. Since it is also certainly locally of finite presentation by [E.G.A. IV 1.4.3(v)] it is covering for the f.p.p.f. topology and hence an epimorphism of groups. Conversely if we assume 2) then we can check

that f_G^i is flat by looking fiber by fiber, since to be faithfully flat or an epimorphism for groups over a field is equivalent by [G.A. III §3, 7.4]. 3) implies 2) because being an epimorphism is a local condition and f_G on G then corresponds to the map $\mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{O}_S[T_1,\dots,T_N]/(T^{p^n}) \longrightarrow$
locally isom to $G^{(p)} = G \times_S S$
$\mathcal{O}_S[T_1,\dots,T_N]/(T^{p^n})$ given by $T_i \longmapsto T_i^p$. Hence f_G is obviously an epimorphism. Finally it remains to see that 1) or 2) implies 3). First, if $S = \mathrm{Spec}\,(k)$ where k is a perfect field, then we can write $G = \mathrm{Spec}\, k[T_1,\dots,T_N]/(T_1^{p^{n_1}},\dots,T_N^{p^{n_N}})$ with $0 < n_i \leq n$ by [G.A. III §3 6.3]. But if n_1 (say) is strictly less than n and if we consider the ring homomorphism corresponding to f_G^{n-1} we will have $1 \otimes T_1 \longmapsto T_1^{p^{n-1}}$ which is zero in $k[T_1,\dots,T_N]/(T_1^{p^{n_1}},\dots,T_N^{p^{n_N}})$. But this violates the hypothesis that $f_G^{n-1} : G \longrightarrow G[1]^{(p^{n-1})}$ is faithfully flat since the corresponding ring mapping would then be injective.

To prove the general implication we proceed to reduce to the special case just treated. eg $S = \mathrm{Spec}(k)$

Since G is by hypothesis finite over S we can certainly assume $G = \mathrm{Spec}\,(B)$ where B is some $S = \mathrm{Spec}\,(A)$ augmented algebra of finite rank. Furthermore, if $\underline{p} \subseteq A$ is any prime ideal, any isomorphism of augmented algebras $B_{\underline{p}} \xrightarrow{\sim} A_{\underline{p}}[T_1,\dots,T_N]/(T_1^{p^n},\dots,T_N^{p^n})$ will certainly extend to a neighborhood of $\underline{p}$. Hence we can assume $S = \mathrm{Spec}\,(A)$ where A is a local ring.

Let us consider $\underline{\omega}_G$ and then look at $\underline{\omega}_G \otimes_A k$ where k is the residue field. This is a finite dimensional vector space over k and hence we can find elements $t_1, \ldots, t_N$ in I, the augmentation ideal of B, which generate $\underline{\omega}_G \underset{A}{\otimes} k$. Let us then define a map

$$A[T_1, \ldots, T_N]/(T_1^{p^n}, \ldots, T_N^{p^n}) \longrightarrow B$$

via $T_i \mapsto t_i$. Note the hypothesis that $G = G[n]$ insures that $t_i^{p^n} = 0$ and hence the mapping is defined. By Nakayama the elements $t_1, \ldots, t_N$ have residue classes which generate I/I^2 and since I is finitely generated (being a direct summand of the A-module B) it is nilpotent because the $p^{n\text{th}}$ power of any element in it is zero. Thus $t_1, \ldots, t_N$ in fact generate I and hence the above mapping is surjective. Thus to conclude that the mapping is an isomorphism it suffices to observe that both source and target are free finitely-generated A-modules and that $\operatorname{rank}_A(B) = \operatorname{rank}_k(B \underset{A}{\otimes} k) = \operatorname{rank}_{\bar{k}}(B \underset{A}{\otimes} \bar{k}) = p^{nN} = \operatorname{rank}_A(A[T_1, \ldots, T_N]/(T_1^{p^n}, \ldots, T_N^{p^n}))$.

Remark (2.1.3) It follows from the proof that the conditions of the proposition are equivalent to the conditions

(1 bis) $\quad G \xrightarrow{f^{n-1}} G[1]^{(p^{n-1})}$ is flat.

(2 bis) $\quad G \xrightarrow{f^{n-1}} G[1]^{(p^{n-1})}$ is an epimorphism.

Remark (2.1.4) It is immediate from condition 3) of the proposition that $\underline{\omega}_G$ is locally free of finite type.

Corollary (2.1.5) The hypotheses being as in (2.1.2) and S being affine, the three conditions of the proposition are equivalent to

4) $\underline{\omega}_G$ is locally free of finite type and G is isomorphic as pointed

scheme to $\operatorname{Spec}(\operatorname{Sym}_{\mathcal{O}_S}[\underline{\omega}_G]/\underline{\omega}_G^{(p^n)})$, $\underline{\omega}_G^{(p^n)}$ denoting the ideal generated by the $p^{n\text{th}}$ powers of sections of $\underline{\omega}_G$.

Proof: That 4) implies the conditions of the proposition is obvious. Conversely if $G = \operatorname{Spec}(B)$ where B is a finite quasi-coherent $\mathcal{O}_S$-algebra, then the conditions of the proposition imply that $\underline{\omega}_G = I/I^2$ is locally free where I is the augmentation ideal of B. Thus we can choose an A-linear section of the morphism $I \to \underline{\omega}_G$ (A being the ring of S). This defines a morphism $\operatorname{Sym}_A(\underline{\omega}_G) \to B$ which factors through the algebra $\operatorname{Sym}_A(\underline{\omega}_G)/\underline{\omega}_G^{(p^n)}$. From the proof of the proposition it follows immediately that localizing the above homomorphism at any prime ideal $\underline{p}$ of A we obtain an isomorphism. Thus the constructed mapping is an isomorphism.

Using the corollary we can prove the theorem (2.1.7) below which gives an alternative definition of a formal Lie group when S is of characteristic p.

Definition (2.1.6) A sheaf of groups G on S is said to be of f-torsion if $G = \varinjlim G[n]$.

Definition (2.1.7) A sheaf of groups G on S is said to be f-divisible if $f_G : G \longrightarrow G^{(p)}$ is an epimorphism.

Theorem (2.1.7): In order that the sheaf of groups G on S be a formal Lie group it is necessary and sufficient that the following three conditions hold:

1) G is of f-torsion.

2) G is f-divisible.

3) The $G[n]$ are finite and locally free S-group schemes.

Proof: The proof is essentially that of proposition 1 in Tate's paper [30]. From the fact that, locally on S, a formal Lie group is given by a power series ring, the necessity follows immediately. Since the definition of a formal Lie group is clearly local on S we can assume S is affine, say $S = \mathrm{Spec}\,(A)$. Then conditions 2) and 3) imply (via proposition (2.1.2) and its corollary (2.1.5)) that $G[n] = \mathrm{Spec}\,(B_n)$ when B_n is a finite locally-free A-algebra. From the fact that $G[1]$ is the kernel of $f_{G[n]}: G[n] \longrightarrow G[n]^{(p)}$, we deduce from the cartesian diagram

$$\begin{array}{ccc} G[n] & \xrightarrow{f} & G[n]^{(p)} \\ \uparrow & & \uparrow \\ G[1] & \longrightarrow & S \end{array}$$

that $B_n/I_n^{(p)} \xrightarrow{\sim} B_1$ where I_n is the augmentation ideal of B_n and $I_n^{(p)}$ is the ideal generated by the p^{th} powers of elements in I_n. But this certainly implies that the natural map $I_n/I_n^2 \longrightarrow I_1/I_1^2$ is an isomorphism. By taking S to be smaller we can assume $\underline{\omega}_{G[1]}$ to be free of finite rank. But now it is obvious that we can choose isomorphisms $\mathrm{Sym}(\underline{\omega}_{G[1]})/\underline{\omega}_{G[1]}^{(p^n)} \xrightarrow{\sim} B_n$ in a compatible way (via corollary (2.15) this simply amounts to choosing inductively liftings of generators of I_n to I_{n+1}). Passing to the inverse limit we find that $B = \varprojlim B_n$ is isomorphic to a power series ring. Hence, for any A-algebra C, the points of G with values in $\mathrm{Spec}(C)$ are the elements of $\varinjlim \mathrm{Hom}\,(B_n, C) = \mathrm{Nil}(C) \times \cdots \times \mathrm{Nil}(C)$ and hence G is a formal Lie variety and therefore a formal Lie group.

Theorem (2.1.8) Let G be a Barsotti-Tate group over S; then $\varinjlim G[n]$ is a formal Lie group and is equal to $\overline{G} \overset{\text{def.}}{=} \varinjlim \mathrm{Inf}^k(G)$.

Proof: By theorem (2.17) it suffices to show $\varinjlim G[n]$ is of f-torsion, f-divisible and that the $G[n]$ are finite and locally free. By its very definition it is obvious that $\varinjlim G[n]$ is of f-torsion. Since by hypothesis $p: G^{(p)} \longrightarrow G^{(p)}$ is an epimorphism and since we have a factorization of this morphism $G^{(p)} \xrightarrow{v} G \xrightarrow{f} G^{(p)}$ it follows that $f: G \longrightarrow G^{(p)}$ is an epimorphism. But now it is clear that $f^{-1}[(\varinjlim G[n])^p]$ is contained in $\varinjlim G[n]$ and hence that $f: \varinjlim G[n] \longrightarrow (\varinjlim G[n])^{(p)}$ is an epimorphism. Thus it remains to show that the $G[n]$ are finite and locally free. Since we have the exact sequences

$$0 \longrightarrow G[i] \longrightarrow G[n] \xrightarrow{f^{n-i}} G[n-i]^{(p^{n-i})} \longrightarrow 0$$

we are reduced to showing that $G[1]$ is finite and locally free. Notice first that because $p = v_G \circ f_G$ we have $G[1] \subseteq G(1)$ and thus $G[1] = G(1)[1]$ is certainly representable. Using the usual references [E.G.A.II 6.1.5(v), E.G.A.IV 1.6.2(v), 1.6.3, 1.4.7] we see that the morphism $G(1) \xrightarrow{f} G(1)^{(p)}$ is finite and of finite presentation and hence it is the same for the morphism $G[1] \longrightarrow S$. Thus to show $G[1]$ is finite and locally free it remains to show it is flat over S. But $v_{G(1)}: G(1)^{(p)} \longrightarrow G[1]$ since $f \circ v = p$, and we must have $v_G^{-1}(G[1]) \subseteq G(1)^{(p)}$ for the same reason. But just like f, v_G is also an epimorphism, and hence the morphism $v_G^{-1}(G[1]) \longrightarrow G[1]$ induced by v is an epimorphism. Thus we have an exact sequence:

$$0 \longrightarrow \mathrm{Ker}\, v \longrightarrow G(1)^{(p)} \xrightarrow{v} G[1] \longrightarrow 0\,.$$

Passing to the fibers we see that for all $s \in S$, v_s is faithfully flat and hence as $G(1)^{(p)}$ is flat over S, it follows that $v: G(1)^{(p)} \longrightarrow G[1]$ is faithfully flat [E.G.A. IV 11.3.11]. This of course implies that $G[1]$ is flat over S and hence we have shown that $\varinjlim G[n]$ is a formal Lie group. To prove the last statement of the theorem we observe that $G' = \varinjlim G[n]$ is ind-infinitesimal since this was part of the definition of a formal Lie group. On the other hand for any $n \geq 0$ we have $\mathrm{Inf}^{p^n-1}(G) \subseteq G[n]$. To see this we observe that for any T over S, $f_G: \Gamma(T, G_1) \longrightarrow \Gamma(T, G^{(p)})$ is simply the mapping arising because G is a contravariant functor from the commutative diagram:

$$\begin{tikzcd} T \arrow[r, "F_T"] \arrow[d] & T \arrow[d] \\ S \arrow[r, "F_S"] & S \end{tikzcd}$$

Finally we observe that if $T' \hookrightarrow T$ is a nilpotent immersion of order $< p^n$, then we have a commutative diagram:

$$\begin{tikzcd} T \arrow[rr, "F_T^n"] \arrow[dr] & & T \\ & T' \arrow[ur] & \end{tikzcd}$$

Thus we have $G' = \overline{G'} = \varinjlim \mathrm{Inf}^k(G') \subseteq \varinjlim \mathrm{Inf}^k(G) \subseteq G'$ which completes the proof.

no longer require S to be of char p.

§3. (3.0) In this section we show that if p is locally nilpotent on S and G is in B.T.(S), then $\overline{G} = \varinjlim \mathrm{Inf}^k(G)$ is a formal Lie group. In order

to do this we make a detailed study of the $G(n)$ making use of the relative cotangent complex $L_{\cdot}^{G(n)/S}$.

(3.1) We begin by studying smoothness properties of pointed schemes. For the next proposition no hypothesis on the base, S, is necessary.

Proposition (3.1.1) Let (G, e_G) be a pointed scheme, locally of finite presentation on S; that is e_G is a section of the structural morphism $G \longrightarrow S$. Then the following two conditions are equivalent:

1) Locally (for the Zariski topology) on S, $\mathrm{Inf}_e^k(G)$ is isomorphic to a pointed scheme of the form $\mathrm{Spec}\,(\mathcal{O}_S[T_1, \ldots, T_n]/(T_1, \ldots, T_n)^{k+1})$; i.e., $\omega_G = e_G^*(\Omega^1_{G/S})$ is locally free of finite type, and $\mathrm{Sym}^i(\omega_G) \xrightarrow{\sim} \mathrm{gr}^i(G, e_G)$ for $i \leq k$. Here the term gr^i has the obvious meaning coming from [E.G.A. IV §16].

2) For any affine scheme X_0 over S, an S-infinitesimal neighborhood X' of X_0 of order k, a sub-scheme X of X' containing X_0 and any S-morphism $f: X \longrightarrow G$ such that $f|X_0$ factors through $S \xrightarrow{e_G} G$, there is a prolongation of f to an $f': X' \longrightarrow G$.

Before giving the proof we exhibit the diagram which (hopefully) makes condition 2) clearer:

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[r, hook, "j"] \arrow[d, "f"] & X' \arrow[dl, dashed, "f'"] \\ S \arrow[r, hook, "e_G"] & G & \end{tikzcd}$$

Proof: We first show that 1) implies 2). Thus consider the data given to us in the above diagram. We can form the new diagram

$$\begin{tikzcd} X_o \arrow[r, hook, "i"] \arrow[d, "j \circ i"'] & X \arrow[r, hook, "j"] \arrow[d] & X' \arrow[dl, dashed] \\ X' \arrow[r, hook] & G_{X'} & \end{tikzcd}$$

Here $X \longrightarrow G_{X'}$ is the morphism (f, j) and $X' \hookrightarrow G_{X'}$ is $e_G \times 1_{X'}$.

It is immediately checked that the left-hand square commutes and and that if we find an X'-morphism to make the diagram commute $g: X' \longrightarrow G_{X'}$, then composing g with the projection $G_{X'} \longrightarrow G$ we find an S-morphism which solves our original problem. Since hypothesis 1) for $G \longrightarrow S$ evidently implies the analogous statement for $G_{X'} \longrightarrow X'$ we have reduced ourselves to the following situation: $X' = S$ and hence S is affine. Notice also that the morphism $f: X \longrightarrow G$ factors through the k^{th} infinitesimal neighborhood of S in G. Thus translating into commutative algebra we have: rings A', A A_o with $A = A'/J$, $A_o = A'/J_o$, $J_o \supseteq J$, $J_o^{k+1} = (0)$, and an A' algebra B and a map of A'-algebras $f: B \longrightarrow A$. Furthermore B is an augmented A'-algebra. Let its augmentation ideal be I. Then under the composite map $B \longrightarrow A \longrightarrow A_o$, I is mapped to (0). Hence under the map $f: B \longrightarrow A$, $f(I) \subseteq J_o/J$. By hypothesis 1) $B = \mathrm{Sym}_{A'}[\underline{\omega}_G]/I^{k+1}$ and hence we have an A'-linear map $\underline{\omega}_G \longrightarrow J_o/J$. Since $\underline{\omega}_G$ is projective, we can lift this to a map $\underline{\omega}_G \longrightarrow J_o$. This allows us to define a map $\mathrm{Sym}_{A'}[\underline{\omega}_G] \longrightarrow A'$ which takes I into J_o. Hence

this map takes I^{k+1} into $J_o^{k+1} = (0)$ and thus we obtain a map $f': B \longrightarrow A'$ which lifts $f: B \longrightarrow A$.

We now turn to the proof that 2) implies 1). Observe that only $\mathrm{Inf}^k(G)$ enters into condition 2) since the given map f certainly factors through $\mathrm{Inf}^k(G)$ and any prolongation to an f' must also factor through $\mathrm{Inf}^k(G)$. To prove that 2) implies 1) we can assume S is affine and hence $\mathrm{Inf}^k(G)$ is also affine. Let the respective rings be A and B where B is an augmented A-algebra whose augmentation ideal I is such that $I^{k+1} = (0)$. In order to show that I/I^2 is locally-free, we are led to the case $k=1$ since condition 2) implies that B/I^2 satisfies the same hypotheses for first order immersions, as B does for k^{th} order immersions. Let us choose, in the notation of condition 2), $X_o = S = \mathrm{Spec}\,(A)$. For any two A-modules M and M' and any surjective homomorphism $M \longrightarrow M'$ we can look at the two rings $D_A(M)$ and $D_A(M')$ (i.e., the ring $D_A(M) = A \oplus M$ with $M^2 = (0), \ldots$). Then we find surjective homomorphisms:

$$D_A(M) \longrightarrow D_A(M') \longrightarrow A$$

But now by definition $B = D_A[\underline{\omega}_G]$ and condition 2) now says that any A-linear map $\underline{\omega}_G \longrightarrow M'$ can be lifted to a map $\underline{\omega}_G \longrightarrow M$. This says that $\underline{\omega}_G$ is projective. But G being locally of finite presentation on S implies that $\underline{\omega}_G$ is of finite presentation [E.G.A. IV 16.4.22] and hence that $\underline{\omega}_G$ is locally-free of finite rank.

It remains to prove that $\mathrm{Sym}^i[\underline{\omega}_G] \xrightarrow{\sim} \mathrm{gr}^i(G, e)$ for $i \leq k$. In order to do this let $B' = \mathrm{Sym}\,[\underline{\omega}_G]/I'^{k+1}$ where I' is the augmentation ideal in

Sym[$\underline{\omega}_G$]. Consider the map $I \longrightarrow \underline{\omega}_G$, and because $\underline{\omega}_G$ is projective choose a section $\underline{\omega}_G \longrightarrow I$. This section allows us to define a map $u: B' \longrightarrow B$ because $I^{k+1} = (0)$. We will show that this map is an isomorphism, which will complete the proof that 2) implies 1). Let us apply hypothesis 2) to $X' = \mathrm{Spec}\,(B')$, $X = \mathrm{Spec}\,(B)$ and $X_o = \mathrm{Spec}\,(A)$. Notice that $u: B' \longrightarrow B$ is surjective modulo squares of the augmentation ideals and hence is surjective. Then by 2) the identity morphism $B \longrightarrow B$ can be lifted to a homomorphism of A-algebras $v: B \longrightarrow B'$ such that $u \circ v = 1_B$. But now from the case k=1, we know $v \circ u$ induces the identity on I'/I'^2. Hence $v \circ u$ induces the identity on $\mathrm{gr}_{I'}(B')$. Thus as I' is nilpotent, $v \circ u$ is an automorphism of B' and hence u is an isomorphism.

<u>Definition</u> (3.1.2) G is said to be smooth along the section e_G up to order k if G satisfies the equivalent conditions of proposition (3.1.1).

(3.2) In this section we define the "naive" relative cotangent complex in the special context in which we shall need it. The word "naive" is used to distinguish the complex which we define, from that defined in general by Illusie [17], although in this special case his definition agrees with the one which we will adopt.

(3.2.1) Let G be a finite and locally-free S group scheme (no hypothesis on S is necessary). Thus $G = \mathrm{Spec}\,(\mathcal{A})$ where $\mathcal{A}$ is a finite and locally-free $\mathcal{O}_S$-algebra. To say G is a group amounts to saying $\mathcal{A}$ is a bi-algebra (i.e., we have algebra homomorphisms $\Delta: \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ and $\epsilon: \mathcal{A} \longrightarrow \mathcal{O}_S$ satisfying well known identities). Thus $\check{\mathcal{A}}$, the linear dual of $\mathcal{A}$, is

equipped with an algebra structure via the transpose of Δ and that of ϵ.

<u>Definition</u> (3.2.2) The situation being as in (3.2.1) the hyperalgebra, U(G), of G is by definition $\check{\mathcal{A}}$ endowed with its algebra structure.

Remark (3.2.3) As is well known $\check{\mathcal{A}}$ is in fact a bi-algebra and $\mathrm{Spec}\,(\check{\mathcal{A}}) = G^*$, the Cartier dual of G.

(3.2.4) Since U(G) is a finite locally free $\mathcal{O}_S$-algebra we obtain a smooth group scheme $U(G)^{\times}$ whose points with values in the S scheme T are by definition the invertible elements in the ring $\Gamma(T, U(G_T)) = \Gamma\left(T, \check{\mathcal{A}} \otimes_{\mathcal{O}_S} \mathcal{O}_T\right)$. Also we have a natural monomorphism $G \hookrightarrow U(G)^{\times}$ which is defined by viewing a T-valued point of G as a homomorphism of $\mathcal{O}_T$-algebras $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_T \longrightarrow \mathcal{O}_T$ and hence as a global section of $\Gamma\left(T, \check{\mathcal{A}} \otimes_{\mathcal{O}_S} \mathcal{O}_T\right)$. The fact that such a homomorphism when viewed as an element of $\Gamma(T, U(G_T))$ is invertible and that the morphism $G \hookrightarrow U(G)^{\times}$ thus defined is indeed a homomorphism of group schemes, results easily from the definitions.

<u>Lemma</u> (3.2.5) The natural monomorphism $G \hookrightarrow U(G)^{\times}$ is a closed immersion.

<u>Proof</u>: Observe $U(G)^{\times}$ is of finite presentation and affine over S. Then by applying [S.G.A. 3 XVI (1.8)] with $G' = S$ (the trivial S-group) and $G'' = G$ and $H = U(G)^{\times}$ we see that $G \longrightarrow U(G)^{\times}$ is proper. Since it is also a monomorphism we use [E.G.A. IV 8.11.5] to finish.

<u>Lemma</u> (3.2.6) The morphism $G \hookrightarrow U(G)^{\times}$ is a regular immersion.

Proof: By [S.G.A.3 III 4.15], G is locally a complete intersection. Hence by [S.G.A. 6 VII 1.2, 1.4], we see $G \longrightarrow U(G)^X$ is a regular immersion.

(3.2.7) The notation being as above, let I be the ideal defining G in $U(G)^X$; then the definition of S.G.A. 6 for $L.^{G/S}$ which we adopt is:

Definition (3.2.8) The relative cotangent complex, $L.^{G/S}$, is the complex of $\mathcal{O}_G$-modules $I/I^2 \longrightarrow \Omega^1_{U(G)^X/S}\big|G$.

Since both terms in this complex are locally-free we have $Le_G^*(L.^{G/S}) = e_G^*(L.^{G/S})$. Let $\pi: G \longrightarrow S$ be the structural morphism. We have the following proposition (which we will need later):

Proposition (3.2.9) $\pi^* e_G^*(L.^{G/S}) \xrightarrow{\sim} L.^{G/S}$.

To prove the proposition we use the following general lemma:

Lemma (3.2.10) Let T be a topos, G be a group of T and P a torseur under G. Assume we are given a fibered category $\mathcal{E}$ over T. Then the category of objects ξ in $\mathcal{E}_P$ given with descent data is equivalent to the category of objects ξ in $\mathcal{E}_P$ on which G acts in a manner compatible with its action on P. The last condition means precisely the following: for any object S' in T and $\varphi \in G(S')$ we are to be given a morphism $\bar{\varphi}: \xi_{P_{S'}} \longrightarrow \xi_{P_{S'}}$ lying over the mapping multiplication by φ on $P_{S'}$ in such a way that it is compatible with multiplication in G and with restriction morphisms.

Proof: Let data of the above type be given for an object $\xi \in \mathcal{E}_P$. First

observe that an equivalent way of stating the condition is to say that for any S' in T and any morphism $\eta: S' \longrightarrow P$ we are to be given an isomorphism $\eta^*(\xi) \xrightarrow{\sim} (\varphi\cdot\eta)^*(\xi)$. The fact that this map must be an isomorphism follows from the requirement that G "acts as a group."

Applying this to $S' = P \times P$ and $\eta = p_1$ we choose $\varphi \in \mathrm{Hom}(P \times P, G)$ to be the unique element such that $\varphi \cdot p_1 = p_2$. Thus we obtain an isomorphism $\theta: p_1^*(\xi) \xrightarrow{\sim} p_2^*(\xi)$. To check the cocycle condition we observe by the above restriction and multiplicativity properties that since $p_{12}^*(\varphi)\cdot p_1' = p_2'$, $p_{23}^*(\varphi)\cdot p_2' = p_3'$, $p_{13}^*(\varphi)\cdot p_1' = p_3'$ (where $p_i': P\times P\times P \longrightarrow P$ is the i^{th} projection), we must have $p_{23}^*(\theta)\circ p_{12}^*(\theta) = p_{13}^*(\theta)$ by the multiplicativity property. Now to check that morphisms between such objects give rise to morphisms between objects with descent data is trivial. Since this is all we actually need for the proposition we omit the easy verification that we indeed have an equivalence of categories.

Proof of proposition (3.2.9) (continued): Take $G = P =$ our finite locally-free group scheme G, $T = \widetilde{Sch/S}$ with the Zariski topology (for example) on Sch/S. Take $\mathcal{E} =$ fibered category of modules. Observe now that if we consider $G \hookrightarrow U(G)^X$, then sections of G induce commutative diagrams via translation:

$$\begin{array}{ccc} G & \hookrightarrow & U(G)^X \\ \downarrow & & \downarrow \\ G & \hookrightarrow & U(G)^X \end{array}$$

Therefore by functoriality they induce an action of the above type (i.e., as in the lemma) on the conormal bundle I/I^2 and on $\Omega^1_{U(G)^\times/S}|G$. These translations are obviously compatible with the morphism $I/I^2 \to \Omega^1_{U(G)^\times/S}|G$ [E.G.A. IV 16.2 and 16.4]. Hence as $G \to S$ has a section it is a morphism of effective descent and thus we can write:

$$L.^{G/S} \xrightarrow{\sim} \pi^* e^*_G(I/I^2) \xrightarrow{\pi^*(v)} \pi^* e^*_G\left(\Omega^1_{U(G)^\times/S}|G\right)$$

for a unique morphism $v\colon e^*_G(I/I^2) \to e^*_G\left(\Omega^1_{U(G)^\times/S}|G\right)$. But as $\pi^* e^*_G \pi^*(v) = \pi^*(v)$ we must have $v = e^*_G \pi^*(v)$ and hence $L.^{G/S} \xrightarrow{\sim} \pi^* e^*_G(L.^{G/S})$.

<u>Lemma</u> (3.2.11) The formation of $L.^{G/S}$ commutes with an arbitrary base change $S' \to S$.

<u>Proof</u>: Since the formation of $U(G)^\times$ obviously commutes with all base changes we must only show that the ideal defining $G_{S'}$ in $U(G)^\times \underset{S}{\times} S'$ is the inverse image of the ideal defining G in $U(G)^\times$. Since this problem is local on S', we can assume $S' = \mathrm{Spec}\,(A')$, $S = \mathrm{Spec}\,(A)$, $U(G)^\times = \mathrm{Spec}\,(B)$, $G = \mathrm{Spec}\,(B/I)$. But since B/I is flat over A the following sequence is exact: $0 \to I \underset{A}{\otimes} A' \to B \underset{A}{\otimes} A' \to B/I \underset{A}{\otimes} A' \to 0$. Hence $I \underset{A}{\otimes} A'$ is identified with its image in $B \underset{A}{\otimes} A'$ which is precisely what we wanted to show.

<u>Recall</u> (3.2.12) The construction of $U(G)$ being functorial in G it follows that if $u\colon G \to H$ is a homomorphism of finite locally-free groups we have a commutative diagram:

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, "u"'] & U(G)^\times \arrow[d] \\ H \arrow[r, hook] & U(H)^\times \end{tikzcd}$$

and hence deduce that there is a morphism $u^*(L.^{H/S}) \to L.^{G/S}$ corresponding to u.

<u>Definition</u> (3.2.13) The co-Lie complex, $\ell.^G$ is by definition $e^*_G(L.^{G/S})$. From (3.2.12) we see that $G \to \ell.^G$ is a contravariant functor.

(3.3) The following propositions give us the necessary results in order to associate a formal Lie group to a Barsotti-Tate group.

<u>Proposition</u> (3.3.1) Let $G(n)$ be an inductive system of finite locally-free groups on an arbitrary scheme S. (The n is just an index which for typographical is not written as a subscript.) Let J be a quasi-coherent ideal of $\mathcal{O}_S$ such that $J^N = (0)$. Let $S_o = \mathrm{Var}(J)$, $G_o(n) = G(n) \underset{S}{\times} S_o$ and in general let the subscript "o" denote the restriction to S_o. Assume we are given a mapping $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that $\varphi(n) \geq n$ for all n. Assume that whenever M is a quasi-coherent module on an affine open set $U_o \subseteq S_o$, $\mathrm{Ext}^1_{\mathcal{O}_{U_o}}(\ell.^{G_o(n)}|U_o, M) \to \mathrm{Ext}^1(\ell.^{G_o(\varphi(n))}|U_o, M)$ is the zero map. (In the rest of the proposition we shall omit the $\mathcal{O}_{U_o}$ and $|U_o$ and simply write S_o, or S depending on the context. This will not lead to any confusion.) Let X' be an affine scheme over S and X be the subscheme defined by an ideal I such that $I^2 = (0)$. Assume we are given an S-morphism $f\colon X \to G(n)$. Then there is an S-morphism $f'\colon X' \to G(\varphi^N(n))$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ G(n) \arrow[r] & G(\varphi^N(n)) \end{tikzcd}$$

<u>Proof</u>: Let $U \subseteq S$ be any affine open set and let M be a quasi-coherent module on U which is killed by J so that M is in a natural way an $\mathcal{O}_{U_o}$-module. By hypothesis the map $\mathrm{Ext}^1(\ell.^{G_o(n)}, M) \longrightarrow \mathrm{Ext}^1(\ell.^{G_o(\varphi(n))}, M)$ is zero. But now since $U_o \hookrightarrow U$ is a closed immersion (and hence an affine morphism) it follows immediately from [S.G.A.4 XVII 4.1.3(iii), 16; I 6.4] that $\mathrm{Ext}^1(\ell.^{G_o(n)}, M)$ is isomorphic to $\mathrm{Ext}^1(\ell.^{G(n)}, M)$. Thus the map $\mathrm{Ext}^1(\ell.^{G(n)}, M) \longrightarrow \mathrm{Ext}^1(\ell.^{G(\varphi(n))}, M)$ is also the zero map.

By induction on the integer j we have M(q. coherent on U) killed by J^j implies $\mathrm{Ext}^1(\ell.^{G(n)}, M) \longrightarrow \mathrm{Ext}^1(\ell.^{G(\varphi^j(n))}, M)$ is the zero map. In fact writing the exact sequence

$$0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0$$

and forming the commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^1(\ell.^{G(n)}, JM) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\ell.^{G(n)}, M) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\ell.^{G(n)}, M/JM) \arrow[d] \\
\mathrm{Ext}^1(\ell.^{G(\varphi(n))}, JM) \arrow[r] & \mathrm{Ext}^1(\ell.^{G(\varphi(n))}, M) \arrow[r] & \mathrm{Ext}^1(\ell.^{G(\varphi(n))}, M/JM)
\end{tikzcd}$$

we see that the image of $\mathrm{Ext}^1(\ell.^{G(n)}, M)$ is contained in that of $\mathrm{Ext}^1(\ell.^{G(\varphi(n))}, JM)$. Therefore we can apply the inductive assumption to the map $\mathrm{Ext}^1(\ell.^{G(\varphi(n))}, JM) \longrightarrow \mathrm{Ext}^1(\ell.^{G(\varphi^j(n))}, JM)$ to conclude.

This tells us that for any quasi-coherent module M on U the map $\mathrm{Ext}^1(\ell.^{G(n)}, M) \longrightarrow \mathrm{Ext}^1(\ell.^{G(\varphi^N(n))}, M)$ is zero since by hypothesis $J^N = (0)$.

Now let us return to the situation of the proposition:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "f"'] & X' \arrow[d] \\
G(n) \arrow[r, "j"] & G(\varphi^N(n))
\end{tikzcd}$$

We cover X' by affine open sets V_i' such that the image of each V_i' in S is contained in an affine U as above. Denote by V_i the affine open subscheme of X having V_i' as its underlying space. If we could prolong $f|V_i$ to an f' on V_i' as desired in the conclusion of the proposition, then because X is affine we could prolong f to X' itself. This follows from the standard obstruction theory, which tells us once we have local solutions of the problem, that the obstruction to having a global solution is an element of $H^1\left(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^* j^* \Omega^1_{G(\varphi^N(n))/S}, I)\right)$ and this group is zero since $\Omega^1_{G(\varphi^N(n))/S}$ is of finite presentation. [E.G.A.IV 16.4.22]. Thus we see that we can assume S to be affine. But now we know from [14; 11.1.5, 11.1.7] that the obstruction to lifting $j \circ f : X \longrightarrow G(\varphi^N(n))$ to X' is the image of the "obstruction element" in $\mathrm{Ext}^1_{\mathcal{O}_X}\left(Lf^* L.^{G(n)/S}, I\right)$ in $\mathrm{Ext}^1_{\mathcal{O}_X}\left(L(j \circ f)^* L.^{G(\varphi^N(n))/S}, I\right)$. But now it follows from (3.2.9) that if $\pi_X : X \longrightarrow S$ is the structural map $Lf^*(L.^{G(n)/S}) = L\pi_X^*(\ell.^{G(n)})$ and similarly of course for $L(j \circ f)^*(L.^{G(\varphi^N(n))/S})$. Thus we want to know that the image of an element in $\mathrm{Ext}^1_{\mathcal{O}_X}(L\pi_X^*(\ell.^{G(n)}), I)$ under the map $\mathrm{Ext}^1(L\pi_X^*(\ell.^{G(n)}), I) \longrightarrow \mathrm{Ext}\left(L\pi_X^*(\ell.^{G(\varphi^N(n))}), I\right)$ is zero. But by adjointness (same reference to Hartshorne and S.G.A. 4 as above) this map is the same as the map

$$\mathrm{Ext}^1_{\mathcal{O}_S}\left(\ell.^{G(n)}, \pi_{X*}(I)\right) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_S}\left(\ell.^{G(\varphi^N(n))}, \pi_{X*}(I)\right)$$

which we've seen is the zero map. This completes the proof.

Corollary (3.3.2) $G = \varinjlim G(n)$ is, under the hypotheses of the proposition, formally smooth.

Proof: If X is any affine scheme over S $\Gamma(X, G) = \varinjlim \Gamma(X, G(n))$, since X is quasi-compact.

Definition (3.3.3) Writing the cotangent complex $L_.^{G/S}$ so that $\Omega^1_{U(G)\times/S}\big|G$ is in degree 0 and I/I^2 is in degree 1, we define $\underline{\omega}_G = H_0(\ell_.^G)$, $\underline{n}_G = H_1(\ell_.^G)$. Notice this use of the symbol $\underline{\omega}_G$ is of course consistent with our previous notation.

Proposition (3.3.4): Consider an exact sequence of finite locally-free S groups: $0 \to G' \to G \to G'' \to 0$. Then there is an exact triangle in the derived category, $D(S)$:

$$\begin{tikzcd} & \ell_.^{G'} \arrow[dl] & \\ \ell_.^{G''} \arrow[rr] & & \ell_.^{G} \arrow[ul] \end{tikzcd}$$

giving rise to an exact sequence of $\mathcal{O}_S$-modules:

(3.3.5) $0 \to \underline{n}_{G''} \to \underline{n}_G \to \underline{n}_{G'} \to \underline{\omega}_{G''} \to \underline{\omega}_G \to \underline{\omega}_{G'} \to 0$

Proof: Let us first observe that by [E.G.A. IV 19.3.9(ii)] the morphism $G \to G''$ makes G locally a complete intersection relative to G'' because after a f.p.q.c. base change G is as G''-scheme isomorphic to $G'_{G''}$. Since G is a finite and locally-free G'' scheme we can certainly embed G in a smooth G''-scheme; for example if G is defined by the $\mathcal{O}_{G''}$-algebra $\mathcal{A}$ then we can use $\mathbb{V}(\mathcal{A})$. [E.G.A.II 1.4.10]. Thus the

relative cotangent complex $L_.^{G/G''}$ is defined [S.G.A. 6 VIII 2.3]. But now by [14; 10.5.21] we have an exact triangle in the derived category $D(G)$

$$\begin{tikzcd} & L_.^{G/G''} \arrow[dl] & \\ \overline{L_.^{G''/S,G}} \arrow[rr] & & L_.^{G/S} \arrow[ul] \end{tikzcd}$$

which comes from the commutative diagram

$$\begin{tikzcd} G \arrow[rr, "\pi"] \arrow[dr] & & G'' \arrow[dl] \\ & S & \end{tikzcd}$$

and where $\overline{L_.^{G''/S,G}} = L\pi^*(L_.^{G''/S})/K[1]$ with K the kernel of $H_1(L\pi^*(L_.^{G''/S})) \to H_1(L_.^{G/S})$ But now if $S' \subseteq S$ is any affine open then it is immediate that the diagram

$$\begin{tikzcd} G \times_S S' \arrow[rr, "\pi_{S'}"] \arrow[dr] & & G'' \times_S S' \arrow[dl] \\ & S' & \end{tikzcd}$$

satisfies the hypothesis of [S.G.A. 6 VIII 2.6]. Thus we find an exact triangle in $D(G \times_S S')$

$$\begin{tikzcd} & L_.^{G\times S'/G''\times S'} \arrow[dl] & \\ L\pi^*_{S'}(L_.^{G''\times S'/S'}) \arrow[rr] & & L_.^{G\times S'/S'} \arrow[ul] \end{tikzcd}$$

Taking the corresponding homology sequence we see that when restricted to $G \times_S S'$ the map

$$H_1(L\pi^*(L_\cdot^{G''/S})) \longrightarrow H_1(L_\cdot^{G/S})$$

is injective (i.e., $K|G \underset{S}{\times} S' = (0)$) Since S' was any affine open subset of S it follows that $K = (0)$ and hence $\overline{L_\cdot^{G''/S,G}} = L\pi^*(L_\cdot^{G''/S})$ and thus we obtain an exact triangle:

$$\begin{tikzcd} & L_\cdot^{G/G''} \arrow[dl] & \\ L\pi^*(L_\cdot^{G''/S}) \arrow[rr] & & L_\cdot^{G/S} \arrow[ul] \end{tikzcd}$$

Let $e_G : S \longrightarrow G$ be the unit section. Then applying Le_G^* to the above exact triangle we obtain an exact triangle:

$$\begin{tikzcd} & Le_G^*(L_\cdot^{G/G''}) \arrow[dl] & \\ \ell_\cdot^{G''} \arrow[rr] & & \ell_\cdot^{G} \arrow[ul] \end{tikzcd}$$

But using that $\pi : G \longrightarrow G''$ is flat and [S.G.A.6 VIII 2.2] we see (just as in 3.2.11) from the cartesian diagram:

$$\begin{tikzcd} G' \arrow[r, hook, "i"] \arrow[d] & G \arrow[d, "\pi"] \\ S \arrow[r, hook, "e_{G''}"'] & G'' \end{tikzcd}$$

that $$Li^*(L_\cdot^{G/G''}) = L_\cdot^{G'/S}$$

and therefore $\ell_\cdot^{G'} = Le_{G'}^*(L_\cdot^{G'/S}) = Le_G^*(L_\cdot^{G/G''})$.

This gives us our desired exact triangle and completes the proof of the proposition.

Lemma (3.3.6) Suppose G is finite and locally-free on S. Then $\underline{\omega}_G$ is locally free if and only if it is flat. If this is the case then $\underline{n}_G$ is also locally free (of finite rank) and rank $(\underline{\omega}_G) = \text{rank}(\underline{n}_G)$.

Proof: Let us write $\ell_\cdot^G = L_1 \longrightarrow L_o$. Then with obvious notation we have exact sequences:

$$0 \longrightarrow \underline{n}_G \longrightarrow L_1 \longrightarrow \text{im} \longrightarrow 0$$

$$0 \longrightarrow \text{im} \longrightarrow L_o \longrightarrow \underline{\omega}_G \longrightarrow 0$$

Hence, since $\underline{\omega}_G$ is of finite presentation, if it is flat, then it is locally-free which implies the image of the morphism $L_1 \longrightarrow L_o$ is locally-free of finite rank. But certainly this implies $\underline{n}_G$ is locally-free of finite rank. In this case we have rank (L_o) - rank (L_1) = rank $(\underline{\omega}_G)$ - rank $(\underline{n}_G)$. Because the relative dimension of G over S is zero it follows that rank $(\underline{\omega}_G)$ = rank $(\underline{n}_G)$.

The following are corollaries of 3.3.4.

Corollary (3.3.7) Given $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$ as in (3.3.4) then:

a) The following conditions are equivalent and if S is of characteristic p, they are implied by $G' \subseteq G[1]$

1) $\underline{\omega}_G \longrightarrow \underline{\omega}_{G'}$ is an isomorphism

2) $\underline{\omega}_{G''} \longrightarrow \underline{\omega}_G$ is the zero map

3) $\underline{n}_{G'} \longrightarrow \underline{\omega}_{G''}$ is surjective

b) Suppose $\underline{\omega}_{G''}$ is locally-free and $\underline{n}_{G'}$ is of finite type and that $\text{rank}_{k(s)}(\underline{n}_{G'} \otimes k(s)) \leq \text{rank}_{k(s)}(\underline{\omega}_{G''} \otimes k(s))$ holds for all $s \in S$. Then the above conditions of a) are equivalent to

3 bis) $\underline{n}_{G'} \longrightarrow \underline{\omega}_{G''}$ is an isomorphism and in this case $\underline{n}_G \longrightarrow \underline{n}_{G'}$ is the zero map.

Proof: That the three conditions of a) are equivalent is immediate from (3.3.5). If S is of characteristic p, then $G' \subseteq G[1]$ implies $G'[1] = G[1]$ and hence since $\mathrm{Inf}^1(G) \subseteq G[1]$ (see the end of the proof of 2.1.8) we have $\underline{\omega}_G = \underline{\omega}_{G'}$ which is condition 1).

b) Assume the additional conditions and that $\underline{n}_{G'} \longrightarrow \underline{\omega}_{G''}$ is surjective. Then because $\underline{\omega}_{G''}$ is locally free we must (locally on S) have a splitting $\underline{n}_{G'} = \mathrm{Ker} \times \underline{\omega}_{G''}$ where Ker denotes the kernel of $\underline{n}_{G'} \longrightarrow \underline{\omega}_{G''}$. By our assumption on the ranks we must have $\mathrm{Ker} \otimes_{\mathcal{O}_S} k(s) = (0)$ for all $s \in S$. Ker is of finite type because $\underline{n}_{G'}$ is. Therefore by Nakayama we must have each stalk of Ker is zero and therefore $\mathrm{Ker} = (0)$. This completes the proof since the last assertion is obvious.

Remark (3.3.8) From (3.3.6) it follows that the inequality in the last corollary can be written $\mathrm{rank}\,(\underline{\omega}_{G'_s}) \leq \mathrm{rank}\,(\underline{\omega}_{G''_s})$ for all $s \in S$.

Corollary (3.3.9) Assume given the exact sequence $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$ as above and assume further that:

1) $\underline{\omega}_G \longrightarrow \underline{\omega}_{G'}$ is an isomorphism

2) $\underline{\omega}_G$ and $\underline{\omega}_{G''}$ are locally free

3) $\mathrm{rank}\,(\underline{\omega}_{G'_s}) \leq \mathrm{rank}\,(\underline{\omega}_{G''_s})$ holds for all $s \in S$

Then for any affine open $U \subseteq S$ and for any quasi-coherent module M on U, the map

$$\mathrm{Ext}^1_{\mathcal{O}_U}(\ell_{\cdot}^{G'}|U, M) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_U}(\ell_{\cdot}^{G}|U, M)$$

is the zero map. In particular for any quasi-coherent M on S the map $\underline{\mathrm{Ext}}^1(\ell_{\cdot}^{G'}, M) \longrightarrow \underline{\mathrm{Ext}}^1(\ell_{\cdot}^{G}, M)$ is zero.

Proof: Because $\underline{\mathrm{Ext}}^1(\ell_{\cdot}^{G}, M)$ is the sheaf associated to the presheaf $U \longmapsto \mathrm{Ext}^1(\ell_{\cdot}^{G}|U, M|U)$ it follows that the last assertion follows from the preceding one. We can obviously assume S is affine. Let $I^{\cdot}$ be an injective resolution of M. By inspecting the complex $\mathrm{Hom}^{\cdot}(\ell_{\cdot}^{G}, I^{\cdot})$ whose component of degree m is

$$\mathrm{Hom}\,(L^{-1}, I^{m-1}) \oplus \mathrm{Hom}\,(L^{o}, I^{m})$$

($\ell_{\cdot}^{G}$ being $L^{-1} \longrightarrow L^{o}$ of course), we find a functorial exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1(\underline{\omega}_G, M) \longrightarrow \mathrm{Ext}^1(\ell_{\cdot}^{G}, M) \longrightarrow \mathrm{Hom}\,(\underline{n}_G, M) \longrightarrow \mathrm{Ext}^2(\underline{\omega}_G, M)$$

and a similar sequence for G'. Hence because $\underline{\omega}_G$ and $\underline{\omega}_{G'}$ are locally free we have isomorphisms:

$$\mathrm{Ext}^1(\ell_{\cdot}^{G}, M) \xrightarrow{\sim} \mathrm{Hom}\,(\underline{n}_G, M)$$

$$\mathrm{Ext}^1(\ell_{\cdot}^{G'}, M) \xrightarrow{\sim} \mathrm{Hom}\,(\underline{n}_{G'}, M)$$

But now we know from (3.3.8) that our hypotheses insure that $\underline{n}_{G'} \longrightarrow \underline{n}_G$ is the zero map and hence we are done.

Corollary (3.3.10) Let S be a scheme such that p^{N+1} kills S, and G be a finite locally-free S-group satisfying the equivalent conditions of lemma (1.1) of Chapter I for an $n \geq N+1$. Then for any affine open subset U of S, the mapping (defined for any q. coherent M) $\mathrm{Ext}^1(\ell_{\cdot}^{G(n-N-1)}, M) \longrightarrow \mathrm{Ext}^1(\ell_{\cdot}^{G}, M)$ is zero.

Proof: Just as in the proof of (3.3.1) we are immediately led, via the adjointness relation recalled there, to the case when $N = 0$, i.e., when S is of characteristic p. Then if $n = 1$, $G(n-N-1) = G(0) = (0)$ and there is nothing to prove. If $n \geq 2$ we must verify the conditions of the previous corollary for $G' = G(n-1)$. Condition 1) is clear as $G'[1] = G[1]$ and hence $\underline{\omega}_G \longrightarrow \underline{\omega}_{G'}$ is an isomorphism. Condition 3) is also trivial as $G'' = G(1)$. To verify that condition 2) holds we use the following proposition.

Proposition (3.3.11) Let S be a scheme of characteristic p and $G = G(n)$ a truncated Barsotti-Tate group on S, or if $n = 1$ assume there is a truncated B. T. group $G(2)$ giving rise to G. Then

a) For all i such that $0 \leq i \leq n$ we have

$$f^{n-i}: G[n] \longrightarrow G[i]^{p^{n-i}} \quad \text{is an epimorphism}$$

b) $\operatorname{Ker} f^n = \operatorname{im} v^n$ and $\operatorname{Ker} v^n = \operatorname{im} f^n$

c) $G[n]$ is finite and locally free on S

Proof: Let us first observe that c) allows us to appeal to [I, (2.1.4)] to conclude the proof of (3.3.10). Condition a) is trivial if $n=1$. If $n \geq 2$ we have (for each i) the following commutative diagram:

$$\begin{tikzcd}
(p^{n-i})^{-1}\left(G[i]^{p^{n-i}}\right) \arrow[rr, "p^{n-i}\ \mathrm{id}_G(p^{n-i})"] \arrow[dr, "v^{n-i}"'] & & G[i]^{(p^{n-i})} \\
& G[n] \arrow[ur, "f^{n-i}"'] &
\end{tikzcd}$$

As the morphism $p^{n-i} \cdot \mathrm{id}_G(p^{n-i})$ is an epimorphism, it follows that f^{n-i}

is also. To prove b) let us first assume that $G(n)$ comes from a $G(2n)$ which is a truncated Barsotti-Tate group. But then we have an exact sequence:

$$0 \longrightarrow G(n) \longrightarrow G(2n) \xrightarrow{p^n} G(n) \longrightarrow 0$$

Therefore $G[n] \subseteq G(n)$ and the map $(p^n)^{-1}(G[n]) \longrightarrow G[n]$ is an epimorphism. But once again we can write $p^n = v^n \circ f^n$ and since obviously $f^n: (p^n)^{-1}(G[n]) \longrightarrow G(n)^{(p^n)}$ we see $v^n: G[n]^{(p^n)} \longrightarrow G[n]$ is an epimorphism. That $\operatorname{im} f^n = \ker v^n$, is of course handled analogously. Thus we have in particular proved b) if $n = 1$. We proceed to establish its truth in general by induction on n. Consider the diagram:

$$\begin{tikzcd}
G(n)^{p^n} \arrow[r, "v^n"] \arrow[d, "p"'] & G[n] \arrow[d, "f"] \\
G(n-1)^{p^n} \arrow[r, "v^{n-1}"'] & G[n-1]^p
\end{tikzcd}$$

By the induction hypothesis $v^{n-1}: G[n-1]^{p^n} \longrightarrow G[n-1]^p$ is an epimorphism. Therefore $f \circ v^n$ is an epimorphism. Hence if we can show $G[1] \subseteq \operatorname{im} v^n$ it will follow that v^n is an epimorphism. But by the case $n = 1$ settled above $G[1] = v(G(1)^p) = v \circ p^{n-1}(G(n)^p) = v^n(f^{n-1}(G(n)^p)) \subseteq v^n(G(n)^{(p^n)})$. The other case is of course handled in the same way. To prove c) we observe that $G[n]$ is certainly finite and of finite presentation over S. Therefore to conclude it is locally free it suffices to show it is flat. But this follows immediately from b) because we have a commutative diagram:

$$\begin{tikzcd} G(n)^{p^n} \arrow[rr, "v^n"] \arrow[dr] & & G[n] \arrow[dl] \\ & S & \end{tikzcd}$$

and v^n being an epimorphism is faithfully flat while $G(n)^{p^n}$ is certainly flat over S.

<u>Remark</u> (3.3.12) The condition that $G(1)$ comes from a $G(2)$ can not be dropped. To see this consider the following example. Let k be a field of characteristic p, $S = \mathrm{Spec}(k[[t]])$ and $G = \mathrm{Spec}(k[[t]]\,[X]/(X^p - tX))$ with addition as group law. Then at the generic point G is étale while at the closed point G becomes α_p. But this implies $\underline{\omega}_G$ has rank 0 at the generic point and rank 1 at the closed point. As S is connected, $\underline{\omega}_G$ certainly can not be locally free.

<u>Theorem</u> (3.3.13) If p is locally nilpotent on S and G is in B.T.(S), then G is formally smooth.

<u>Proof</u>: Let X' be an affine scheme over S and X a closed subscheme defined by an ideal of square zero. Let $f: X \longrightarrow G$ be in $\Gamma(X, G)$. We must show f can be lifted to an $f': X' \longrightarrow G$. As X is quasi-compact we have $\Gamma(X, G) = \varinjlim \Gamma(X, G(n))$ and hence can assume $f: X \longrightarrow G(n)$ for some n. We cover X by a finite number of affine opens U_i $i=1,\dots,m$ such that the image of U_i in S is contained in an affine open V_i. Since p is nilpotent on each V_i there is an integer N such that p^{N+1} is zero on $\cup V_i$. Replacing S by $S' = \cup V_i$ and G by $G_{S'}$ we are led to the

case when p is nilpotent on S, $p^{N+1} \cdot 1_S = 0$. But now by (3.3.10) and (3.3.1) we see f can be lifted to an f' and the theorem is proved.

The following example shows that if p is not locally nilpotent on S, a Barsotti-Tate group need not be formally smooth.

<u>Example</u> (3.3.14) Let $S = \mathrm{Spec}\,(A)$ and consider the Barsotti-Tate group μ (I, 3.8.1). To say μ is formally smooth means that whenever C is an A-algebra, I a nilpotent ideal of C (of square zero) and $x \in C/I$ is such that $x^{p^n} = 1$, then there is an $x' \in C$ which reduces modulo I to x and has the property that for some $m \geq n$, $x'^{p^m} = 1$.

Writing $x' = u + y$ where u is some definite lifting of x and $y \in I$ we have $x'^{p^n} = u^{p^n} + p^n y$ (since I has square zero) and since $x^{p^n} = 1$ we see $u^{p^n} = 1 + \zeta$ for some $\zeta \in I$. Since $x'^{p^m} = 1$ this means $(1+\zeta)^{p^{m-n}} + p^m y = 1$ and hence $p^{m-n}\zeta + p^m y = 0$, or equivalently $p^{m-n}(\zeta + p^n y) = 0$. Since given x we have chosen u and thus ζ, we see that we can find an x' so as to lift the point of μ with values in Spec (C/I) to Spec (C) if and only if we can find a $y \in I$ and integer m' such that $p^{m'}(\zeta + p^n y) = 0$. But this is true if and only if the element $\frac{\zeta}{p^n}$ in I_p belongs to the image of I. But now (n being fixed) we have $\zeta = u^{p^n} - 1$ and $\zeta^2 = 0$ and hence there is a homomorphism of the ring $A[T]/(T^{p^n} - 1)^2$ into C which takes $T \mapsto u$ and induces a homomorphism $A[T]/(T^{p^n} - 1) \longrightarrow C/I$ taking $T \mapsto x$. Thus we see it suffices to check our condition for the ring $C = A[T]/(T^{p^n} - 1)^2$ and the element $\zeta = T^{p^n} - 1$ in this ring. Since $C/(T^{p^n} - 1)$ is a free A-module we have

$C = (T^{p^n} - 1) \oplus C/(T^{p^n} - 1)$ as A-module. Thus we see $\frac{\zeta}{p^n}$ is in the image of the ideal $(T^{p^n} - 1)$ if and only if it is in the image of C in C_p. But since C is a free A-module with base $1, T, \ldots, T^{2p^n-1}$ we see this will be true if and only if $\frac{1}{p^n}$ belongs to the image of A in A_p (i.e., if and only if $\frac{1}{p}$ belongs to the image of A in A_p). Thus we can conclude that μ is formally smooth if and only if $\frac{1}{p}$ belongs to the image of A in A_p. Therefore if $A = \mathbb{Z}$ or $\mathbb{Z}_p$ (the p-adic integer and not the localization of $\mathbb{Z}$ with respect to the element p) we see μ is not formally smooth over A.

Finally we shall prove that if p is locally nilpotent on S then $\overline{G}$ is a formal Lie group. We begin with a lemma.

<u>Lemma</u> (3.3.15) Let G be a p-torsion group on S with all G(n) representable. Assume we are given an S-scheme X' and a subscheme X defined by an ideal I such that $I^{k+1} = (0)$ and $p^N \cdot I/I^2 = 0$. Then if $f' : X' \longrightarrow G$ is such that $f = f'|X : X \longrightarrow G(n)$, we have $f' : X' \longrightarrow G(n+kN)$.

<u>Proof</u>: The problem is local on X' and hence we can assume X' to be affine and thus quasi-compact. But then $f' \in \Gamma(X', G) = \varinjlim \Gamma(X', G(m))$ and hence we can assume $f' : X' \longrightarrow G(n')$ for some n'. Therefore we can assume that G is representable. We use induction on k. If we could show $f'|\mathrm{Var}(I^k) : \mathrm{Var}(I^k) \longrightarrow G(n + (k-1)N)$, then by the case k=1 we would know $f' : X' \longrightarrow G(n+kN)$. Thus it suffices to treat the case k=1, i.e., $I^2 = 0$. Since $f : X \longrightarrow G(n)$ we have $p^n \cdot f = 0$ and thus $p^n f' \in G(X')$ has the property that its restriction to G(X) is zero. Since

I^2 is zero and G is representable we know the group of sections of G over X' whose restriction to X is zero is isomorphic to the group $\mathrm{Hom}_{\mathcal{O}_X}(\omega_G \otimes_{\mathcal{O}_S} \mathcal{O}_{X'}, I)$ [S.G.A. 3 III 0.9]. But since p^N kills I, we certainly have $p^N(p^n f') = 0$ which implies $f' \in G(n+N)(X)$.

<u>Corollary</u> (3.3.16) Let p^N kill S and let G be as in the proposition. Then the k^{th} infinitesimal neighborhood of G(n) in G is the same as that of G(n) in G(n+kN). In particular $\mathrm{Inf}^k(G) = \mathrm{Inf}^k(G(kN))$ and is therefore representable.

<u>Proof</u>: If $f : T' \longrightarrow G$ belongs to the k^{th} infinitesimal neighborhood of G(n) in G, then there is a covering family $\{T_i' \longrightarrow T'\}$ and schemes T_i such that $T_i \hookrightarrow T_i'$ is a nilpotent immersion of order k and $f|T_i : T_i \longrightarrow G(n)$. But then by the proposition $f|T_i' : T_i' \longrightarrow G(n+kN)$ and hence $f \in \Gamma(T', G(n+kN))$ which proves the corollary.

<u>Corollary</u> (3.3.17) If p^N kills S and if $k < p^n$ we have $\mathrm{Inf}^k(G) \subseteq G(n+N-1)$ and hence $\mathrm{Inf}^k(G) = \mathrm{Inf}^k(G(n+N-1))$.

<u>Proof</u>: Let X' be an S-scheme and $X \hookrightarrow X'$ be a nilpotent immersion of order k. Denote with the subscript "o" the object obtained by reducing a given object modulo p. Given $f' : X' \longrightarrow G$ whose restriction to X is zero, then we have $f_o' : X_o' \longrightarrow G_o$ belongs to $\mathrm{Inf}^k(G_o)$. By the reasoning at the end of the proof of (2.1.8) (where we show that $\varinjlim G[n] = \varinjlim \mathrm{Inf}^k(G)$) we know that $\mathrm{Inf}^k(G_o) \subseteq G_o[n] \subseteq G_o(n)$. But this means that $f' \in G(X')$ has its restriction to $G(X_o') = G_o(X_o')$ belonging to $G(n)(X_o')$.

If we now apply the proposition (3.3.15) with $I = p\mathcal{O}_{X'}$, $k = N-1$ and $N = 1$, we find $f' \in G(n+N-1)(X')$

Theorem (3.3.18) Let p be locally nilpotent on S and G be a Barsotti-Tate group on S. Then $\overline{G} = \varinjlim \mathrm{Inf}^k(G)$ is a formal Lie group.

Proof: By (1.1.6) we know $\overline{G}$ is a subgroup of G and hence we must show it is a formal Lie variety. By (3.3.16) $\mathrm{Inf}^k(G)$ is, locally on S, representable and therefore since it is a sheaf it is representable.

By Theorem (3.3.13) we know G is formally smooth and obviously this implies that $\overline{G}$ is formally smooth. This tells us that $\mathrm{Inf}^k(G)$ satisfies the lifting condition 2) of (3.1.1) and hence, since locally on S $\mathrm{Inf}^k(G) = \mathrm{Inf}^k(G(m))$ for an appropriate m, it follows from (3.1.1) that locally on S $\mathrm{Inf}^k(G)$ satisfies condition 1) of that proposition. But now it is obvious that $\overline{G}$ satisfies condition 2) of Definition (1.1.4) and hence is a formal Lie group.

Definition (3.3.19) We define for a Barsotti-Tate group G on S (p locally nilpotent on S) $\underline{\omega}_G$ to be $\underline{\omega}_{\overline{G}}$.

Remark (3.3.20) It follows immediately from (3.3.18) that $\underline{\omega}_G$ is locally-free of finite rank and from (3.3.16) or, for a better estimate, (3.3.17) that locally on S $\underline{\omega}_G = \underline{\omega}_{G(m)}$ if m is sufficiently large. If p^N kills S, then $\underline{\omega}_G = \underline{\omega}_{G(N)}$ as follows from (3.3.17).

Remark (3.3.21) Let p be nilpotent on S and G, H two B.T. groups on S. Let $S_o = \mathrm{Var}(p)$. The map $\mathrm{Hom}(G,H) \longrightarrow \mathrm{Hom}(G_o, H_o)$ is

injective.

Proof: Let $u: G \longrightarrow H$ be such that $u_o = 0$. If p^N kills S, this implies $\mathrm{im}(u) \subseteq \mathrm{Inf}^{p^N}(H)$. By (3.3.17) $\mathrm{im}(u) \subseteq H(n)$ for n sufficiently large. But then $u \circ p^n \cdot \mathrm{id}_G = p^n \cdot \mathrm{id}_H \circ u = 0$. Since G is p-divisible, $u = 0$.

§4. In this section we study the relation between formal Lie groups and Barsotti-Tate groups on a scheme S, with p locally nilpotent on S.

Lemma (4.1) Let B be a ring in which p is nilpotent and I be a nilpotent ideal of B. Define a sequence of ideals $I_1 = pI + I^2, \ldots, I_{n+1} = pI_n + (I_n)^2$. Then for n sufficiently large $I_n = (0)$.

Proof: Let $J = pB + I$. Then one checks immediately that $I_n \subseteq J^{n+1}$, from which the result follows.

Lemma (4.2) If p is locally nilpotent on S and $\overline{G}$ is a formal Lie group over S, then $\overline{G}$ is of p-torsion.

Proof: We must show $\overline{G} = \varinjlim \overline{G}(n)$ and since this is a statement about sheaves it suffices to check it locally on S. Thus we can assume $S = \mathrm{Spec}(A)$ with p nilpotent on A and $\overline{G}$ is given by a power series ring $A[[X_1, \ldots, X_N]]$. If T is any affine S-scheme, say $T = \mathrm{Spec}(B)$, then an element of $\overline{G}(T)$ will be an N-tuple $(b_1, \ldots, b_N)$ with each b_i nilpotent. Let I be the ideal generated by $\{b_1, \ldots, b_N\}$. Then each component of $p \cdot (b_1, \ldots, b_N)$ belongs to $pI + I^2$ and $p^2(b_1, \ldots, b_N) \in p\cdot(pI + I^2) + (pI + I^2)^2, \ldots$. Thus by (4.1) we see $\overline{G}$ is of p-torsion.

Let S be the spectrum of an artin local ring with residue field of characteristic p and let G be a p-divisible formal Lie group on S. We shall show that G is a Barsotti-Tate group. To do this it suffices to prove that $G(1)$ is finite and locally-free.

Proposition (4.3) Let A be an artin local ring with residue field k of characteristic p and let $u: G \longrightarrow G$ be an epimorphism of a formal Lie group to itself. Then $\mathrm{Ker}(u)$ is finite and locally-free.

Proof: Denote by the subscript "o" the result of reducing an object defined over A to k the residue field of A. Let $(Sch/k)'$ denote the full sub-category of (Sch/k) consisting of those schemes which are locally of finite type over k. Endow $(Sch/k)'$ with the topology induced by the f.p.p.f. topology on (Sch/k). Observe that the restriction of $u: G \longrightarrow G$ to $(Sch/k)'$ is obviously an epimorphism. By [S.G.A.3 VII B 1.3.4] there is a morphism of sites $v: (Vf/k) \longrightarrow (Sch/k)'_{f.p.p.f.}$ given by the "inverse image" functor $v^{-1}: (Sch/k)' \longrightarrow Vf/k$ which is defined by $v^{-1}(X) = \widehat{X/k}$ in the notation of [S.G.A.3 VII B 1.2.6]. Since the extension of v^{-1} to sheaves is exact it transforms G into itself thought of as a "formal variety" in the sense of [S.G.A.3 VII B] and transforms u into an epimorphism in the category of formal groups over k. Since k is a field [S.G.A.3 VII B 1.4] implies that the map $\tilde{u}_o: k[[X]] \longrightarrow k[[X]]$ corresponding to u_o is "topologically flat" and because the power series ring is Noetherian this means it is flat.

Because the category of commutative formal groups over k is antiequivalent to the category of commutative affine groups over k

[S.G.A.3 VII B 2.2.2] it follows from [G.A. III §3, 5.5 a)] that $\mathrm{Ker}(u_o)$ is finite over k. Since A is artin this implies $\mathrm{Ker}(u)$ is finite over A. To see $\mathrm{Ker}(u)$ is flat over A it suffices to show the map $\tilde{u}: A[[X]] \longrightarrow A[[X]]$ corresponding to u is flat. Let $\underline{m}_A$ denote the maximal ideal of A. Consider the following sequence of maps

$$A \longrightarrow A[[X]] \xrightarrow{\tilde{u}} A[[X]] .$$

Because the first map and the composite are flat, the map

$$m_A \cdot A[[X]] \otimes_{A[[X]]} A[[X]] \longrightarrow \tilde{u}\,(m_A \cdot A[[X]]) \cdot A[[X]]$$

is bijective. Since we already know $\tilde{u}_o$ is flat we can because $\underline{m}_A$ is nilpotent apply the criterion of flatness [4, Chap. 3 §5 Theorem 1] to conclude $\tilde{u}$ is flat. This completes the proof.

Proposition (4.4) Let p be locally nilpotent on S and let G be in B. T. (S). Then the following conditions are equivalent:

1) $G = \overline{G}$.

2) G is a formal Lie group.

3) For all n $G(n)$ is radiciel.

4) $G(1)$ is radiciel.

Proof: 1) implies 2) by (3.3.18). 2) implies 1) because, by definition, a formal Lie group is ind-infinitesimal. Since property 2) is stable under base change by (1.03), to prove that 2) implies 3) we must show the map $G(n) \longrightarrow S$ is injective. Clearly we can assume S is affine and hence $G(n)$ is quasi-compact and therefore $G(n) \subseteq \mathrm{Inf}^k(G)$ for some k. But since the map $\mathrm{Inf}^k(G) \longrightarrow S$ is obviously injective it follows that $G(n)$ is radiciel. Conversely if $G(n)$ is radiciel, then since the closed immersion $S \longrightarrow G(n)$ is surjective, the ideal of this immersion must be locally nilpotent. Obviously this implies that $G(n)$ is ind-infinitesimal (and in fact is equal to one of its infinitesimal neighborhoods locally on S) and hence $G = \varinjlim G(n)$ is ind-infinitesimal. Thus $G = \overline{G}$ and 3) implies 1) (and 2)). Finally as 3) obviously implies 4) we must see that 4) implies 3). But this follows immediately via using the exact sequence $0 \longrightarrow G(n-1) \longrightarrow G(n) \xrightarrow{p^{n-1}} G(1) \longrightarrow 0$.

Corollary (4.5) If p is locally nilpotent on S, there is an equivalence of categories between that of Barsotti-Tate groups on S, with G(1) radiciel, and the category of formal Lie groups G with $p: G \longrightarrow G$ an epimorphism and G(1) finite and locally-free.

Proof: By (4.2) and (4.4) both categories are identified with the same full sub-category of f.p.p.f. sheaves of groups on S.

Corollary (4.6) If S is artin, a p-divisible formal Lie group is a Barsotti-Tate group with G(1) radiciel and conversely

Proof:(4.3) and (4.5).

Proposition (4.7) Let p be loc.nilpotent on S and G in B.T.(S). In order that $\overline{G} = 0$ it is necessary and sufficient that G be ind-étale.

Proof: If G is ind-étale, then for all k, $\mathrm{Inf}^k(G) = (0)$, since locally a point of $\mathrm{Inf}^k(G)$ with values in an S-scheme T must be a point of G(n) with values in T for some n, and hence must be zero since G(n) is

étale. Conversely if $\overline{G} = 0$, then for any $s \in S$, $\overline{G}_s = 0$, which implies $(G(n))_s$ has no connected (= radiciel) part and hence is étale. But G(n) being flat over S, this implies that G(n) is étale.

Apologies are offered in advance to the reader for the proof in the following lemma.

Lemma (4.8) Let $X \xrightarrow{f} S$ be finite and locally-free. Then the function $s \longmapsto$ separable rank (X_s) is locally constant if and only if there are morphisms $i: X \longrightarrow X'$, $f': X' \longrightarrow S$ which are finite and locally-free with i radiciel and surjective, f' étale and $f = f' \circ i$. The factorization is "unique" up to unique isomorphism and is functorial in X/S.

Proof: Observe that because of the uniqueness assertion it suffices to prove the lemma locally on S. Thus we can assume S affine and separable rank $(X_s) = n$ for all $s \in S$ (n some integer). Also observe that the "if" assertion is trivial. Now, by [E.G.A. IV 8.9.1, 8.10.5 (x), 11.2.6 (ii), 9.7.8, 9.3.3] we can assume S is noetherian in order to prove the existence uniqueness and functoriality assertions. We proceed in several steps which we outline:

1) existence and uniqueness when S is a field.
2) existence and uniqueness when S is a complete (Noetherian) local ring.
3) uniqueness for arbitrary S = Spec (A), A Noetherian.
4) existence of $f': X' \longrightarrow S$ when S is a local ring.
5) existence of $i: X \longrightarrow X'$ when S is a local ring.
6) existence for arbitrary S = Spec (A), A Noetherian.

7) functoriality.

1) If $S = \mathrm{Spec}\,(k)$ and $X = \mathrm{Spec}\,(B)$, we write $B = B_1 \times \ldots \times B_t$ as a product of finite dimensional local k-algebras. For each i we let k_i' be the maximal separable extension of k in the residue class field of B_i. Since B_i is artin we see there is a unique homomorphism $k_i' \longrightarrow B_i$ lifting the natural inclusion. Thus we can take $X' = \mathrm{Spec}(k_1' \times \ldots \times k_t')$. This gives us existence. Uniqueness is clear because of the uniqueness of a field of representatives for k_i' in B_i.

2) Using the constancy of the separable rank to show $X \xrightarrow{i} X'$ is radiciel and surjective we see from [E.G.A. IV 18.3.2, 18.5.14, 18.5.12, 11.3.11, 1.5.4 (v)] that there is a unique solution of the problem when $S = \mathrm{Spec}\,(A)$, A a complete Noetherian local ring.

3) Let $X \xrightarrow{i} X' \xrightarrow{f'} S$ and $X \xrightarrow{i'} X'' \xrightarrow{f''} S$ be two solutions. To construct a unique isomorphism between them it suffices to show that there is a unique isomorphism between their localizations at any point $s \in S$. For, if this were done, then for each $s \in S$, there would be a neighborhood U_s of s to which this isomorphism extends, since we are really dealing with modules of finite type over a Noetherian ring. But by the uniqueness these isomorphisms would have to agree on $U_s \cap U_{s'}$ for any two points $s, s' \in S$. Therefore they could be patched together to give the desired uniqueness statement for all of S. Hence we have reduced ourselves to the case when S is a Noetherian local ring A. Let $S' = \mathrm{Spec}\,(\hat{A})$ and $S'' = \mathrm{Spec}\,(\hat{A} \otimes_A \hat{A})$. The morphism $S' \longrightarrow S$ is faithfully flat and quasi-

compact and hence we will apply descent [S.G.A. 1 VIII 2.1]. By 2) we have a commutative diagram:

$$\begin{tikzcd} X'_{S'} \arrow[rr, "\eta", "\sim"'] & & X''_{S'} \\ & X_{S'} \arrow[ul, "i_{S'}"] \arrow[ur, "i'_{S'}"'] & \end{tikzcd}$$

We must show that η is a morphism of objects with descent data. To see this let τ_X(resp. $\tau_{X'}$, $\tau_{X''}$) denote the canonical isomorphism $p_1^*(X_{S'}) \xrightarrow{\sim} p_2^*(X_{S'})$ (resp. ...) We must show $\tau_{X''} \circ p_1^*(\eta) = p_2^*(\eta) \circ \tau_{X'}$.

But we know $p_2^*(\eta) \circ \tau_{X'} \circ p_1^*(i_{S'}) = p_2^*(\eta) \circ p_2^*(i_{S'}) \circ \tau_X = p_2^*(i'_{S'}) \circ \tau_X = \tau_{X''} \circ p_1^*(i'_{S'}) = \tau_{X''} \circ p_1^*(\eta) \circ p_1^*(i_{S'})$. But $i: X \longrightarrow X'$ is faithfully flat and hence $p_1^*(i_{S'})$ is faithfully flat. But this implies it is an epimorphism of schemes and hence $p_2^*(\eta) \circ \tau_{X'} = \tau_{X''} \circ p_1^*(\eta)$, completing the proof.

4) To show $f': X \longrightarrow S$ exists we shall show that, using the notation of 3) above, the X' which we know from 2) to exist over S' can be descended to S. Thus we have the standard situation $S'' \rightrightarrows S' \longrightarrow S$ and we have a solution of our problem for $X_{S'}$, call this solution Y. We want to descend Y to S. But by the uniqueness proved in 3) we see there is an isomorphism $p_1^*(Y) \xrightarrow{\sim} p_2^*(Y)$ (since both are solution for $X_{S''}$). But using the uniqueness of isomorphisms between solutions we see the isomorphism $p_1^*(Y) \xrightarrow{\sim} p_2^*(Y)$ must satisfy the cocycle condition and hence Y can be descended to an X' étale and finite over S [E.G.A. IV 17.7.3(ii), 2.7.1].

5) From 2) and 4) we know that over S' we have a morphism $i_{S'}: X_{S'} \longrightarrow X'_{S'}$. We want to see that this implies the existence of a similar morphism $i: X \longrightarrow X'$. When we pull back to S'' we find a commutative diagram

$$\begin{tikzcd}
 & X_{S''} \arrow[dl, "p_1^*(i_{S'})"'] \arrow[dr, "p_2^*(i_{S'})"] & \\
X'_{S''} \arrow[rr, "\sim", "\mu"'] & & X'_{S''}
\end{tikzcd}$$

because of the uniqueness established in part 3). While μ need not be the identity it follows again from part 3) that the isomorphism μ must satisfy the cocycle condition $p_{13}^*(\mu) = p_{23}^*(\mu) \circ p_{12}^*(\mu)$. But now this implies (same reference to S.G.A. 1 as above) that there is a scheme T finite and étale over S and an isomorphism $\varphi: X'_{S'} \xrightarrow{\sim} T_{S'}$ such that φ establishes an isomorphism between X' with descent datum μ, and $T_{S'}$ with its canonical descent datum. If we now replace $i: X_{S'} \longrightarrow X'_{S'}$ by $\varphi \circ i: X_{S'} \longrightarrow T_{S'}$ we see easily from the following calculation that $\varphi \circ i$ is a morphism between objects equipped with descent data and hence can be descended.

Consider the following diagram in which all morphisms σ, ρ have obvious meaning:

$$\begin{tikzcd}
p_1^*(X_{S'}) \arrow[d, "p_1^*(i_{S'})"'] & X_{S''} \arrow[l, "\rho_X"'] \arrow[r, "\sigma_X"] & p_2^*(X_{S'}) \arrow[d, "p_2^*(i_{S'})"] \\
p_1^*(X'_{S'}) & X'_{S''} \arrow[l, "\rho_{X'}"'] \arrow[r, "\sigma_{X'}"] \arrow[d, "\mu"] & p_2^*(X'_{S'}) \\
 & X'_{S''} &
\end{tikzcd}$$

We are told that $\mu \circ \rho_{X'}^{-1} \circ p_1^*(i_{S'}) \circ \rho_X = \sigma_{X'}^{-1} \circ p_2^*(i_{S'}) \circ \sigma_X$ and that $\sigma_{X'} \circ \mu \circ \rho_{X'}^{-1} : p_1^*(X'_{S'}) \xrightarrow{\sim} p_2^*(X'_{S'})$ satisfies the cocycle condition. Thus we find a T/S and an isomorphism $\varphi: X'_{S'} \xrightarrow{\sim} T_{S'}$ which makes the following diagram commute:

$$\begin{tikzcd}
p_1^*(X'_{S'}) \arrow[r, "\rho_{X'}^{-1}"] \arrow[d, "p_1^*(\varphi)"'] & X'_{S''} \arrow[r, "\mu"] & X'_{S''} \arrow[r, "\sigma_{X'}"] & p_2^*(X'_{S'}) \arrow[d, "p_2^*(\varphi)"] \\
p_1^*(T_{S'}) \arrow[rrr, "\sigma_T \circ \rho_T^{-1}"] & & & p_2^*(T_{S'})
\end{tikzcd}$$

Therefore we have $\sigma_{X'} \circ \mu \circ \rho_{X'}^{-1} \circ p_1^*(i_{S'}) \circ \rho_X = p_2^*(i_{S'}) \circ \sigma_X$ giving $p_2^*(\varphi) \circ \sigma_{X'} \circ \mu \circ \rho_{X'}^{-1} \circ p_1^*(i_{S'}) \circ \rho_X = p_2^*(\varphi) \circ p_2^*(i_{S'}) \circ \sigma_X$. But by the diagram above, the left-hand side is

$$\sigma_T \circ \rho_T^{-1} \circ p_1^*(\varphi \circ i_{S'}) \circ \rho_X$$

Therefore $\sigma_T \circ \rho_T^{-1} \circ p_1^*(\varphi \circ i_{S'}) = p_2^*(\varphi \circ i_{S'}) \circ \sigma_X \circ \rho_X^{-1}$ which tells us $\varphi \circ i_{S'}$ is a morphism between objects with descent data. This completes the proof of 5).

6) We observe that the solution which we know from 4) and 5) to exist at each local ring $\mathcal{O}_s$ can be prolonged to some neighborhood of s. See [E.G.A. IV 10.8.5]. By the uniqueness proved in part 3) these solutions can be patched together to give us a solution valid over all of S.

7) Since functoriality is obvious when S is a field, it follows formally from [E.G.A. IV 18.5.12] for the case when S is a complete Noetherian local ring. To know we can descend the morphism from S' to S (in the notation used above) we use the fact that all faces except possibly the bottom one in the following diagram commute and that $p_1^*(i)$ is an epimorphism.

$$\begin{tikzcd}
& p_1^*(X) \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & p_1^*(Y) \arrow[dl] \arrow[dd] \\
p_2^*(X) \arrow[rr, crossing over] \arrow[dd] & & p_2^*(Y) & \\
& p_1^*(X') \arrow[rr, dashed] \arrow[dl, dashed] & & p_1^*(Y') \arrow[dl] \\
p_2^*(X') \arrow[rr] & & p_2^*(Y') \arrow[from=uu, crossing over] &
\end{tikzcd}$$

This gives us functoriality when S is a local ring. Now just as above, we extend this first to an open neighborhood of any point and finally to all of our scheme S.

Proposition (4.9) Let p be locally nilpotent on S and G be in B.T.(S). The following conditions are equivalent

1) $\overline{G}$ is a B.T. group.

2) G is an extension of an ind-étale B.T. group G'' by an ind-infinitesimal B.T. group G'.

2 bis) G is an extension of an ind-étale B.T. group by a p-divisible formal Lie group.

3) For all n $G(n)$ is an extension of a finite étale group by a finite locally-free radiciel group.

3 bis) $G(1)$ is an extension of a finite étale group by a finite locally free radiciel group.

4) $s \longmapsto$ separable rank $(G(1)_s)$ is a locally constant function.

Proof: 4) implies 3 bis) by the lemma, since the functoriality assertion together with the obvious fact that the construction of the scheme X' in the lemma commutes with products, tells us that if G is a group then the associated scheme G' is also a group and $G \longrightarrow G'$ is a homomorphism which is an epimorphism since it is faithfully flat.

It is clear that 3 bis) implies 4) since the separable rank of $G(1)$ at s will then be equal to the rank of the étale quotient of $G(1)$ at s and hence is certainly a locally constant function.

Obviously 3) implies 3 bis) and conversely 3 bis) implies 3) via 4) since the separable rank of $G(n)_s$ = (separable rank $G(1)_s)^n$ as follows from the exact sequences: $0 \longrightarrow G(n-1) \longrightarrow G(n) \xrightarrow{p^{n-1}} G(1) \longrightarrow 0$.

Now let us assume that 3) holds so that for each n we have an exact sequence: $0 \to G'(n) \to G(n) \to G''(n) \to 0$ with $G'(n)$ finite, locally free and radiciel, $G''(n)$ finite and étale. We will show that the systems $(G'(n))$ and $(G''(n))$ give us Barsotti-Tate groups. To do this it suffices to see that if $0 \to G \to H \to K \to 0$ is an exact sequence of finite locally-free groups satisfying the condition of the lemma, then the corresponding sequences of étale quotients or radiciel kernels are exact. But this will follow from (4.10) proved below. For by (4.10) and [E.G.A. IV 2.61(i), 2.7.1 (viii), 2.2.11 (iv)] it then suffices to check the corresponding when S is the spectrum of an algebraically closed field. But in this case it is obvious because for any finite group H we have $H = H^o \times H^{ét}$. Therefore by applying the above discussion to the sequences

$0 \to G(i) \to G(n) \xrightarrow{p^i} G(n-i) \to 0$ we see that $G' = \varinjlim G'(n)$ and $G'' = \varinjlim G''(n)$ are Barsotti-Tate groups. Furthermore by passage to the limit we see we have an exact sequence $0 \to G' \to G \to G'' \to 0$ with G' ind-infinitesimal and G'' ind-étale. This shows that 3) implies 2).

2) implies 3) is trivial because from the exactness of the sequence $0 \to G' \to G \to G'' \to 0$ follows that of $0 \to G'(n) \to G(n) \to G''(n) \to 0$ [I, 2.4.3]. 2) implies 2 bis) is also clear because G' being ind-infinitesimal we have $G' = \overline{G'}$ and is therefore a formal Lie group by (3.3.18). Conversely let $0 \to G' \to G \to G'' \to 0$ be an exact sequence with G' a p-divisible formal Lie group and G'' an ind-étale B.T. group. In order to show 2 bis) implies 2) we must prove G' is a B.T. group. To do this

it suffices (because of (4.2)) to show $G'(1)$ is finite and locally free. But from our exact sequence we obtain a sequence $0 \to G'(1) \to G(1) \to G''(1) \to 0$ since G' is p-divisible. Therefore $G'(1)$ is finite and locally-free.

Finally it remains to show 1) and 2) are equivalent. To do this we shall utilize (4.11) which is proved below. Assuming 2) we have from (4.11) and (4.7) that $G' = \overline{G}$ and hence $\overline{G}$ is a Barsotti-Tate group. Conversely if $\overline{G}$ is a B.T. group then we can form the sequence $0 \to \overline{G} \to G \to G/\overline{G} \to 0$. From [I 2.4.3] we know $G/\overline{G}$ is a Barsotti-Tate group. But by (4.11) $\overline{(G/\overline{G})} = (0)$ and hence by (4.7) $G/\overline{G}$ is ind-étale. This completes the proof.

Lemma (4.10) Let $0 \to G \xrightarrow{i} H \xrightarrow{u} K \to 0$ be a complex of finite locally-free groups on S. The sequence is exact if and only if for all $s \in S$, the sequence $0 \to G_s \to H_s \to K_s \to 0$ is exact.

Proof: Only the "if" part requires proof. By the criterion for checking flatness fiber by fiber [E.G.A. IV 11.3.11] we know that $H \to K$ is an epimorphism if all maps $H_s \to K_s$ are epimorphisms. Thus it remains to prove the map $G \to \mathrm{Ker}(u)$ is an isomorphism. This can be checked locally on S and hence we can assume S is affine, say $S = \mathrm{Spec}(A)$, $G = \mathrm{Spec}(C)$, $\mathrm{Ker}(u) = \mathrm{Spec}(B)$ where B and C are projective (finitely generated) A-modules. To show $B \to C$ is an isomorphism it suffices to prove this at each point. Hence we can assume A is a local ring with maximal ideal $\underline{m}$. By hypothesis $B/\underline{m}B \to C/\underline{m}C$ is an isomorphism. By Nakayama (since C is finitely generated) the map $B \to C$ is surjective.

To show it is injective let W be its kernel. Then because C is projective the sequence $0 \longrightarrow W \longrightarrow B \longrightarrow C \longrightarrow 0$ splits and hence W is finitely generated since B is. But then we have $W/\underline{m}W = (0)$ and hence using Nakayama again we see $W = (0)$.

Lemma (4.11) Let p be locally nilpotent on S and let $0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$ be an exact sequence of Barsotti-Tate groups on S. Then $0 \longrightarrow \overline{G}_1 \longrightarrow \overline{G}_2 \longrightarrow \overline{G}_3 \longrightarrow 0$ is also exact.

Proof: The exactness of $0 \longrightarrow \overline{G}_1 \longrightarrow \overline{G}_2 \longrightarrow \overline{G}_3$ is trivial. We must prove $\overline{G}_2 \longrightarrow \overline{G}_3$ is an epimorphism. Let T be an S-scheme and $y \in \overline{G}_3(T)$ be given. Then because $G_2 \longrightarrow G_3$ is an epimorphism we can find a covering $\{T_i \longrightarrow T\}$ such that for each i, there is an x_i in $G_2(T_i)$ with the image of x_i being $y|T_i$. By passing to a covering of each T_i we can assume $y|T_i$ has the property that $y|\overline{T}_i = (0)$ where $\overline{T}_i \hookrightarrow T_i$ is a nilpotent immersion and T_i is affine. But this tells us that $x_i|\overline{T}_i$ belongs to $G_1(\overline{T}_i)$. Since G_1 is formally smooth by (3.3.13), we know there is an x_i' in $G_1(T_i)$ which lifts $x_i|\overline{T}_i$. Then $x_i - x_i' \in G_2(T_i)$ maps to $y|T_i$ and has its restriction to $\overline{T}_i$ equal to zero. Therefore $x_i - x_i' \in \overline{G}_2(T_i)$ and hence the map $\overline{G}_2 \longrightarrow \overline{G}_3$ is indeed an epimorphism.

Let A be an adic ring for the topology defined by the powers of an ideal I with I/I^2 of finite type over A/I. Set $A_n = A/I^{n+1}$, $S = \mathrm{Spec}\,(A)$, $S_n = \mathrm{Spec}\,(A_n)$.

Lemma (4.13): The natural functor Formal Lie Varieties (S) $\longrightarrow \varprojlim$ Formal Lie Varieties (S_n) is an equivalence of categories. In particular it induces an equivalence of categories between the category of formal Lie groups on S and the inverse limit of the categories of formal Lie groups on the various S_n's.

Proof: First observe that a formal Lie variety X on S can be thought of as the sheaf corresponding to the formal scheme $\mathrm{Spf}\,(\widehat{\mathrm{Sym}}[\underline{\omega}_X])$ where the symmetric algebra is completed with respect to the topology defined by powers of the augmentation ideal. Under this identification $X|S_n$ is identified with $\mathrm{Spf}\left(\widehat{\mathrm{Sym}}\,[\underline{\omega}_X \underset{A}{\otimes} A_n]\right)$. But a mapping $Y \longrightarrow X$ of formal Lie varieties on S then corresponds to an A-linear map $\underline{\omega}_X \longrightarrow \mathrm{Sym}[\underline{\omega}_Y]^+ = \prod_{i=1}^{\infty} S^i[\underline{\omega}_Y]$ where the "+" denotes the augmentation ideal and $S^i[\underline{\omega}_Y]$ is the ith symmetric power of $\underline{\omega}_Y$. Thus $\mathrm{Hom}\,(Y, X)$ is identified with $\prod_{i \geq 1} \mathrm{Hom}_A\,(\underline{\omega}_X, S^i[\underline{\omega}_Y]) = \prod_{i \geq 1} \varprojlim \mathrm{Hom}_{A_n}\,(\underline{\omega}_X \otimes A_n, S^i\,(\underline{\omega}_Y \otimes A_n))$ [E.G.A. 0_{I} 7.2.10] $= \varprojlim \prod_{i \geq 1} \mathrm{Hom}_{A_n}\,(\underline{\omega}_X \otimes A_n, S^i\,(\underline{\omega}_Y \otimes A_n))$, as inverse limits commute. But this last is via the above simply identified with $\varprojlim \mathrm{Hom}\,(Y|S_n, X|S_n)$. Thus the functor is fully faithful. To show it is essentially surjective assume we are given a compatible family of formal Lie varieties X_n. Then we obtain a family of finite locally-free A_n-modules $\underline{\omega}_{X_n}$ with $\underline{\omega}_{X_n} \otimes A_{n-1} = \underline{\omega}_{X_{n-1}}$. But let $\underline{\omega} = \varprojlim \underline{\omega}_{X_n}$. $\underline{\omega}$ is of finite type over A by [E.G.A. 0 I 7.2.9]. But by [E.G.A. IV 18.3.2.1 (ii)] we know $\underline{\omega}$ is projective. But now it is clear that setting $X = \mathrm{Spf}\,(\widehat{\mathrm{Sym}}[\underline{\omega}])$ we have $X|S_n = X_n$ and hence our functor is indeed an equivalence of categories.

(4.14) Let us assume that p is nilpotent on S_o and let G belong to B.T. (S). Then for each n, we have the formal Lie group $\overline{G}_n$ and since these obviously are compatible we can by (4.13) define a formal Lie group $\overline{G}$. Thus we obtain a functor $G \mapsto \overline{G}$ from B.T. (S) $\longrightarrow$ Form Lie Gr. (S). But notice that we can not in general define a homomorphism $\overline{G} \longrightarrow G$ inducing the given map $\overline{G}_n \hookrightarrow G_n$ on S_n. To see this take $S =$ Spec $(\mathbb{Z}_p)$ and let G be any Barsotti-Tate group on S. Then since $\overline{G} = \varinjlim \mathrm{Inf}^k(G)$, any homomorphism $\overline{G} \longrightarrow G$ induces for each k a morphism of pointed sheaves $\mathrm{Inf}^k(G) \longrightarrow G$. But as S is affine, any such morphism must factor through some $G(n)$. Thus we are led to examine a morphism of pointed schemes $\mathrm{Inf}^k(G) \xrightarrow{\varphi} G(n)$. Thus we have a commutative diagram

$$\begin{tikzcd} \mathrm{Inf}^k(G) \arrow[rr, "\varphi"] & & G(n) \\ & S \arrow[ul, "e_k"] \arrow[ur, "e"'] & \end{tikzcd}$$

and restricting to the generic fiber we still have such a diagram. But we know:

1) $\mathrm{Inf}^k(G)$ is an infinitesimal thickening of S.

2) $G(n) \mid \mathbb{Q}_p$ is étale.

3) $\varphi \circ e_k = e \circ \pi_{\mathrm{Inf}^k(G)} \circ e_k$.

Therefore $\varphi \mid \mathbb{Q}_p$ is the trivial map. But since φ corresponds to a mapping of free $\mathbb{Z}_p$-modules, it is determined by its restriction to the generic fiber. Thus φ is trivial and we see there is no non-zero way to map $\overline{G}$ to G.

(4.15) Keeping the same notation as above, we see that the obvious functor from finite locally free groups on S to $\varprojlim$ fin. loc. free gr. (S_n) is an equivalence of categories, because such groups are given in terms of modules of finite type over the appropriate ring with additional structure making them into bi-algebras also defined in terms of the underlying modules. Thus using [E.G.A. 0_I 7.2.9, 7.2.10] and the criterion of flatness we obtain the equivalence. This equivalence preserves exact sequences. To see this observe that it suffices to treat the case of epimorphisms and show it is kernel preserving. For epimorphisms this follows from the fact that they can be expressed in terms of the bi-algebras of the groups. Thus if $G_n \longrightarrow H_n$ is an epimorphism of groups on S_n, then we have C_n is a finite locally free B_n-module for all n where $G_n = \mathrm{Spec}\,(C_n)$, $H_n = \mathrm{Spec}\,(B_n)$. But $B = \varprojlim B_n$ satisfies the same ring theoretic hypotheses made on A (for the ideal IB). Thus C is a finite locally free B-module ($C = \varprojlim C_n$), and hence the morphism $G \longrightarrow H$, between the two groups associated to the families (G_n) and (H_n), is an epimorphism. The kernel preserving property is proved in an analogous manner using the fact that the kernel is expressed in terms of a tensor product of rings. But now it is clear, since exact sequences of the form $0 \longrightarrow G(i) \longrightarrow G(n) \xrightarrow{p^i} G(n-i) \longrightarrow 0$ are preserved under the equivalence, that we have the following lemma.

Lemma (4.16) The natural functor establishes an equivalence of categories B.T. $(S) \xrightarrow{\sim} \varprojlim$ B.T. (S_n).