Conventions

1. "p" denotes a prime number fixed once and for all.

2. "group" always means commutative group.

3. "group over S, S-group, ... " will always mean a f.p.p.f. sheaf of (commutative) groups on the site $(Sch/S)_{f.p.p.f.}$. Groups which are representable will be referred to as such or via a modifying adjective (i.e., flat, finite, and locally-free, ...) which makes clear that they are group-schemes.

4. The references [8], [9], [12], [13] which are frequently cited in the text are referred to as:

   a) [8] G.A.

   b) [9] S.G.A.3

   c) [12] E.G.A.

   d) [13] S.G.A.

Following standard conventions E.G.A.IV denotes a particular reference in the 4th chapter of E.G.A. Similarly $S.G.A._*$ **, ... refers to a particular place in expose ** of the *th seminar held at Bures.

Chapter I. Definitions and Examples

§1. (1.0). Let S be a scheme and G a group on S (i.e., following the conventions introduced above, G is a commutative f.p.p.f. sheaf of groups on the site Sch/S) such that $p^n G = (0)$. Then we have the following lemma:

Lemma (1.1). The following conditions are equivalent

(i) G is a flat $\mathbb{Z}/p^n\mathbb{Z}$-module

(ii) $\mathrm{Ker}(p^{n-i}) = \mathrm{Im}(p^i)$ for $i = 0, \ldots, n$

Proof: First we show (i) implies (ii). From (i) it follows that $gr^{\cdot}(\mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p\mathbb{Z}} gr^o(G) \simeq gr^{\cdot}(G)$ (the associated graded group being taken with respect to the filtration defined by powers of p). Because of this we know that p^i induces an isomorphism from G/pG to $p^iG/p^{i+1}G$ for $i \leq n-1$. Thus $\mathrm{Ker}(p^{n-1}) \subseteq \mathrm{Im}(p)$ and hence $\mathrm{Ker}(p^{n-i}) \subseteq \mathrm{Ker}(p^{n-1}) \subseteq \mathrm{Im}(p)$ which implies that $\mathrm{Ker}(p^{n-i}) = p \cdot \mathrm{Ker}(p^{n-i+1}) = p(p^{i-1}G) = p^iG$ (by induction on i).

To prove that (ii) ⇒ (i), we observe by taking $i=1$ that $pG = \mathrm{Ker}(p^{n-1})$ and hence that p^{n-1} induces an isomorphism $G/pG \xrightarrow{\sim} p^{n-1}G$. Since this map factors as $G/pG \longrightarrow pG/p^2G \longrightarrow \cdots \longrightarrow p^{n-1}G$ we see that each of these maps is an isomorphism. Thus, since

$$gr^o(\mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p\mathbb{Z}} gr^o(G) \xrightarrow{\sim} gr^o(G),$$

we have

$$gr^{\cdot}(\mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p\mathbb{Z}} gr^o(G) \xrightarrow{\sim} gr^{\cdot}(G).$$

To complete the proof we want to utilize a version of the "criterion of flatness."

By [16;II §4] to prove the flatness of G it suffices to show

$\mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(M,G) = 0$ for any $\mathbb{Z}/p^n\mathbb{Z}$-module M. But the reasoning of [4; Chap. III §5 #3 remark 1 and prop. 1] is completely formal and hence applies in the general context. Consider the exact sequence:

$$M \otimes \mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p, G) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(M,G) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}/p\mathbb{Z}}(M, G/pG) \longrightarrow 0$$

which arises from the terms of low degree in the spectral sequence for "associativity" of Tor.

$\mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, G) = 0$ by hypothesis (as noted above).

$\mathrm{Tor}_1^{\mathbb{Z}/p\mathbb{Z}}(M, G/pG) = 0$ since $\mathbb{Z}/p\mathbb{Z}$ is a field. This shows that $\mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(M,G)$ is zero for any $\mathbb{Z}/p\mathbb{Z}$-module M and thus completes the proof.

Definition (1.2) If $n \geq 2$, a truncated Barsotti-Tate group of level n is an S-group G such that:

1) G is a finite locally-free group scheme

2) G is killed by p^n and satisfies the equivalent conditions of lemma (1.1).

Remark (1.3) For completeness let us define a truncated Barsotti-Tate group of level 1 (on a scheme S where p is locally nilpotent) as a group G which satisfies:

1) G is finite and locally free and killed by p.

2) Denoting by S_o the closed subscheme $\mathrm{Var}(p.1_S)$ of S and $G_o = G \times_S S_o$, $\mathrm{im}\, v_{G_o} = \mathrm{Ker}\, f_{G_o}$, $\mathrm{im}\, f_{G_o} = \mathrm{Ker}\, v_{G_o}$ (see [II 3.3.11, 3.3.12]).

Notation (1.4) If G is a group, we will write $G(n)$ for the kernel of p^n. If G is killed by p^n, we write $G = G(n)$.

Lemma (1.5) a) If $G(n)$ is a flat $\mathbb{Z}/p^n\mathbb{Z}$-module then $G(n)$ is a finite locally-free group scheme if and only if $G(1)$ is and then all the $G(i)$ are.

b) If $G(n)$ is finite and locally free then $p^i: G(n) \longrightarrow G(n-i)$ is an epimorphism if and only if it is faithfully flat.

Proof: We prove b) first. Clearly if $p^i: G(n) \longrightarrow G(n-i)$ is faithfully flat it is an epimorphism. Conversely if it is an epimorphism, then by using the criterion for checking flatness fiber by fiber [E.G.A.IV 11.3.11] we are reduced to the case when S is the spectrum of a field and here the result is standard. [G.A. III, §3, 7.4]

To prove a) we observe that if $G(n)$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ and $G(1)$ is finite and locally free, then we have exact sequences:

$0 \longrightarrow G(1) \longrightarrow G(2) \xrightarrow{p} G(1) \longrightarrow 0$

$0 \longrightarrow G(1) \longrightarrow G(3) \xrightarrow{p} G(2) \longrightarrow 0, \ldots$. Therefore by induction we see that all the $G(i)$ are finite locally-free (since an extension of two such groups is another one by descent theory [G.A. III §4, 1.9]). Conversely, if $G(n)$ is finite and locally-free, then each $G(i)$ is certainly finite of finite presentation over S [E.G.A. II 6.15. (iii) and (v) for "finite," E.G.A. IV 1.6.2. (iii) and (v) for "of finite presentation"]. From part b) (proved above) we know each $G(i)$ is flat over S. Hence each $G(i)$ is finite and locally-free.

§2. (2.0) Let S be a scheme and G a group on S. Denote by $G(n)$ the kernel of multiplication by p^n on G. G is said to be of p-torsion if $\varinjlim G(n) = G$. G is said to be p-divisible if $p \cdot id_G : G \longrightarrow G$ is an epimorphism.

Definition (2.1): G is a Barsotti-Tate group if it satisfies the following three conditions:

(2.1.1) G is of p-torsion

(2.1.2) G is p-divisible

(2.1.3) G(1) is a finite, locally-free group scheme

Notation (2.2): We write B.T.(S) for the category of Barsotti-Tate groups on S, whose objects are the Barsotti-Tate groups and whose morphisms are simply homomorphisms of S-groups.

Remarks (2.3): Let G be a Barsotti-Tate group

1) $G(n) = G(n+1)(n)$

2) For any i such that $0 \leq i \leq n$, p^{n-i} induces an epimorphism $G(n) \longrightarrow G(i)$ (because multiplication by p^{n-i} is an epimorphism of G).

3) From remarks 1) and 2) and the fact that G(1) is finite and locally-free it follows from 1.5 that the G(n) for $n \geq 2$ are truncated Barsotti-Tate groups and that we have exact sequences:

(2.3.1) $$0 \longrightarrow G(n-i) \longrightarrow G(n) \xrightarrow{p^{n-i}} G(i) \longrightarrow 0$$

4) It follows from the elementary theory of finite group schemes over a field that the rank of the fiber of G(1) at a point $s \in S$ is of the form $p^{h(s)}$ where h is a locally constant function on S. It also follows from remark 3) that the rank of the fiber of G(n) at s is $p^{nh(s)}$ [G.A. IV §3, 5].

5) Assume we have a system of groups G(n) with G(n) finite and locally-free such that:

a) $G(n) = G(n+1)(n)$

b) The rank of the fiber of G(n) at s is $p^{nh(s)}$ where h is a locally constant function on S.

We consider the exact sequence

$$0 \longrightarrow G(n-i) \longrightarrow G(n) \xrightarrow{p^{n-i}} G(i) \ .$$

By looking at each fiber and using the multiplicativity of the ranks, $G(n)_s \xrightarrow{p^{n-i}} G(i)_s$ is faithfully flat. Therefore since G(n) is flat over S, it follows

that $G(n) \xrightarrow{p^{n-i}} G(i)$ is faithfully flat

and hence an epimorphism. Thus we see that $G = \varinjlim G(n)$ is a Barsotti-Tate group and therefore (using also remarks 1) and 4)) it follows that our definition of Barsotti-Tate group is equivalent to that of Tate [30].

6) From remark 5) it follows that our definition of Barsotti-Tate group is essentially independent of the fact that we choose to work with f.p.p.f. sheaves. Nevertheless it will be quite convenient to view the category B.T.(S) as a full sub-category of the category of abelian sheaves (for the f.p.p.f. topology) on S.

Sorites (2.4)

(2.4.1) If $S' \xrightarrow{f} S$ is a morphism and G is in B.T.(S), then $f^*(G)$ is in B.T.(S').

That $f^*(G)$ is of p-torsion and p-divisible follows immediately from the fact that f^* is exact as does the formula $f^*(G)(n) = f^*(G(n))$. But since $f^*(G(1)) = G(1) \times_S S'$, it is immediate that $f^*(G)(1)$ is finite and locally-free and hence $f^*(G)$ is in B.T.(S').

Remark (2.4.2): The assignment $S \mapsto$ B.T.(S) gives a fibered category over the category of schemes. It is in fact a stack when (Schemes) is endowed with the f.p.q.c. topology. This follows easily from the definitions and descent theory [11, I 3.2].

(2.4.3) If $0 \to G_1 \to G_2 \to G_3 \to 0$ is exact and

a) G_1 and G_2 are in B.T.(S)

b) G_1 and G_3 are in B.T.(S),

then in either case the third group is in B.T.(S) also.

i.e. that $G_3(1)$ is a scheme, not just a sheaf for fppf.

Proof: In both cases it follows from the serpent lemma that the sequence $0 \to G_1(1) \to G_2(1) \to G_3(1) \to 0$ is exact. The representability of $G_3(1)$ (in case a)) is given by [G.A. III, §2 3.2] while the representability of $G_2(1)$ (in case b)) is given by [G.A. III, §4, 1.9] which also tells us $G_2(1)$ is finite and locally-free. The fact that $G_3(1)$ is finite and locally-free (in case a)) is somewhat more involved. By [E.G.A. II 5.4.3(i), E.G.A. IV 1.6.2(v)] $G_1(1) \hookrightarrow G_2(1)$ is a proper monomorphism of finite presentation and hence it is a closed immersion [E.G.A. IV 8.11.5]. By [S.G.A. 3 VI B 9.2 (x), (xi), (xii), (xiii)] $G_3(1)$ is separated, flat, quasi-finite and of finite presentation over S. That $G_3(1)$ is finite and locally-free now follows from [E.G.A. II, 5.4.3 (i), IV 8.11.1, 1.4.7].

In either case a) or b) it is immediate that the relevant group is of p-torsion and p-divisible and thus a Barsotti-Tate group.

(2.4.4) By considering the exact sequences (2.3.1) we see that the family of Cartier duals $G(n)^*$ together with the maps $p^*: G(n)^* \to G(n+1)^*$

(coming from the exact sequences) give us a Barsotti-Tate group G^*, the Cartier dual of G, with $G^*(n) = G(n)^*$. The assignment $G \mapsto G^*$ extends to morphisms so that we obtain a duality on the category of Barsotti-Tate groups. Just as with ordinary Cartier duality, this duality is compatible with all base changes.

Remark (2.4.5). The category B.T.(S) is not abelian. Kernels do not exist in this category since the kernel of the morphism $G \xrightarrow{p} G$ must be killed by p and hence can not be a Barsotti-Tate group (unless G = 0).

§3. (3.0) We give in this paragraph several examples of Barsotti-Tate groups. First though, we recall some terminology about finite, locally-free group schemes, G, on an arbitrary base S.

(3.1) G is said to be of multiplicative type if the following three equivalent conditions hold:

(3.1.1) Locally on S for the étale topology, G is isomorphic to a group of the form Spec $(\mathcal{O}_S[M])$ where M is an ordinary finite abelian group.

(3.1.2) G^*, the Cartier dual of G, is étale.

(3.1.3) Locally, for the Zariski topology, there is a monomorphism $G \to T$ where T is a torus (i.e., a group which locally for the étale topology is isomorphic to $\mathbb{G}_m^\ell$).

That (3.1.1) implies (3.1.2) is obvious. The reverse implication is an immediate consequence of [S.G.A. 3 X 4.5, 4.8]. To see that (3.1.1) implies (3.1.3) we can, since the question is local on S, assume S to be affine and then by the standard arguments assume S to be Noetherian [E.G.A. IV 8.9.1, 8.5.5, 8.10.5(x), 17.7.8(ii)]. We decompose S into a disjoint union of open sub-schemes S_λ, such that for all s belonging to a given S_λ, $G_{\bar{s}}$, the geometric fiber, is isomorphic to $\mathrm{Spec}(\overline{k(s)}[M_\lambda])$ for a well-defined finite group M_λ. Replacing S by S_λ and then by a connected component of S_λ (which is open since S is locally noetherian) we reduce to the case where S is connected. Then by [S.G.A3 X 7.2] we see that to give G is equivalent to giving a finite π-module M where π is the "enlarged" fundamental group corresponding to the choice of some geometric point of S. But now writing M as a quotient of $\mathbb{Z}[M]$ when π operates on $\mathbb{Z}[M]$ in the obvious way, we see that the group corresponding to the abelian group $\mathbb{Z}[M]$ is a torus and hence we achieve the desired embedding.

Finally to prove that (3.1.3) implies (3.1.2) it suffices to check the implication when S is the spectrum of a field. But now as G is a finite closed sub-group of a torus, the implication follows immediately from [C.A. IV §1 2.4(a)].

(3.2) G is said to be <u>infinitesimal</u> if the structural morphism $G \longrightarrow S$ is <u>radiciel.</u> This is of course a condition which is verified fiber by fiber. We shall see in [II 4.4], that this use of the word "infinitesimal" is consistent with a later meaning we shall give it in connection with formal Lie groups.

(3.3) Recall finally the following definition.

<u>Definition</u> (3.3.1) An abelian scheme $A \xrightarrow{f} S$ is a commutative group scheme such that:

1) f is proper
2) f is smooth
3) f has geometrically connected fibers.

<u>Example</u> (3.4). If A/S is an abelian scheme then $\varinjlim A(n)$ is a Barsotti-Tate group of rank 2d where d is the relative dimension of A/S and hence a locally constant function on S. Since the group $\varinjlim A(n)$ is obviously of p-torsion, we must show it is p-divisible and A(1) is finite and locally-free of rank p^{2d}. To know it is p-divisible it obviously suffices to check that $p: A \longrightarrow A$ is an epimorphism. Thus it suffices to know $p: A \longrightarrow A$ is faithfully flat and hence we are by [E.G.A IV 11.3.11] reduced to the case when we are over an algebraically closed field. But as is well known, multiplication by p on an abelian variety is surjective, and hence by the lemma of generic flatness [21; 6.12] we are done. Finally, since A(1) has zero-dimensional and hence finite fibers and since $A(1) \longrightarrow S$ is proper (being obtained by the base change $S \longrightarrow A$ from the map $p: A \longrightarrow A$ which is proper by [E.G.A. II 5.4.3(i)]), it follows that $A(1) \longrightarrow S$ is finite [E.G.A. IV 8.11.1]. Since $A(1) \longrightarrow S$ is flat, finite and of finite presentation it is finite and locally free [E.G.A. IV 1.4.7]. Finally the statement about the rank follows immediately from [22; §6]. Later we shall have much more to say about Barsotti-Tate groups of this type which we denote by $\bar{A}$ or $A(\infty)$.

<u>Example</u> (3.5) Let T be <u>a torus on S</u> and consider $\varinjlim T(n) \overset{\text{def.}}{=\!=} T(\infty)$. Then $T(\infty)$ is a Barsotti-Tate group of rank d when d is the relative dimension of T/S. That $T(\infty)$ is of p-torsion is obvious. To see that $T(\infty)$ is p-divisible and T(1) is finite and locally-free of rank d, we are by descent reduced to the case where $T \cong \mathbb{G}_m^d$ where both are completely trivial.

<u>Example</u> (3.6) We assume in this example that p is locally nilpotent on S. Also, this example will be treated in more detail later. Let G be a <u>formal Lie group on S.</u> We will verify in [II 4.2], that G is automatically of p-torsion. Assume that multiplication by p is an epimorphism of G and that G(1) is finite and locally-free. This last condition follows from the hypothesis that $G \xrightarrow{p} G$ is an epimorphism when the base S is artin. These assumptions imply that G is a Barsotti-Tate group with G(1) and hence all G(n) infinitesimal. As we shall verify in detail later [II, 4.5] we have an equivalence of categories between the category of Barsotti-Tate groups on S with G(1) infinitesimal and that of formal Lie groups G such that $G \xrightarrow{p} G$ is an epimorphism and G(1) is finite and locally-free.

<u>Example</u> (3.7) Let G be a Barsotti-Tate group on S such that G(1), and hence all G(n), is étale. We call such a group ind-étale. Associated to G is the projective system $T_p(G)$: $G(1) \xleftarrow{p} G(2) \xleftarrow{p} G(3) \longleftarrow \dots$. By the very definition of the phrase, $T_p(G)$ is a "faisceau p-adique constant tordu sans torsion." [S.G.A. 5 VI 1.2] It is immediately checked that any homomorphism between $T_p(G)$ and $T_p(H)$ must come from a

homomorphism from G to H. Thus it is essentially a tautology that the category of Barsotti-Tate groups on S which are ind-étale is equivalent (via the functor $G \longmapsto T_p(G)$) to the category of "faisceaux p-adique constant tordu sans torsion." If S is connected and $\bar{s}$ is a geometric point of S, this last category is equivalent to that of continuous representations of $\pi_1(S, \bar{s})$ in finite free $\mathbb{Z}_p$-modules.

<u>Example</u> (3.8) A Barsotti-Tate group G on S is said to be <u>toroidal</u> if G(1) is of multiplicative type. Of course this implies that all G(n) are of multiplicative type. From 3.1 we know that G^*, the Cartier dual will be ind-étale and hence that the functor: $G \longmapsto T_p(G^*)$ induces an anti-equivalence between the category of <u>toroidal</u> Barsotti-Tate groups on S and that of "faisceaux p-adiques constant tordu sans torsion." To obtain a covariant equivalence between these two categories, let us first introduce the following notation.

(3.8.1) $$\mu \overset{\text{def.}}{=\!=} \mathbb{G}_m(\infty) = \varinjlim \mu_{p^n} .$$

Consider the functor on toroidal groups defined by

$$G \longmapsto \underline{\mathrm{Hom}}_{S\text{-}gr}(\mu, G) \overset{\text{def.}}{=\!=}$$

the inverse system $(\underline{\mathrm{Hom}}_{S\text{-}gr}(\mu_{p^n}, G(n)))_{n\geq 1}$. But this last inverse system is identified via Cartier duality with the inverse system

$$(\underline{\mathrm{Hom}}_{S\text{-}gr}(G(n)^*, \mathbb{Z}/p^n\mathbb{Z}))_{n\geq 1} = T_p(G^*)^{\vee}$$

[see S.G.A. 3 X 5.8].