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INTRODUCTION

The object of this paper is to prove these results announced by Grothendieck.

Theorem 1. If A is an abelian scheme over S its universal extension is crystalline in nature and its Lie algebra is isomorphic to the one-dimensional crystalline cohomology of A^* over S, $R^1f_{*,\mathrm{crys}}(\mathcal{O}_{A^*,\mathrm{crys}})$.

Theorem 2. If G is a Barsotti-Tate (= p-divisible) group on S, a base such that p is locally nilpotent, then its universal extension is crystalline in nature, and its Lie algebra provides a generalization of the classical Dieudonné module theory for Barsotti-Tate groups.

UNIVERSAL EXTENSIONS

If A is an abelian variety over a field k, the universal extension of A is defined to be an extension of algebraic groups

$$(*) \qquad 0 \to V(A) \to E(A) \to A \to 0$$

where $V(A)$ is a vector group over k and such that $(*)$ is universal for extensions of A by vector groups.

Rosenlicht [22] defined this notion and showed that any abelian variety A possesses a universal extension. The key to his construction are the isomorphisms

$$\mathrm{Ext}(A,G_a) \cong H^1(A,\mathcal{O}_A) \cong \mathrm{Hom}_k(\underline{\omega}_{A^*},G_a)$$

which gives

$$\mathrm{Ext}^1(A,V) \cong \mathrm{Hom}_k(\underline{\omega}_{A^*},V)$$

where V is an arbitrary vector group over k, A^* is the dual abelian variety, with zero section

$$e\colon \mathrm{Spec}\, k \to A^*$$

and $\underline{\omega}_{A^*} = e^*\Omega^1_{A^*/S}$. Taking $V = \underline{\omega}_{A^*}$, the universal extension is the element in $\mathrm{Ext}(A,V)$ corresponding to $1 \in \mathrm{Hom}(\underline{\omega}_{A^*}, \underline{\omega}_{A^*})$.

In the same paper, Rosenlicht described the relationship between differentials of the 2^{nd} kind and rational cross-sections of the universal extension.

In [27] Weil observed that when working on an abelian variety A over an arbitrary field, consideration of extensions of A by a vector group replaces the study of differentials of the second kind, while consideration of extensions of A by a torus replaces the study of differentials of the third kind. He attributes these ideas (in the classical case) to Severi. Over $\mathbb{C}$, Barsotti in [1 bis] established algebraically the isomorphism $\mathrm{Ext}(A,G_a) \cong H^1(\mathcal{O}_A)$

$$\cong \frac{\text{differentials of second kind}}{\text{holomorphic differentials} + \text{exact differentials}}$$

(See Serre's [24] and [25] for a beautiful account of these ideas).

Another approach to the universal extension is provided by Tate's definition of generalized Picard varieties [26]. He considers the group of divisors on A not containing the zero e, which are algebraically equivalent to zero, modulo the subgroup of principal divisors (f) where $f \equiv 1 \mod \underline{m}_e^2$ ($\underline{m}_e$ = maximal ideal at e). (See also [15 bis]). Both Tate and Lang ask whether this abstract group carries a natural algebraic structure. This algebraic structure was provided by Murre [18] and also by Oort (unpublished).

Grothendieck, more recently, provided still another viewpoint on the universal extension (by means of the theory of group extensions with integrable connections - which he called $\natural$-extensions). In a letter to Tate, Grothendieck announced that the universal extension over $\mathbb{C}$ is crystalline in nature and conjectured that the same is true over any base. In his Montreal lectures he discussed the relation between the universal extension and "the generalized Dieudonné theory" [13].

A discussion of the crystalline nature of the universal extension and applications to the deformation theory of abelian varieties and Barsotti-Tate groups is given in [16] via the theory of the exponential. Previously Cartier had in [5] solved these problems (at least when the base is a perfect field) for p-divisible formal Lie groups. His approach also yielded the result that the Lie algebra of the universal extension is the Dieudonné module (a result which we generalize below).

We shall treat alongside the theory for abelian varieties the corresponding theory for p-divisible (= Barsotti-Tate) groups. Amusingly enough, we repeat the complicated history sketched above.

Thus, in Chapter I §1 we introduce the universal extension (in a more general context, but) in the spirit of Rosenlicht, and Serre. In Chapter I § 2 we identify the universal extension with something we call Extrig (rigidified extensions) which is modelled on Tate's approach.

In Chapter I §3 we identify Extrig with $\mathrm{Ext}^\natural$ ($\natural$-extensions) and thus pass to Grothendieck's.

From $\mathrm{Ext}^\natural$ one may establish the crystalline nature in a lengthy, but straightforward way (c.f. Chapter II), and also pass to a hypercohomological interpretation of the universal extension (Chapter I §4) thereby establishing the link with De Rham cohomology.

In Chapter I §5 we mention some connections between the constructions we have dealt with and the Mordell-Weil group of an elliptic curve over $\mathbb{Q}$.

In chapter II we discuss the crystalline nature of the universal extension, i.e. its relation to "generalized" Dieudonné Theory. The results of §9 and 13 and 15 imply that for A, an abelian variety over a perfect field k (char k = p > 0), and G, its associated p-divisible group, there is a canonical isomorphism between the Dieudonné module of G and the crystalline H^1 of A . The reduction modulo p of this statement was proven by Oda [18 bis].

Throughout this chapter we rely heavily on the work of Berthelot, Grothendieck and Illusie.

We refer the reader to the introduction to Chapter II for more precision on its contents.

OPEN QUESTIONS

a) Give a comparison of our theory of Dieudonné crystals associated to p-divisible formal Lie groups (over S) with Cartier's theory.

b) Find a Dieudonné crystal theory for finite, locally-free p-groups over S_o (a base of characteristic p).

c) Determine whether the functor $G \mapsto \mathbb{D}^*(G)$ on a base S_o , of characteristic p , is fully-faithful.

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